Some problems related to eigenvalues of graphs

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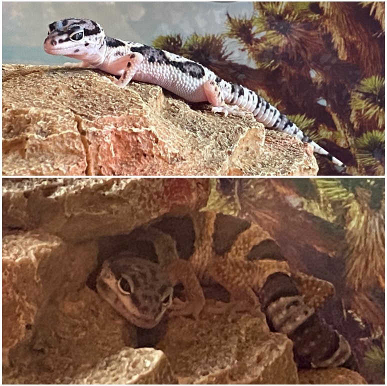
What keeps me up at night

$\lambda_1 = ?$

$\lambda_2 = ?$

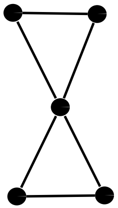
$\lambda_{min} = ?$

What keeps me up at night



Graphs, Adjacency Matrices and Eigenvalues

- G **graph** on n vertices
- The **adjacency matrix** A is an $n \times n$ matrix where $A(x, y)$ equals the number of edges between x and y.
- The **eigenvalues** of A: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{min}$.



$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\frac{1+\sqrt{17}}{2}$, 1, -1, -1, $\frac{1-\sqrt{17}}{2}$

Eigenvalues of Regular Graphs

1. If G is a connected d-regular graph with n vertices, then
$$d = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_{min} \geq -d.$$
2. If G is d-regular, then G is bipartite iff $\lambda_{min} = -d$.

λ_2

Algebraic connectivity, expansion constant, expanders.

λ_{min}

Independence number, chromatic number, max-cut and bipartiteness.

Claw-free graphs

Aharoni, Alon and Berger 2016

$K_{1,3}$ free

If G is a d-regular claw-free, then

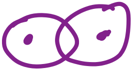
$$d + \lambda_n(G) \geq \frac{t(d,3)}{d-1} \approx \frac{d}{4},$$

$k = 3$, claw-free graphs

Take an even cycle and blow up each vertex into K_s.
We get a d-regular claw-free graph with $d = 3s - 1$ and

$$\lambda_{min} = -s - 1$$

$$d + \lambda_{min} = 2s - 2 \approx \frac{2d}{3}.$$



Close the gap between the bounds and the examples.

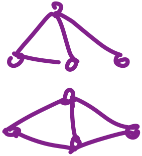
Claw-free graphs

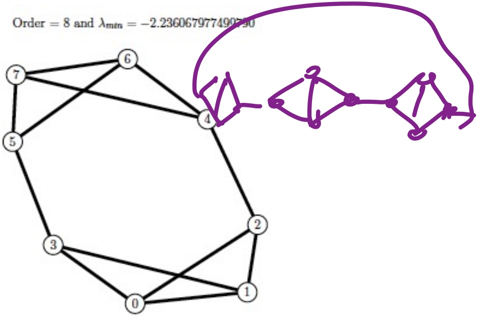
Aharoni, Alon and Berger 2016

If G is a 3-regular claw-free, then $\lambda_{min}(G) \geq -2.5$.

Cioabă, Gregory and Elzinga 2020

$\lambda_{min} \geq -2.272$
The lower bound is the smallest root of $x^3 + x + 14$.





$K_{1,k}$-free graphs

$e(N(x)) \geq t(d,k)$

$\alpha < k$

Aharoni, Alon and Berger 2016

If G is a d-regular graph that contains no induced $K_{1,k}$, then

$$d + \lambda_{min}(G) \geq \frac{t(d,k)}{d-1},$$

The proof does not use $K_{1,k}$-free, but a consequence of it: each vertex is contained in at least $t(d,k)$ triangles.

A generalization

Cioabă 2021

Let K be a k-regular graph and G a d-regular graph. If there exists a collection $\mathcal{K}$ of subgraphs of G isomorphic to K such that

1. each vertex of G is contained in at least m copies of K,
2. each edge of G is contained in at most t copies of K,

then

$\geq t(d,k)$

$$d + \lambda_{min}(G) \geq \frac{m}{t} \cdot \underbrace{(k + \overset{1}{\lambda_{min}(K)})}_{d-1}$$

Aharoni, Alon and Berger:
$K = K_3$, $k + \lambda_{min}(K) = 2 - 1 = 1$, $m = t(d,k)$, $t = d - 1$.

Knox and Mohar 2019: similar results using fractional decomposition of graphs.

A generalization

Cioabă 2021

Let K be a k-regular graph and G a d-regular graph. If there exists a collection $\mathcal{K}$ of subgraphs of G isomorphic to K such that

1. each vertex of G is contained in at least m copies of K,
2. each edge of G is contained in at most t copies of K,

then

$$d + \lambda_{min}(G) \geq \frac{m}{t} \cdot (k + \lambda_{min}(K)).$$

Proof: Take a unit eigenvector x for $\lambda_{min}(G)$.

$$t(d + \lambda_{min}(G)) = t \sum_{ij \in E} (x_i + x_j)^2 \geq \sum_{H \in \mathcal{K}} \sum_{ij \in E(H)} (x_i + x_j)^2$$

$$\geq \sum_{H \in \mathcal{K}} (k + \lambda_{min}(K)) \sum_{\ell \in V(K)} x_\ell^2$$

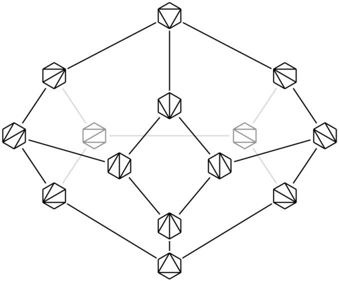
$$\geq (k + \lambda_{min}(K))m.$$

Associahedron graph $\mathcal{A}_n$

Triangulations of a convex n-gon

Tamari lattice 1951, Stasheff polytope 1963

$n = 6$

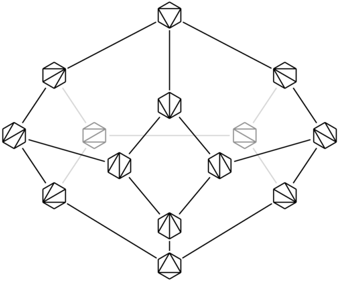


Sleator, Tarjan and Thurston 1988: $\mathrm{diam}(\mathcal{A}_n) \leq 2n - 10$ for $n > 12$
Pournin 2014: $\mathrm{diam}(\mathcal{A}_n) = 2n - 10$ for $n > 12$

Associahedron graph $\mathcal{A}_n$

Fabila-Monroy, Flores-Penaloza, Huemer, Hurtado, Urrutia and Wood 2009:
Conjectures

1. $\chi(\mathcal{A}_n) \to \infty$ as $n \to \infty$.
2. $\chi(\mathcal{A}_n) = O(\log n)$.



Associahedron graph $\mathcal{A}_n$

Addario-Berry, Reed, Scott and Wood 2018

$\chi(\mathcal{A}_n) = O(\log n)$.

1st conjecture evidence ??

$\chi(\mathcal{A}_n) = 3$ for $5 \leq n \leq 9$; $\chi(\mathcal{A}_{10}) = 4$.

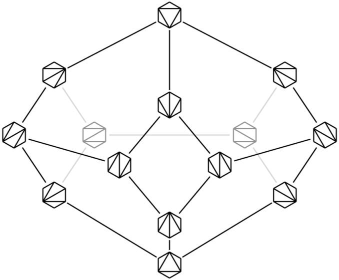
Associahedron graph $\mathcal{A}_n$

Hoffman

$$\chi(G) \geq 1 + \frac{\lambda_1(G)}{|\lambda_{min}(G)|} \Rightarrow \chi(\mathcal{A}_n) \geq 1 + \frac{n-3}{|\lambda_{min}(\mathcal{A}_n)|}.$$

Is $|\lambda_{min}(\mathcal{A}_n)| = o(n)$?

Associahedron graph $\mathcal{A}_n$



$\mathcal{A}_n$

Each vertex is in at least $n-4$ cycles C_5.

Each edge is in at most 4 cycles C_5.

$\lambda_{min}(\mathcal{A}_n) \geq \frac{-5-\sqrt{5}}{8}(n-3) - \frac{3-\sqrt{5}}{8}$.

Associahedron graph $\mathcal{A}_n$

$\mathcal{A}_n$

Computations

$n-3$	2	3	4	5	6	7	8	9
λ_{min}	-1.618	-2.414	-3.177	-3.912	-4.667	-5.409	-6.157	-6.904

Cauchy interlacing and Fekete's lemma

Fix a diagonal: $\lambda_{min}(\mathcal{A}_{k+\ell}) \leq \lambda_{min}(\mathcal{A}_k) + \lambda_{min}(\mathcal{A}_\ell)$.

$-0.6904 \geq \lim_{n\to\infty} \frac{\lambda_{min}(\mathcal{A}_n)}{n-3} \geq -0.9045$.

What is

$$\lim_{n\to\infty} \frac{\lambda_{min}(\mathcal{A}_n)}{n-3}?$$

Associahedron graph $\mathcal{A}_n$

Conjecture (Aldous)

$$n - 3 - \lambda_2(\mathcal{A}_n) = \Theta(n^{-1/2})$$

From: David ALDOUS
To: Cioaba, Sebastian

Well it's just this back-of-envelope calculation that I never bothered to write more carefully.
The general lower bound for relaxation time \tau is that for any functional X of the chain

\tau \ge var(X) / E[(X_1-X_0)^2]

Apply this to X = size of a given branch from the centroid.
It's known that X/n has limit density on [0,1] as n \to \infty, so
(in estimates below, = means order of magnitude)

(a) var(X) = n^2.

A move only changes X if it involves an edge at the centroid, chance = 1/n.
When this happens the change in X is the size Y of a typical branch, which is

P(Y = i) = i^{-3/2} for i = o(n) , so E[Y^2] = n^{3/2}
and then

(b) E[(X_1-X_0)^2] = 1/n \times n^{3/2} = n^{1/2}

Regards
David A

Aldous's Conjecture

Theorem (Caputo, Liggett, Richthammer 2010)

If T is a set of transpositions in S_n and H is the graph with vertex set $[n]$ and edge set T, then the spectral gap of $Cay(S_n, T)$ equals the algebraic connectivity of H.

Examples

1. $T_1 = \{(i,j) : 1 \leq i < j \leq n\}$, $Cay(S_n, T_1)$ has $\lambda_2 = \binom{n}{2} - n$.
2. $T_2 = \{(1,k) : 2 \leq k \leq n\}$, $Cay(S_n, T_2)$ has $\lambda_2 = n - 2$.
3. $T_3 = \{(j, j+1) : 1 \leq j \leq n-1\}$, $Cay(S_n, T_3)$ has $\lambda_2 = n - 1 - 4\sin^2\left(\frac{\pi}{2n}\right)$.

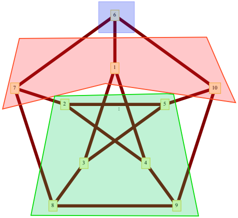
Equitable Partitions

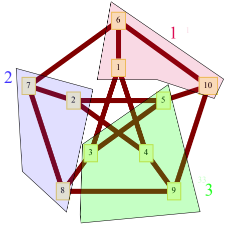
Definition

Let G be a graph with vertex set V. A partition $V = X_1 \cup \cdots \cup X_r$ is called **equitable** if there exist numbers $b_{j,\ell}, 1 \le j, \ell \le r$ such that

$$\forall j, \ell \in \{1, \ldots, r\}, \forall x \in V_j, |N(x) \cap V_\ell| = b_{j,\ell}.$$

The matrix $B = (b_{j,\ell})_{1 \le j,\ell \le r}$ is **the quotient matrix** of the partition.





Equitable Partitions

Proposition

If G has an equitable partition, then the eigenvalues of the quotient matrix are eigenvalues of the adjacency matrix of G.
The eigenvectors of $A(G)$ are either:

1. *constant on the cells of the partition and the corresponding eigenvalues are the eigenvalues of the quotient matrix.*
2. *orthogonal to the characteristic vectors of the sets in the equitable partition.*

Generalized Pancake graphs

Pancake graph $\mathcal{P}_n$

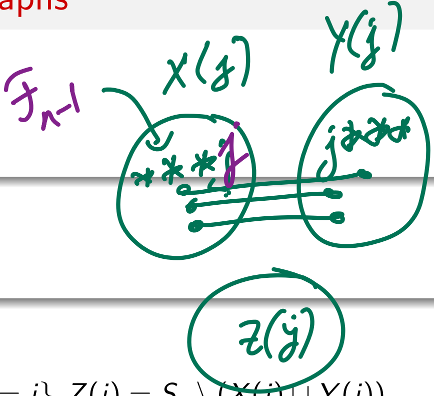
Vertices: the permutations in S_n.
$\sigma \sim \tau$ iff there is $k \geq 2$:
$(\tau_1, \tau_2, \dots, \tau_k, \tau_{k+1}, \dots, \tau_n) = (\sigma_k, \sigma_{k-1}, \dots, \sigma_1, \sigma_{k+1}, \dots, \sigma_n)$.

Generalized Pancake graphs $\mathcal{F}_n$

Chung and Tobin 2017: for each $k \geq 2$, σ has exactly one neighbor $(\sigma_k, \alpha_2, \dots, \alpha_{k-1}, \sigma_1, \sigma_{k+1}, \dots, \sigma_k)$.

Examples of graphs $\mathcal{F}_n$

Pancake graphs $\mathcal{P}_n$
$Cay(S_n, \{(1, k) : 2 \leq k \leq n\})$ (Flatto, Odlyzko, Wales 1985).



Cesi 2009

$\lambda_2(\mathcal{P}_n) = n-2.$

Chung and Tobin 2017

$\lambda_2(\mathcal{F}_n) = n-2.$

Equitable Partition of $\mathcal{F}_n$

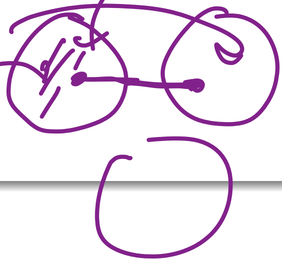
$X(j) = \{\tau_n = j\}, Y(j) = \{\tau_1 = j\}, Z(j) = S_n \setminus (X(j) \cup Y(j)).$

$$B = \begin{bmatrix} n-2 & 1 & 0 \\ 1 & 0 & n-2 \\ 0 & 1 & n-2 \end{bmatrix}, \text{ eigs: } n-1, n-2, 1.$$

$\lambda_2(\mathcal{F}_n) \geq n-2.$

Generalized Pancake graphs

$X(j)$ $\mathcal{F}_{n-1}$ $Y(j)$



f sums to 0

Chung and Tobin 2017

$\lambda_2(\mathcal{F}_n) = n - 2.$

$\lambda_2(\mathcal{F}_n) \leq n - 2$

Induction on n: $n = 3$ $\mathcal{F}_3 = C_6, \lambda_2(C_6) = 1$.
Take an eigenvalue/eigenvector $Af = \mu f$ such that f sums to zero on each $X(j)$ and $Y(j)$.
Prove $\mu \leq \underline{\lambda_2(X(j))} + 1 = n - 3 + 1 = n - 2$.

$\downarrow$

$\lambda_2(\mathcal{F}_{n-1})$

Pancake graphs

$\lambda_{min}(\mathcal{P}_n) =?$
$n = 3 \ : -2$
$n = 4 \ : -2.56$
$n = 5 \ : -3.29$
$n = 6 \ : -4.0033.$

Reversal graphs

Reversal graphs R_n

Vertices: the permutations in S_n
Edges: $\sigma \sim \tau$ if one obtained from the other by substring reversal: $(\sigma_1, \dots, \sigma_n)$ is adjacent to $(\tau_1, \dots, \tau_n)$ if there exist $1 \leq i < j \leq n$ such that
$(\tau_1, \dots, \tau_{i-1}, \tau_i, \dots, \tau_j, \tau_{j+1}, \dots, \tau_n) = (\sigma_1, \dots, \sigma_{i-1}, \sigma_j, \dots, \sigma_i, \sigma_{j+1}, \dots, \sigma_n)$.
R_n is $\binom{n}{2}$-regular.

Bafna and Pezner 1996

$\mathrm{diam}(R_n) = n - 1$.

Chung and Tobin 2017

$\lambda_2 = \binom{n}{2} - n$.

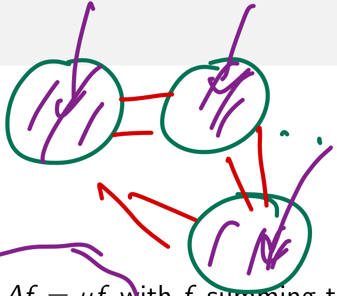
Reversal graphs

$\lambda_2(R_n) \geq \binom{n}{2} - n$

For $1 \leq i, j \leq n$, $U_i(j) = \{\tau_i = j\}$.
For given j, the partition $U_1(j) \cup \cdots \cup U_n(j)$ is equitable
The quotient matrix has eigs $\binom{n}{2} - \lfloor k/2 \rfloor n + 2\binom{\lfloor k \rfloor/2\rfloor}{2}, 1 \leq k \leq n$.
$k = 2, 3$ gives $\binom{n}{2} - n$.

Reversal graphs

$\mu \leq \binom{n}{2} - n$



$\lambda_2(R_n) \leq \binom{n}{2} - n$

Induction on n.
Take eigenvector/eigenvalue pair $Af = \mu f$ with f summing to zero on each $U_i(j)$.
Take the partition $U_1(1), U_2(1), \ldots, U_n(1)$, where $U_1(j) = \{\tau_1 = j\}$.
Each subset $U_1(j)$ induces R_{n-1}.
The remaining subgraph of R_n with edges $\{\sigma\tau : \sigma_1 \neq \tau_1\}$ is the pancake graph $\mathcal{P}_n$.

Open Problems

$\lambda_{min} = ?$.
Total variation distance bounds.
Chung-Tobin conjecture $O(n \log n)$ bound for $\mathcal{F}_n$.

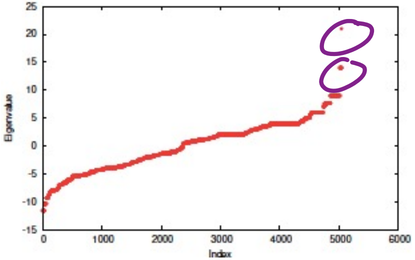


Figure: Eigenvalues of R_7

Followup

Huang, Huang, Cioabă 2019

The equitable partition into left cosets of stabilizer of a point works for other Cayley graphs of S_n for finding λ_2.
Union of conjugacy classes that move at most 5 points.

$C_2 = \{(i,j) : 1 \leq i < j \leq n\}$, $Cay(S_n, C_2)$ has $\lambda_2 = \binom{n}{2} - n$.
$C_3 = \{(i,j,k) : 1 \leq i,j,k \leq n\}$, $Cay(S_n, C_3)$ has $\lambda_1 = \lambda_2 = 2\binom{n}{3}$.
$Cay(S_n, C_2 \cup C_3)$ has $\lambda_1 = n(n-1)(2n-1)/6, \lambda_2 = n(n-1)(2n-7)/6$.

It does not work for λ_{min}.

Signed reversal graphs

The signed reversal graph SR_n

Vertices: the $2^n \cdot n!$ signed permutations of $\{1, \ldots, n\}$
Edges: $(\sigma_1, \ldots, \sigma_n)$ is adjacent to $(\tau_1, \ldots, \tau_n)$ if there exist $1 \leq i \leq j \leq n$ such that

$$(\tau_1, \ldots, \tau_{i-1}, \tau_i, \ldots, \tau_j, \tau_{j+1}, \ldots, \tau_n) = (\sigma_1, \ldots, \sigma_{i-1}, -\sigma_j, \ldots, -\sigma_i, \sigma_{j+1}, \ldots, \sigma_n).$$

$\mathrm{SR}(n)$ is regular with degree $\binom{n+1}{2}$.

Signed reversal graphs

$\chi(SR_n) \leq n$

Chromatic number

n	1	2	3	4	5	6
$\chi(\mathrm{SR}_n)$	2	2	3	4	4	≤ 5

Table: Chromatic number of SR_n for $n \leq 6$

The behavior of $\chi(\mathrm{SR}_n)$ for larger n ?
Independence number of $\cancel{\chi}(\mathrm{SR}_n)$?

Signed reversal graphs

Equitable partition

For $\pi \in S_n$, $V_\pi = \{\pi_1^{\epsilon_1}, \dots, \pi_n^{\epsilon_n}, \epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$.
The partition into the $n!$ sets V_π has quotient matrix

$$nI_{n!} + A(R_n).$$

Another equitable partition

For $1 \le j \le n$, $U_j(+) = \{\tau_j = +n\}, U_j(-) = \{\tau_j = -n\}$. Quotient matrix $2n \times 2n$

$\binom{n}{2}$ appears in both spectra.

Signed reversal graphs

Conjecture (Cioabă, Royle and Tan 2020)

$\lambda_2(\mathrm{SR}_n) = \binom{n}{2}$.

Known

True for $n \geq 5$
$\lambda_2(\mathrm{SR}_n) \geq \binom{n}{2}$ for any n.

Pipedreams

Eigenvalue proof of Hilton-Milner.

The end: no elephants, no zebras