What is a state?

1) Density matrix D

$$D \succeq 0, \quad tr(D) = 1$$

pure state
$rk(D)=1$

vs mixed state

e.g $\|z\| = 1$, $D = zz^*$

2) D density $\Rightarrow$ $D = \sum_r w_r z_r z_r^*$

$\|z_r\| = 1$
$w_r \geq 0, \sum_r w_r = 1$

3) $D_a := e_a e_a^T$ vertex state

3) **2x2 density matrices:**

2x2 Hermitian, $tr(H)=1$: $H = \begin{pmatrix} \frac{1}{2}+a & b+ic \\ b-ic & \frac{1}{2}-a \end{pmatrix}$ $\quad a,b,c \in \mathbb{R}$

$\det(H) = \frac{1}{4} - a^2 - b^2 - c^2$

Then $H \succeq 0 \iff a^2+b^2+c^2 \leq 1.$

Pure states map to points on ~~unit~~ 1/2 sphere in $\mathbb{R}^3$

Measurement: $M_1, \dots, M_d$: $M_r \succeq 0$, $\sum M_r = I$

outcome r with prob. $\langle D, M_r \rangle$

standard basis $e_r e_r^T$: prob $\langle D, e_r e_r^T \rangle = D_{r,r}$

We can represent the outcome of a measurement of D using $M_1, \dots, M_k$ by a diagonal matrix:

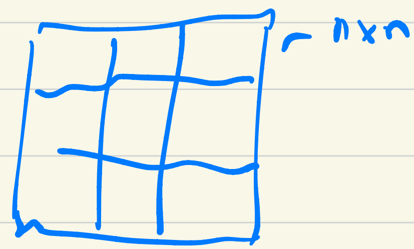
$$\sum_r \langle D, M_r \rangle e_r e_r^T = D \circ I$$

This is a density matrix.

Channels Let $\mathcal{M}_n$ denote $\mathrm{Mat}_{n\times n}(\mathbb{C})$. A **quantum channel** is a linear map $\Phi : \mathcal{M}_n \to \mathcal{M}_d$ which maps density matrices to density matrices. So Φ is

(a) trace preserving

(b) completely positive



Theorem If Φ is a channel from $\mathcal{M}_n \to \mathcal{M}_d$, there are $d\times n$ matrices $A_1, \ldots, A_k$ such that

$$\Phi(D) = \sum_i A_i D A_i^* \qquad \left(\sum_i A_i^* A_i = I\right)$$

Examples:

(a) $D \to UDU^{-1}$, U unitary

(b) $D \to \sum_r \langle D, M_r \rangle e_r e_r^T$

(c) convex combinations of channels

(d) any linear map $\Psi: M_n \to M_d$ has an adjoint $\Psi^*: M_d \to M_n$. If Ψ is a channel & $\Psi(I) = I$, then Ψ^* is a channel.

Orbits

density D

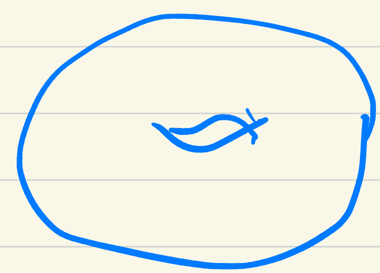
$$D(t) := U(t) D U(-t)$$

$$\mathcal{U} = \{ U(t) : t \in \mathbb{R} \} \leq U_d(\mathbb{C})$$

$\{ D(t) : t \in \mathbb{R} \}$ is an orbit of $\mathcal{U}$.

pst: D_2 in orbit of D_1

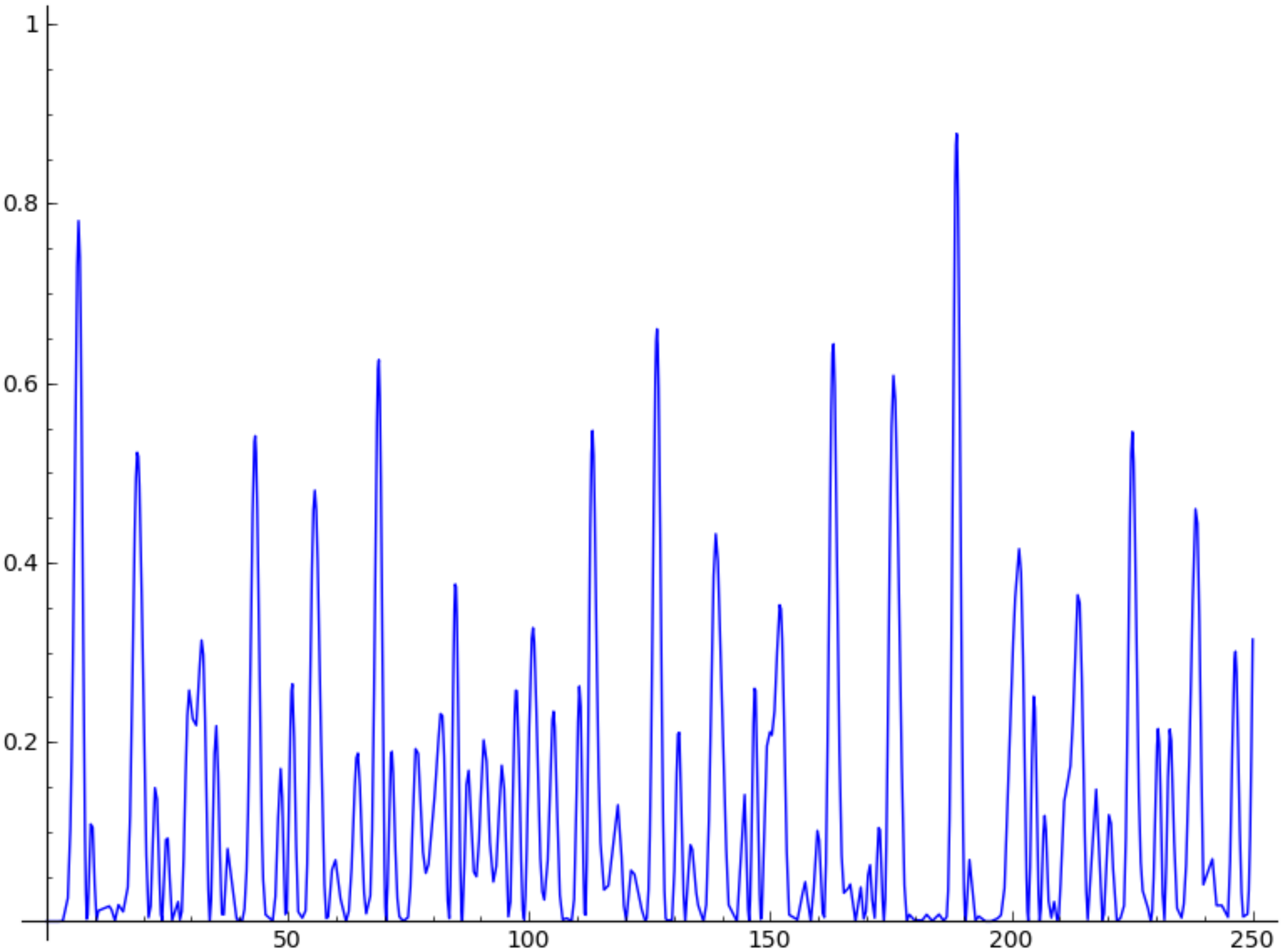
psgt: D_2 in closure of orbit of D_1



$$D(t) = \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r D E_s$$

$$E_k D(t) E_\ell = e^{it(\theta_k - \theta_\ell)} E_k D E_\ell$$

- D & $D(t)$ real
  $\Rightarrow e^{it(\theta_r - \theta_s)}$ real
  $\Rightarrow \sin t(\theta_r - \theta_s) = 0$
- $E_k D(t) E_\ell = \pm E_k D E_\ell$



$M(t)_{1,11}$ for P_{11}

Recurrence

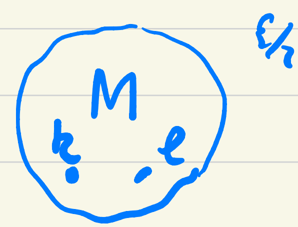
$\mathcal{U}_1 = \{ U(k) : k \in \mathbb{Z} \}$ — sequence of pts in a compact space

Two cases:

(a) $\mathcal{U}_1$ finite, walk is periodic

(b) $\mathcal{U}_1$ infinite, sequence has a limit point. M say.

$\| U(k) - U(l) \| < \epsilon$



$\| U(l-k) - I \| < \epsilon$

almost periodic

If D is a density and U is unitary, the sequence $U^k D U^{-k}$ is either periodic, or has a limit point, M say.

So there exist k, l with $\|D(k) - M\|, \|D(l) - M\| < \varepsilon$

$$\Rightarrow \|D(k) - D(l)\| < \varepsilon \Rightarrow \|D(k-l) - D\| < \varepsilon$$

almost periodic

Averages

$$\Psi(D) := \lim_{T\to\infty} \frac{1}{T}\int_0^T D(t)\,dt$$ average state

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{r,s} e^{it(\theta_r-\theta_s)} E_r D E_s \,dt$$

$$= \sum_r E_r D E_r$$

(Discrete: $\Psi(D) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} U^k D U^{-k} = \sum_r F_r D F_r$)

Averages are projections

Average states for K_n

$$E_1 = \frac{1}{n} J, \qquad E_2 = I - \frac{1}{n} J$$

$$E_1 e_a e_a^T E_1 = \frac{1}{n^2} J$$

$$E_2 e_a e_a^T E_2 = \left(I - \frac{1}{n} J\right) e_a e_a^T \left(I - \frac{1}{n} J\right)$$

$$= e_a e_a^T - \frac{1}{n} \mathbf{1} e_a^T - \frac{1}{n} e_a \mathbf{1}^T + \frac{1}{n^2} J$$

$$\boxed{\Psi(e_a e_a^T) = e_a e_a^T + O\left(\tfrac{1}{n}\right)}$$

- D_1 & D_2 in same orbit $\Rightarrow \Psi(D_1) = \Psi(D_2)$
- D_2 in closure of D_1-orbit $\Rightarrow \bar{\Psi}(D_1) = \Psi(D_2)$
- $\Psi(D)$ is a density matrix
- Ψ is a quantum channel.
- If D is rational, so is $\bar{\Psi}(D)$

Eigenvalues of $\Psi(D_a)$

- $D_a = e_a e_a^T$

$$\Psi(D_a) = \sum_r E_r e_a e_a^T E_r$$

- $$(E_r e_a e_a^T E_r)^2 = E_r e_a \underbrace{e_a^T E_r^2 e_a}_{(E_r)_{a,a}} e_a^T E_r$$

$\Rightarrow$ $\frac{1}{(E_r)_{aa}} E_r e_a e_a^T E_r$ is idempotent

$(E_r)_{aa}$ is eigenvalue of $\Psi(D_a)$

spectral decomp: $\Psi(D_a) = \sum_r (E_r)_{aa} \cdot \frac{1}{(E_r)_{aa}} E_r e_a e_a^T E_r$

Corollaries

(a) a & b cospectral $\iff$ $\psi(D_a)$ & $\psi(D_b)$ are cospectral

(b) a & b strongly cospectral $\iff$ $\psi(D_a) = \psi(D_b)$

Projections

- $D_1 \mapsto \Psi(D_1)$ is linear

- maps Hermitian matrices onto commutant of A

- self-adjoint

$$\langle D_1, \Psi(D_2) \rangle = \mathrm{tr}\left(D_1 \sum E_r D_2 E_r\right) = \sum_r \mathrm{tr}(D_1 E_r D_2 E_r)$$

$$= \sum_r \mathrm{tr}(E_r D_1 E_r D_2)$$

$$= \langle \Psi(D_1), D_2 \rangle$$

$$D = \Psi(D) \Leftrightarrow D(t) = D \ \forall t \Leftrightarrow DA = AD$$

$$D(t) - \Psi(D) = U(t)\big(D - \Psi(D)\big)U(-t)$$

$$\Rightarrow \| D(t) - \Psi(D) \| = \| D - \Psi(D) \|$$

Mixing matrix

$$M(t) := U(t) \circ \overline{U(t)} = U(t) \circ U(-t)$$

$$= \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r \circ E_s$$

$M(t)$ is doubly stochastic

$M(t)_{a,b} = 1 \Rightarrow ab\text{-pst}$

$M(t) = \frac{1}{d} J \Rightarrow$ uniform mixing average mixing

$$\hat{M} := \lim_{T\to\infty} \frac{1}{T}\int_0^T M(t)\,dt$$

$$= \sum_r E_r \circ E_r$$

K_d:

$$E_1 = \frac{1}{d}J, \quad E_2 = I - \frac{1}{d}J$$

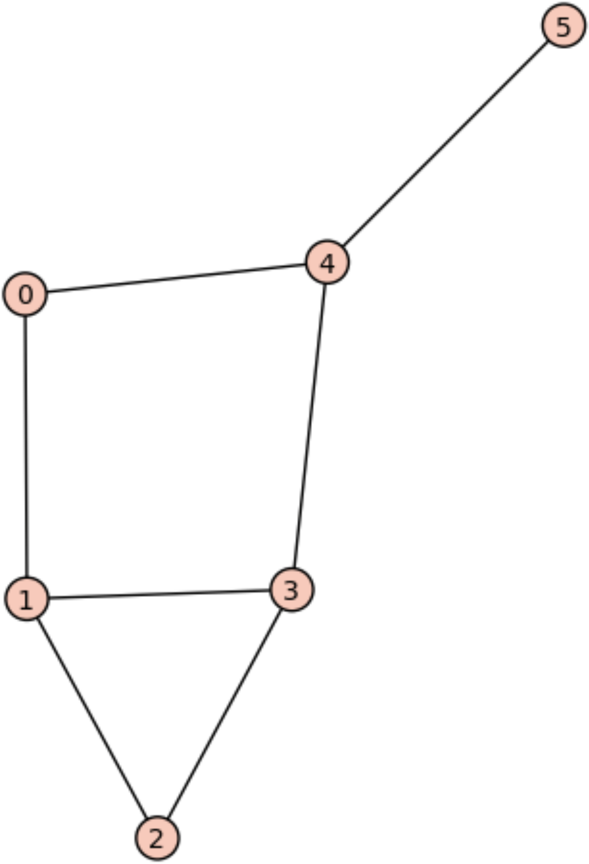
$$E_1^{\circ 2} = \frac{1}{d^2}J \qquad E_2^{\circ 2} = \left(1 - \frac{2}{d}\right)I + \frac{1}{d^2}J$$

$$\boxed{\hat{M}_{K_d} = I - \frac{2}{d}I + \frac{2}{d^2}J}$$

If n is odd, the average mixing matrix of C_n is

$$\frac{1}{n} I + \frac{n-1}{n^2} J$$

Theorem $\hat{M}$ is rational.


5
4
0
1
3
2

$$\begin{pmatrix} 0.355121 & 0.139699 & 0.106773 & 0.162288 & 0.117103 & 0.119016 \\ 0.139699 & 0.197285 & 0.204627 & 0.178789 & 0.160203 & 0.119397 \\ 0.106773 & 0.204627 & 0.313482 & 0.131082 & 0.123480 & 0.120557 \\ 0.162288 & 0.178789 & 0.131082 & 0.210383 & 0.159043 & 0.158414 \\ 0.117103 & 0.160203 & 0.123480 & 0.159043 & 0.258548 & 0.181622 \\ 0.119016 & 0.119397 & 0.120557 & 0.158414 & 0.181622 & 0.300994 \end{pmatrix}$$

Theorem $I \succeq M(t) \succeq 2\hat{M} - I$

$\hat{M}_{aa} > \frac{1}{2} \Rightarrow M(t)_{a,a} > 0 \ \forall t$

Proof,

$$M(t) = \sum_r E_r^{\circ 2} + \sum_{r<s} \left(e^{it(\theta_r - \theta_s)} + e^{-it(\theta_r - \theta_s)}\right) E_r \circ E_s$$

$$M(t) = \hat{M} + 2 \sum_{r<s} \cos t(\theta_r - \theta_s) \, E_r \circ E_s$$

$$I = \hat{M} + 2 \sum_{r<s} E_r \circ E_s$$

$$\Rightarrow I - M(t) \succeq 0, \quad I + M(t) \succeq 2\hat{M}$$

vertex state $D_u = e_u e_u^T$

$$\langle \Psi(D_u), \Psi(D_v)\rangle = \langle D_u, \Psi(D_v)\rangle$$

$$= \sum_r \mathrm{tr}\left(e_u \underbrace{e_u^T \epsilon_r e_v} e_v^T \epsilon_r\right)$$

$$= \sum_r (\epsilon_r)_{uv}^2$$

$$= \left(\sum_r \epsilon_r \circ \epsilon_r\right)_{u,v}$$

$$\doteq \hat{M}_{u,v}$$

Traces

$$\operatorname{tr}(\hat{M}) = \sum_r \operatorname{tr}(E_r \circ E_r)$$

$$\operatorname{tr}(E_r) = \sum_{a \in V} (E_r)_{a,a} = m_r$$

$$\operatorname{tr}(E_r \circ E_r) = \sum_{a \in V} (E_r)_{a,a}^2$$

$$\operatorname{tr}(E_r \circ E_r) \geq \frac{m_r^2}{n}$$

$$\operatorname{tr}(\hat{M}) \geq \frac{1}{n} \sum_r m_r^2 \geq 1$$

- right here $\Leftrightarrow$ X is walk regular
- $\operatorname{tr}(\hat{M}) = 1 \Rightarrow$ walk regular, all eigenvalues simple

  $\Rightarrow X = K_2$

Suppose $\mathrm{rk}(\hat{M}) = \underline{1}$. Then $\hat{M} = xx^T$ and

as $\hat{M}\underset{\sim}{1} = \underset{\sim}{1}$, we have $x_a \cdot x^T 1 = 1$.

Thus $x = \frac{1}{\sqrt{n}} \underset{\sim}{1}$ and hence $|V(x)| \leq 2$.

Entries of $\hat{M}$:

$$\Psi(D_a)_{uu} = \langle \psi(D_a), e_u e_u^T \rangle$$

$$= \langle \psi^2(D_a), e_u e_u^T \rangle$$

$$= \langle \psi(D_a), \psi(D_u) \rangle$$

$$= \hat{M}_{a,u}$$

? Is there a graph on more than two vertices such that $\hat{M}$ has a constant row?

?

Are there infinitely many graphs X such that $\mathrm{rk}(\hat{M}) = 2$?

The End(s)