AGT&QC

Assignment 3

1. Prove or disprove: if π is an equitable partition of X, it is an equitable partition for $\overline{X}$.

2. Show that if X is a controllable graph, the only equitable partition of X is the discrete partition (with all cells of size one).

3. Let π be an equitable partition of the graph X with m cells. Let M be the characteristic matrix of π and set $B = A/\pi$. Show that there is a basis for $\mathbb{R}^{V(X)}$ with respect A has the form

$$A = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}.$$

   Deduce that the characteristic polynomial of B divides that of A.

4. Let π be an equitable partition of a graph X with characteristic matrix M. Let A be the adjacency matrix of X and let B be the quotient A/π, i.e., $AM = MB$. Show that each eigenvector of A in $\operatorname{col}(M)$ determines an eigenvalue of B with the same eigenvalue.

5. If π is a partition of $V(X)$ and let $F(\pi)$ denote the space of functions on $V(X)$ that are constant on the cells of π. Show that if π and ρ are equitable partitions of X, the subspaces

$$F(\pi) \cap F(\rho), \qquad F(\pi) + F(\rho).$$

   Show that these subspaces may be expressed as $F(\sigma)$ and $F(\tau)$ (for suitable equitable partitions σ and τ, and express these partition in terms of π and ρ. [Look up 'meet' and 'join'.]

6. If $\mathcal{P} = \{P_1, \ldots, P_m\}$ and $\mathcal{Q} = \{Q_1, \ldots, Q_n\}$ are projective measurements on X and Y respectively, prove that $\mathcal{P} \otimes \mathcal{Q}$ is projective measurement on $X \times Y$.

7. Prove that $\mathcal{M}(X, d) \times \mathcal{M}(Y, e)$ is isomorphic to a subgraph of $\mathcal{M}(X \times Y, de)$.

8. Show that the distance partition with respect to a vertex of a strongly regular graph is equitable.

9. Prove that the direct sum $\mathcal{P} \oplus \mathcal{Q}$ of quantum homomorphisms $\mathcal{P}$ and $\mathcal{Q}$ is a quantum homomorphism.

10. Prove that the coproduct $\mathcal{P} \star \mathcal{Q}$ of quantum homomorphisms $\mathcal{P}$ and $\mathcal{Q}$ is a quantum homomorphism. Show further that it is classical if $\mathcal{P}$ and $\mathcal{Q}$ are.

11. Prove that the coproduct of quantum homomorphisms is associative.

12. Assume R is doubly stochastic, and determines a directed graph X. Is it true that R and RR^T give rise to the same partition of $V(X)$ into strong components?

13. Show that if $X \xrightarrow{q} K_2$, then X is bipartite.

14. Show that a 3×3 unitary derangement is a monomial matrix, i.e., has exactly one non-zero entry in each row and in each column. Deduce that $\chi_q^{(1)}(X) = 3$, then $\chi(X) = 3$.

15. Let W be a flat unitary matrix of order $c \times c$. Let $P_1, \ldots, P_c$ be projections summing to I_d and define unitary matrices $U_1, \ldots, U_c$ by
$$U_i := \sqrt{c} \sum_{j=1}^{c} W_{i,j} P_j.$$
Prove that $\sum_{i=1}^{c} P_i \otimes P_i = \frac{1}{c} \sum_{i=1}^{c} U_i \otimes U_i^{-1}$.

16. Let $\Omega(d)$ be the orthogonality graph on the unit vectors in $\mathbb{C}^d$ and let $\Phi(d)$ be the orthogonality graphs on the 1-dimensional subspaces of $\mathbb{C}^d$. Prove these two graphs are homomorphically equivalent.

17. Prove that $\chi_{sv}(K_n) \leq n$.

18. Prove that $\omega(X) \leq \chi_v(X)$.

19. If X is the graph of an $n \times n$ Latin square and $n \geq 3$, prove that $\omega(X) = n$.

20. [withdrawn]

21. [withdrawn]

22. [withdrawn]

23. Suppose $\mathcal{P} : X \to Y$ and $\mathcal{Q} : Y \to Z$ are quantum homomorphisms of indices d and e respectively, and that C and D are density matrices of orders $d \times d$ and $e \times e$ respectively. Prove that

$$\langle \mathcal{P} \star \mathcal{Q}, C \otimes D \rangle = \langle \mathcal{P}, C \rangle \langle \mathcal{Q}, D \rangle.$$

24. Assume W is a flat unitary matrix and D_1 and D_2 are diagonal matrices, all of order $m \times m$. Prove that $\langle D_1, W^* D_2 W \rangle = \frac{1}{m} \operatorname{tr}(D_1^*) \operatorname{tr}(D_2)$.

25. [withdrawn]

26. [withdrawn]

27. Assume that the eigenvalues of X are $\theta_1, \ldots, \theta_m$ and that Y is regular with eigenvalues $\sigma_1, \ldots, \sigma_n$. (So $m = |V(X)$ and $n = |V(Y)|$.) Determine the eigenvalues of the homomorphic product $X \ltimes Y$.

28. Prove that a commutative coherent algebra is homogeneous.

29. Assume W is invertible and Schur-invertible. Prove that W is a type-II matrix if and only if $J \in \mathcal{N}_W$.

30. Let V be the $n \times n$ Vandermonde matrix (with ij-entry $\theta^{(i-1)(j-1)}$). Prove that $\mathcal{N}_V$ is the algebra of all polynomials in the permutation matrix of an n-cycle.

31. If W_1 and W_2 are type-II matrices, prove that

$$\mathcal{N}_{W_1 \otimes W_2} = \mathcal{N}_{W_1} \otimes \mathcal{N}_{W_2}.$$