AGT&QC

Assignment 2

1. If A and B are non-zero complex matrices, prove that $A \otimes B$ is flat if and only if A and B are flat.

2. Prove that if X and Y admit uniform mixing at time τ, then $X \square Y$ admits uniform mixing at time τ.

3. Prove that the diameter of $X \square Y$ is the sum of the diameters of X and Y.

4. Prove that $M_{X \square Y} = M_X \otimes M_Y$.

5. Show that K_3 and C_4 admit uniform mixing. [Note that C_4 is a Cartesian product.]

6. Prove that if the permutation matrix P commutes with A, it commutes with $U(t)$ for all t. Hence show that if P commutes with A and $Pe_a = e_a$ and there is perfect state transfer from a to b, then $Pe_b = e_b$. [Hence any automorphism of X that fixes a must also fix b.] Use this to show that K_n does not admit perfect state transfer when $n \geq 3$.

7. If E is a spectral idempotent of the bipartite graph X corresponding to the eigenvalue θ, show that
$$\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} E \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$
is the spectral idempotent corresponding to $-\theta$.

8. Show that if an $n \times n$ Hadamard matrix exists, either $n = 2$ or $4|n$.

9. Assume H is a complex Hadamard matrix with entries from $\{\pm 1, \pm i\}$. There are unique matrices H_0 and H_1 with entries from $\{0, \pm 1\}$ such that $H_0 \circ H_1 = 0$ and $H = H_0 + iH_1$. Show that the matrix
$$\begin{pmatrix} H_0 + H_1 & H_0 - H_1 \\ -H_0 + H_1 & H_0 + H_1 \end{pmatrix}$$
is a Hadamard matrix, and deduce that the order of H is even.

10. Let H be an $n \times n$ complex Hadamard matrix, and let S_i denote the sum of the entries in the i-th row of H.

    (a) Prove that $\sum_i |S_i|^2 = n^2$.

    (b) Use the inequality

    $$\frac{1}{n}\left(\sum_{i=1}^{n} x_i^2\right) \geq \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$$

    prove that $\sum_i |S_i| \leq n\sqrt{n}$ and that equality holds if and only if $|S_i| = \sqrt{n}$ for all i.

    (c) Prove that if $|S_i|$ is the same for all i, then n is the sum of two integer squares.

11. Determine the mixing matrix for $K_{m,n}$.

12. Let A be an $n \times n$ Hermitian matrix with spectral decomposition $A = \sum_r \theta_r E_r$. If $z \in \mathbb{C}^n$, show that the dimension of the space W spanned by

    $$\{A^k z : k \geq 0\}$$

    is equal to the size of the eigenvalue support of z, i.e., to the size of the set

    $$\{\theta_r : E_r z \neq 0\}.$$

    Show further that the eigenvalues of the 'restriction' of A to W are all simple.

13. Let X and Y be graphs with disjoint vertex sets and assume $a \in V(X)$ and $b \in V(Y)$. Let Z be the graph we get by adding an edge joining a to b. Prove that $\phi(Z,t) = \phi(X,t)\phi(Y,t) - \phi(X \setminus a, t)\phi(Y \setminus b, t)$.

14. If H is an $n \times n$ Hermitian matrix, $\mathbb{C}^n$ has an orthogonal basis consisting of eigenvectors of H. Using this, show that same is true when H is normal.

15. Assume we have perfect state transfer from vertex a in X to vertex b at a time τ. Let $A = A(X)$. Prove that

    $$AU(\tau)e_a e_a^T U(-\tau)A = Ae_b e_b^T A$$

    and, from this, deduce that a and b have the same valency.

16. Assume $S \subseteq \{1, \ldots, n\}$ and let D_S be the diagonal 01-matrix with $(D_S)_{u,u} = 1$ if and only if $u \in S$. Then $|S|^{-1} D_S$ is a density matrix. (We call it a subset state.) Show that if we have perfect state transfer between two subset states, the corresponding subsets have the same size.

17. Assume that $A = A(X)$ and $B = B(X)$ have spectral decompositions
$$A = \sum_r \theta_r E_r, \quad B = \sum_s \tau_s F_s$$
If X and Y are walk-regular, show that $X \times Y$ and $X \,\square\, Y$. are walk-regular. [Hint: as a first step show that $A \otimes B$ can be expressed as a linear combination of idempotents $E_r \otimes E_s$.]

18. Show that the complement of a walk-regular graph is walk-regular.

19. Define the support $\sigma(D)$ of the density matrix D to be the set of indices a such that $D_{a,a} \neq 0$. If $a \in V(X)$, use D_a to denote the vertex state $e_a e_a^T$. Prove that if D_S is a subset state, the support of $U(t) D_S U(-t)$ contains the support of $U(t) D_a U(-t)$ for each a in S.

20. Show that a connected regular bipartite graph with at most five distinct eigenvalues is walk-regular.

21. Suppose a and b are strongly cospectral vertices in a strongly regular graph and let Q be an orthogonal matrix such that:
    (a) Q is a rational polynomial in A.
    (b) $Qe_a = e_b$.
    (c) $Q^2 = I$.

    Show that Q is a linear combination of I, $A(X)$ and $A(\overline{X})$, and then deduce that X or $\overline{X}$ is isomorphic to mK_2 for some m.

22. If X is a graph, we have seen that
$$\phi'(X, t) = \sum_{u \in V(X)} \phi(X \setminus u, t)$$
Using this and the partial fraction expansion of $p'(t)/p(t)$ (which you may quote without proof), prove that the eigenvalue support of a vertex in a vertex-transitive graph is the set of eigenvalues of X. [You may assume the the eigenvalue support of a vertex a is the set $\{\theta_r : E_r e_a \neq 0\}$.]

23. Let Y be a connected component of the graph X. Prove that the eigenvalue support of a, viewed as a vertex of X, equals the eigenvalue support of a, viewed as a vertex of X. [Bonus: does this hold for all pure states?]

24. If $a \in V(X)$, prove that the eigenvalue support of a is equal to the set of poles of the rational function $\phi(X \setminus a, t)/\phi(X, t)$.

25. Assume $a \in V(X)$ and $\gamma(t)$ is the greatest common divisor of $\phi(X, t)$ and $\phi(X \setminus a, t)$. Show that the eigenvalue support of a is the set of zeros of the polynomial $\phi(X, t)/\gamma(t)$. Show further that the eigenvalues of this quotient are simple.

26. Prove that a vertex a in X is controllable if and only if $\phi(X \setminus a, t)$ and $\phi(X, t)$ are coprime.

27. Show that a vertex of valency one in a path is controllable. The eigenvalues of P_n are
$$2 \cos \left(\frac{\pi k}{n+1} \right), \quad k = 1, \dots, n.$$
From this deduce that there is no perfect state transfer between vertices of valency one on the path P_n when $n \geq 5$.

28. Prove that if X is connected and has a controllable vertex, all eigenvalues of X are simple.

29. Let Δ be the diagonal matrix of valencies of the vertices of X. Assume $n = |V(X)$. The *Laplacian* L of X of the matrix $\Delta - A$. Using the fact that $L(X)$ is a sum of Laplacians of edges of X, show that
$$x^T L x = \sum_{ij \in E(X)} (x_i - x_j)^2,$$
whence $L \succcurlyeq 0$. Deduce that if the vector x is constant on a connected component of X, then $L_x = 0$. Hence show that 0 is an eigenvalue of L with multiplicity equal to the number of connected components of X. [In fact, the characteristic vectors of the components of X are an orthogonal basis forker(L).]

30. Let X be a graph with spectral idempotents $E_1, \dots, E_m$. Two vertices a and b of X are *parallel* if the projections $E_r e_a$ and $E_r e_b$ are parallel. Show

that if a and b are parallel, any automorphism of X that fixes a must also fix b. [Remark: strongly cospectral vertices are parallel (but this is not a hint).]

31. Prove that two vertices in X are strongly cospectral if and only if they are both cospectral and parallel.

32. Consider a quantum walk on X with $U(t) = \exp(itL)$. Assume X is connected and that 0 is a simple eigenvalue of L, with $\mathbf{1}$ as an eigenvector. Prove that the eigenvalue support of any vertex contains 0, and then show that the eigenvalue support of a periodic vertex consists entirely of integers.

33. Let T be a tree with invertible adjacency matrix A.

    (a) Use the 1-sum formula to show that $\det(A) = \pm 1$.

    (b) Using information about the eigenvalues in the eigenvalue support of a periodic vertex in a bipartite graph, prove that the eigenvalue support of a periodic vertex in T is a subset of $\{-1, 1\}$.

    (c) Show that if T has a periodic vertex, then $T \cong P_2$.

34. Let X be a graph on n vertices. Show that if X has only simple eigenvalues and is periodic, then $|V(X)| \leq 11$.

35. Assume X is regular and both X and $\overline{X}$ are connected. Show that if vertices a and b are strongly cospectral in X, they are strongly cospectral in $\overline{X}$.

36. Prove that $U_{P_5}(t)_{1,5} = 1$ if and only if $t = 0$. (Compute the characteristic polynomial of P_5 using the 1-sum formula.)

37. Prove that if D_1 and D_2 are pure states and $\widehat{D_1} = \widehat{D_2}$, then D_1 and D_2 are strongly cospectral.

38. Prove that a reflection on a real vector space is an orthogonal mapping.

39. Prove that if E_r is a spectral idempotent of X and $\mathrm{rk}(E_r) = 1$, then $\mathrm{rk}(E_e \circ E_r) = 1$. Deduce from this that, if X is bipartite on n vertices and has only simple eigenvalues, then $\mathrm{rk}(\widehat{M}) \leq n/2$.

40. If X is a strongly regular graph, prove that $\widehat{M}$ is a linear combination of A, I and J.

41. If P and Q are $n \times n$ projections, prove that $\operatorname{rk}(PQP) = \operatorname{rk}(QPQ)$.

42. Let X be a graph on n vertices, with vertex degrees $v_d, \ldots, v_1 n$. Let $LD(X)$ denote the line digraph of X and define the matrix

$$\hat{A}(LD(X))_{ab,cd} = \begin{cases} \frac{1}{\deg(b)}, & \text{if } b = c; \\ 0, & \text{otherwise.} \end{cases}$$

If C is the matrix

$$\begin{pmatrix} \frac{2}{d_1}J - I & & & \\ & \frac{2}{d_2}J - I & & \\ & & \ddots & \\ & & & \frac{2}{d_n}J - I \end{pmatrix}$$

and R is the usual arc-reversal operator, show that

$$RC = \hat{A}(LD(X)) - R.$$

43. If $u, v \in \mathbb{R}^d$ and $\|u\| = \|v\|$, show that there is a reflection that fixes a hyperplane and swaps u and v. Using this, deduce that any $d \times d$ orthogonal matrix of a product of d reflections.

44. let X be a directed graph. Prove that $D_h^T D_t = A(LD(X))$ and $D_t D_h^T = A(X)$.

45. If $\widehat{M}$ is the average mixing matrix for a discrete walk based on the unitary operator U, prove that

$$\widehat{M} = \lim_{K \to \infty} \frac{1}{K} \sum_{m=0}^{K-1} U^m \circ \overline{U}^m.$$

46. Show that the average mixing matrix of a discrete walk on the arcs of a graph is the Gram matrix of the average arc-states. [If α is an arc, the corresponding arc-state is $e_\alpha e_\alpha^T$.]

47. Asssume α, β are arcs in X with corresponding states D_α, D_β. Let $U = \sum_r e^{i\theta_r} F_r$ be the spectral decomposition of U. Prove that if $\widehat{D_\alpha} = \widehat{D_\beta}$, then $F_r e_\beta = \pm F_r e_b e$ for all r. [In other words, α and β are strongly cospectral.]

48. Let U be the transition matrix of an arc-reversal walk. Show that if U is rational, then $\widehat{M}$ is rational.