AGT&QC

Assignment 1

1. Prove that if the matrices A and B commute, then $\exp(A+B) = \exp(A)\exp(B)$. Give a counterexample to show that this fails if $AB \neq BA$.

2. If A is an $n \times n$ matrix, prove there is a real scalar γ such that, for all $k \geq 0$.
$$|(A^k)_{i,j}| \leq \gamma^k.$$

3. If $\|\cdot\|$ is the induced norm on the algebra of operators on the normed vector space V, prove that $\|AB\| \leq \|A\|\|B\|$.

4. If A is a square matrix, prove that $\det(\exp(A)) = \exp(\operatorname{tr}(A))$. (You may assume that A is complex, but the result holds over any field of characteristic zero.)

5. Let $x_1, \ldots, x_n$ be a set of unit vectors in $\mathbb{R}^d$ such for some scalar α with $0 \leq \alpha < 1$, we have $|\langle x_i, x_j\rangle| = \alpha$ whenever $i \neq j$. Let $P_i = x_i x_i^T$. Prove that the Gram matrix of the matrices P_i (using the usual trace inner product) is invertible. Deduce that $n \leq \binom{d+1}{2}$. [For a bonus, derive the analogous bound in the complex case.]

6. Prove that complex square matrix is nilpotent if and only if all its eigenvalues are zero.

7. Show that if A and B are symmetric matrices of rank one and neither matrix is a scalar multiple of the other, then $\operatorname{rk}(A + B) = 2$.

8. Show that a matrix M is normal if and only if, for all z,
$$\langle Mz, Mz\rangle = \langle M^*z, M^*z\rangle.$$

9. If A is a square complex matrix show that there exists unique Hermitian matrices R and S such that $A = R + iS$. Show further that A is normal if and only R and S commute.

10. If S is a skew symmetric matrix (necessarily real), show that the eigenvalues of S are purely imaginary, and are symmetric about the real axis in the complex plane.

11. If S is a skew symmetric matrix, prove that $\exp(S)$ is a real orthogonal matrix. Is it true that every real orthogonal matrix can be expressed as the exponential of a skew symmetric matrix?

12. If $P, Q \succcurlyeq 0$ and $\langle P, Q \rangle = 0$, show that $PQ = QP = 0$.

13. If A is adjacency matrix of a graph X and Δ is the diagonal matrix with $\Delta_{i,i}$ equal to the valency of the i-th vertex of X, then $L := \Delta - A$ is the *Laplacian matrix* of X. Express the Laplacian of $X \,\square\, Y$ in terms of the Laplacians of X and Y.

14. The *d-cube* Q_d has the 01-vectors of length d as its vertices, with two vectors adjacent if and only if they differ in exactly one coordinate. Prove that Q_d is isomorphic to the Cartesian product of d copies of K_2 (or of Q_2).

15. Let $\mathcal{P}$ be a set of $n \times n$ permutation matrices. Show that the set of matrices A that commute with each element of $\mathcal{P}$ is Schur-closed.

16. Let $\mathcal{A}$ be a matrix-algebra that is Schur-closed and contains J. Prove that $\mathcal{A}$ has a unique basis of 01-matrices.

17. If φ is a homomorphism from X to Y and $u, v \in V(X)$ show that $\mathrm{dist}_Y(\varphi(u), \varphi(v)) \le \mathrm{dist}_X(u, v)$.

18. A subgraph Y of a graph X is *geodetic* if for each pair of vertices u, v in Y we have $\mathrm{dist}_Y(u, v) = \mathrm{dist}_X(u, v)$. Show that the core of a graph is a geodetic subgraph.

19. Define the projections π_X and π_Y on the direct product $X \times Y$ by

$$\pi_X(x, y) = x, \quad \pi_Y(x, y) = y.$$

Show that these are homomorphisms. If f_X, f_Y are homomorphisms from the graph Z to X and Y respectively, show that there is a unique homomorphism φ from Z to $X \times Y$ such that

$$f_X \circ \varphi = h = f_Y \circ \varphi.$$

[This shows that, in some sense, $X \times Y$ is the "smallest" graph that admits homomorphisms to X and Y. What is the largest?]

20. If X is a circulant graph on n vertices and θ is a complex n-th root of unity, show that the function that takes k in X to θ^k is an eigenvector of X. Show that if θ has order n, the eigenvectors arising from the distinct powers of θ are linearly independent.

21. Assume X is a cubelike graph with vertex set $\mathbb{Z}_2^d$ and connection set $\mathcal{C}$. (View $\mathbb{Z}_2^d$ as a vector space.) If $a \in \mathbb{Z}_2^d$, define a map $\tau_a : \mathbb{Z}_2^d \to \{1, -1\} \subseteq \mathbb{Z}$ by
$$\tau(a)(u) = (-1)^{a^T u}.$$
Show that τ_a is an eigenvector and construct an orthogonal basis of eigenvectors. Prove that all eigenvalues of X are integers.

22. Show that connected cubelike graph on 2^d vertices contains a spanning subgraph isomorphic to the d-cube Q_d.

23. Show that a cubelike graph is connected if and only if its connection set is a spanning subset of the vector space $\mathbb{Z}_2^d$.

24. A permutation group G on a set S is *generously transitive* if, for each pair of elements of S, there is an element of G that swaps them. (The automorphism groups of cycles form one class of examples.) Prove that if the automorphism group of X is generously transitive, so is the automorphism group of its core.

25. Show that if G is abelian and $X(G, \mathcal{C})$ is a Cayley graph for G with valency at least three, then the girth of X is at most four.

26. Show that an automorphism of the d-cube that fixes a vertex and each of its neighbours is the identity.

27. Let r be fixed, and suppose that for each pair of distinct vertices u and v in X, there is an r-coloring of X where u and v get different colors. Show that X is a subgraph of a direct product of some number of copies of K_r.

28. Prove that a cubelike graph that contains a triangle contains a copy of K_4.

29. If X is a graph, let $\mathcal{N}(X)$ denote the multiset of neighborhoods of X. (Here a neighborhood is just a set of vertices.) If X and Y are graphs with the same vertex set, show that $X \times K_2 \cong Y \times K_2$ if and only $\mathcal{N}(X) \cong \mathcal{N}(Y)$.

30. Show that Q_3 is a Cayley graph for both $\mathbb{Z}_2^3$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$. (A drawing will suffice for a proof.) [For a bonus, generalize this.]

31. Let Ω denote the set of partitions of $\{1, \dots, 9\}$ into three disjoint triples. The symmetric group Sym(9) acts on Ω.

    (a) Show that Sym(9) acts transitively on Ω. (You may be brief.)

    (b) Compute the size of a stabilizer of a partition, and so determine $|\Omega|$.

    (c) Determine the number of orbitals of Sym(9), and determine which orbitals are graphs.

    (d) [bonus] Show that the subgroup Sym(8) of Sym(9) acts transitively on Ω. (Your proof should **not** require extensive computation.)

32. Determine the eigenvalues and eigenspaces of the complete bipartite graph $K_{m,n}$.

33. Let X and Y be the Latin square graphs corresponding to the groups $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$. Prove that they are not isomorphic.

34. Let X be the complete graph K_{n^2}. A *parallel class* in X is defined to be a subgraph isomorphic to nK_n. Two parallel classes are *skew* if they do not have an edge in common.

    (a) Show that an $n \times n$ Latin square determines a set of three pairwise-skew parallel classes in K_{n^2}.

    (b) If S_1 and S_2 denote the adjacency matrices of two skew parallel classes, prove that

    $$(S_i + I)^2 = n(S_i + I), \quad (S_1 + I)(S_2 + I) = J.$$

    (c) Assume $S_1, \dots, S_r$ are the adjacency matrices of r pairwise skew parallel classes. Determine the eigenvalues of the graph with adjacency matrix $A = S_1 + \cdots + S_r$.

    (d) Show that the graph in (c) is strongly regular.

35. Suppose we have two series

    $$A(x) = \sum_{n \geq 0} a_n x^n, \quad B(x) = \sum_{n \geq 0} b_n x^n$$

    where $B(x)^2 = A(x)$. The problem is to compute the coefficients of $B(x)$ in terms of the a_r's.

(a) Show that $2A(x)B(x)' = A(x)'B(x)$ (where the prime denotes derivative).

(b) Derive a recurrence for b_n where the coefficients depend on the a_r.

(c) [bonus] Check your recurrence by computing the coefficients of $\sqrt{1-4x}$ using the binomial theorem.

36. Let L denote the set of strings formed the elements of the alphabet $\{a, b\}$ that do not contain aa as a substring. Let M denote the set of strings over $\{a, b\}$ that contain exactly one copy of aa, as the last two symbols. Let ϵ denote the empty string of length zero. Verify the equations

$$\epsilon \cup L\{a, b\} = L \cup M, \qquad Laa = M \cup Ma.$$

Assume that weight is length. If $L(t)$ and $M(t)$ respectively denote the generating functions for L and M, we have equations

$$1 + 2tL(t) = L(t) + M(t), \qquad t^2 L = M(t)(1 + t).$$

Solve these and express the generating functions for L and M as rational functions.

37. Suppose $p(t)$ is a polynomial with zeros $\theta_1, \dots, \theta_d$ and respective multiplicities $m_1, \dots, m_d$. Derive the partial fraction decomposition

$$\frac{p'(t)}{p(t)} = \sum_{r=1}^{d} \frac{m_r}{t - \theta_r}.$$

38. Let A be an $m \times n$ matrix and B and $n \times m$ matrices. From the equations

$$\begin{pmatrix} I & 0 \\ -B & I \end{pmatrix} \begin{pmatrix} I & A \\ B & I \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I - BA \end{pmatrix}$$

and

$$\begin{pmatrix} I & A \\ B & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & I \end{pmatrix} = \begin{pmatrix} I - AB & A \\ 0 & I \end{pmatrix},$$

deduce that $\det(I - AB) = \det(I - BA)$. Using this, prove that AB and BA have the same non-zero eigenvalues with the same multiplicities.

39. Show that the direct product of connected graphs X and Y is disconnected if and only if X and Y are both bipartite.

40. If X and Y are connected and bipartite, show that the two components of $X \times Y$ have the same nonzero eigenvalues with the same multiplicities.

41. The adjacency matrix of the complement $\overline{X}$ of X is $J - I - A$. Hence

$$\phi(\overline{X}, t) = \det(tI - J + I + A) = \det((t+1)I + A)\det(I - ((t+1)I + A)^{-1}J).$$

Assume that $n = |V(X)|$ and that $\theta_1, \ldots, \theta_d$ are the distinct eigenvalues of X. Show that

$$\det(I - ((t+1)I + A)^{-1}J) = 1 - \mathbf{1}^T((t+1)I + A)^{-1}\mathbf{1} = 1 - \sum_r \frac{\mathbf{1}^T E_r \mathbf{1}}{t + 1 + \theta_r}$$

and hence deduce that

$$(-1)^n \frac{\phi(\overline{X}, t)}{\phi(X, -t-1)} = 1 - . \sum_r \frac{\mathbf{1}^T E_r \mathbf{1}}{t + 1 + \theta_r}$$

[Bonus] Simplify the right side when X is k-regular and connected.

42. Assume z is an eigenvector X with eigenvalue λ. If $u \in V(X)$ and $z(u) = 0$, show that the restriction of z to $V(X) \setminus u$ is an eigenvector $X \setminus u$ with eigenvalue λ.

43. Assume that $|V(X)| = n$ and b is a 01-vector of length n. If $A = A(X)$, define

$$A^b = \begin{pmatrix} 0 & b^T \\ b & A \end{pmatrix}$$

and let X^b be the graph with adjacency matrix A^b. Starting with the equation

$$\begin{pmatrix} I & 0 \\ 0 & (tI - A)^{-1} \end{pmatrix} \begin{pmatrix} t & -b^T \\ -b & tI - A \end{pmatrix} = \begin{pmatrix} t & -b^T \\ (tI - A)^{-1}b & I \end{pmatrix},$$

prove that

$$\frac{\det(tI - A^b)}{\det(tI - A)} = 1 - b^T(tI - A)^{-1}b.$$

if the distinct eigenvalues of A are $\theta_1, \ldots, \theta_d$ and the corresponding spectral idempotents are $E_1, \ldots, E_d$, show that

$$\frac{\phi(X^b, t)}{\phi(X, t)} = t - \sum_{r=1}^{d} \frac{b^T E_r b}{t - \theta_r}.$$

[Bonus] If X is regular and connected, show that the rational function

$$\frac{\phi(X^b,t)}{\phi(X,t)} - \frac{\phi(X^{\mathbf{1}-b},t)}{\phi(X,t)}$$

has only one pole.

44. Assume the graph Z is formed by taking graphs X and Y with disjoint vertex sets and adding an edge joining vertex a in X to vertex b in Y. Prove that

$$\phi(Z,t) = \phi(X,t)\phi(Y,t) - \phi(X \setminus a,t)\phi(Y \setminus b,t).$$

45. Let a and b be distinct vertices in X and let Z be formed from two copies of X by merging the vertex b in the first copy with the vertex a in second. Prove that if a and b are cospectral in X, then a and b are cospectral in Z. [Remark: the second b is **not** the merged vertex.]

46. Assume X is a cubelike graph on 2^d vertices with valency m. We can view the connection set as the columns of a $d \times m$ matrix M over $\mathbb{Z}_2$. So two binary vectors are adjacent in X if and only if their difference is a column of M. Prove that if Q is an invertible $d \times d$ matrix over $\mathbb{Z}_2$, the cubelike graph determined by QM is isomorphic to X.

47. Assume $X = X(G,\mathcal{C})$ and let ψ be a function from G to the non-zero complex numbers $\mathbb{C}\setminus 0$. Prove that if ψ is a homomorphism, then it is an eigenvector. Determine the eigenvalue.

48. [Work over the reals for this question, though it holds over $\mathbb{C}$ as well.] Let P in $\mathrm{End}(V \otimes V)$ be defined by the condition $P(u \otimes v) = v \otimes u$. (So $P^2 = I$ and P is a permutation operator.) If $A \in \mathrm{End}(V)$, prove that $P(A \otimes A^T) = (A^T \otimes A)P$ and deduce that $P(A \otimes A^T)$ is symmetric.

49. A *Hadamard matrix* is an $n \times n$ matrix with entries ± 1 such that $HH^T = nI$. (It follows that either $n = 2$ or four divides n.) With the matrix P as in the previous exercise, prove that $P(H \otimes H^T)$ is a symmetric Hadamard matrix with constant diagonal.