Type-II Matrices

Chris Godsil

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Outline

1. Matrices
   - Definitions
   - Examples: Unitary
   - Examples: Combinatorial
   - Examples: Geometric

2. Link Invariants
   - Algebra
   - Braids

3. Association Schemes
   - DFT
   - Schemes
   - Questions

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Schur Product

**Definition**

If $A$ and $B$ are $m \times n$ matrices, their **Schur product** $A \circ B$ is the $m \times n$ matrix given by

$$(A \circ B)_{i,j} = A_{i,j}B_{i,j}.$$
The matrix \( J \) with all entries equal to 1 is the identity for Schur multiplication.
Inverses

- The matrix $J$ with all entries equal to 1 is the identity for Schur multiplication.
- If no entry of $A$ is zero, there is a unique matrix $A^{(-)}$ such that

$$A \circ A^{(-)} = J;$$

we call $A^{(-)}$ the **Schur inverse** of $A$. 
Type II

**Definition**

A $v \times v$ complex matrix $W$ is a type-II matrix if

$$WW^{(-)T} = vI.$$
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$$WW^{(-)T} = vI.$$ 

So if $W$ is a type-II matrix then

$$W^{-1} = \frac{1}{v}W^{(-)T}.$$
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The Cyclic Spin Model

Example

Choose $\theta$ so that $\theta^2$ is a primitive complex $v$-th root of 1, and let $W$ be the $v \times v$ matrix given by

$$W_{i,j} := \theta^{(i-j)^2}, \quad 0 \leq i, j < v.$$
Examples: Unitary

Check:

\[(WW^{(-)T})_{i,j} = \sum_r \theta^{(i-r)^2-(j-r)^2} = \theta^{i^2-j^2} \sum_r \theta^2(j-i)r = v\delta_{i,j}.\]
Flat Matrices

**Definition**

A complex matrix $M$ is flat if its entries all have the same absolute value.
Theorem

If \( W \) is a \( v \times v \) matrix over \( \mathbb{C} \), then any two of the following statements imply the third:

(a) \( W \) is type II.
(b) \( W \) is flat.
(c) \( W \) is unitary.
Quantum Physics

**Definition**

Two orthogonal bases $x_1, \ldots, x_v$ and $y_1, \ldots, y_v$ of $\mathbb{C}^v$ are **unbiased** if all inner products $\langle x_i, y_j \rangle$ have the same absolute value. Two unitary matrices $X$ and $Y$ are unbiased if $X^*Y$ is flat.
If $X$ and $Y$ are unitary matrices, then $X^*Y$ is unitary. So $X$ and $Y$ are unbiased if and only if $X^*Y$ is a flat type-II matrix.
If $X$ and $Y$ are unitary matrices, then $X^*Y$ is unitary. So $X$ are $Y$ are unbiased if and only if $X^*Y$ is a flat type-II matrix.

Unitary matrices $X$ and $Y$ are unbiased if and only if the (unitary) matrices $I$ and $X^*Y$ are unbiased.
If $X$ and $Y$ are unitary matrices, then $X^*Y$ is unitary. So $X$ are $Y$ are unbiased if and only if $X^*Y$ is a flat type-II matrix.

- Unitary matrices $X$ and $Y$ are unbiased if and only if the (unitary) matrices $I$ and $X^*Y$ are unbiased.

- Hence each flat type-II matrix determines an unbiased pair of bases.
A Problem

A Question

What is the maximum size of a set of mutually unbiased bases in $\mathbb{C}^v$?
What We Know

(a) The maximum is at most $v + 1$. 
What We Know

(a) The maximum is at most \( v + 1 \).

(b) This bound can be realized if \( v \) is a prime power.
What We Know

(a) The maximum is at most $v + 1$.

(b) This bound can be realized if $v$ is a prime power.

(c) In general, the best we can do is three. :-(

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**Type-II Matrices**
The Potts Model

Let $J$ be the $v \times v$ matrix with all entries equal to 1 and set

$$W := (\gamma - 1)I + J.$$  

Then $J^2 = vJ$ and so

$$WW^{(-)T} = (2 - \gamma - \gamma^{-1})I + (v - 2 + \gamma + \gamma^{-1})J.$$  

Hence $W$ is type II if and only if

$$\gamma^2 + (v - 2)\gamma + 1 = 0.$$  

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Symmetric Designs

Definition

For today’s purposes, a **symmetric design** is given by a $v \times v$ 01-matrix $N$ such that, for suitable integers $k$ and $\lambda$,

$$NJ = N^T J = kJ, \quad NN^T = (k - \lambda)I + \lambda J.$$
The Fano Plane

Example

\[ N = \begin{pmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \]
If

$$W := (\gamma - 1)N + J$$

then

$$WW^{(-)T} = (2 - \gamma - \gamma^{-1})(k - \lambda)I + (k(\gamma + \gamma^{-1} - 2) + v)J$$

and therefore $W$ is type II if and only if

$$(k - \lambda)(\gamma - 1)^2 + v(\gamma - 1) + v = 0.$$
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Equiangular Lines

**Definition**

Let \( x_1, \ldots, x_n \) be a set of unit vectors in \( \mathbb{C}^d \). The lines spanned by these vectors are **equiangular** if there is a scalar \( a \) such that if \( i \neq j \), then

\[
|\langle x_i, x_j \rangle|^2 = a.
\]
A Bound

**Lemma**

Suppose the lines spanned by $x_1, \ldots, x_n$ are equiangular and the matrices $X_i$ are defined by $X_i = x_i x_i^*$. Then the matrices $X_i$ are linearly independent elements of the space of Hermitian matrices.
A Bound

Lemma

Suppose the lines spanned by $x_1, \ldots, x_n$ are equiangular and the matrices $X_i$ are defined by $X_i = x_ix_i^*$. Then the matrices $X_i$ are linearly independent elements of the space of Hermitian matrices.

Corollary

$n \leq d^2$. 

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The Construction

Theorem

Suppose \( n = d^2 \) and \( x_1, \ldots, x_n \) spans a set of equiangular lines in \( \mathbb{C}^d \). Let \( G \) be the Gram matrix of this set of vectors. Then \( G^2 = dG \) and if

\[
\gamma^2 + (d + 2)\gamma + a^2 = 0,
\]

then \((\gamma - 1)I + G\) is a type-II matrix.
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The Nomura Algebra

Let $W$ be a complex $v \times v$ Schur invertible matrix. Then $W_{i/j}$ is the vector in $\mathbb{C}^v$ given by:

$$(W_{i/j})_r := \frac{W_{r,i}}{W_{r,j}}.$$
The Nomura Algebra

Let $W$ be a complex $v \times v$ Schur invertible matrix. Then $W_{i/j}$ is the vector in $\mathbb{C}^v$ given by:

$$(W_{i/j})_r := \frac{W_{r,i}}{W_{r,j}}.$$ 

**Definition**

The **Nomura Algebra** $N_W$ of a Schur-invertible matrix is the set of complex matrices $M$ such that each vector $W_{i/j}$ is an eigenvector for $M$. 

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Type-II Matrices
An Example

Consider the cyclic spin model:

\[ W_{i,j} = \theta^{(i-j)^2} \]

(\text{where } \theta^2 \text{ is a primitive complex } n\text{-th root of 1}). Then

\[ (W_{i/j})_r = \theta^{i^2-j^2} \theta^{2(i-j)r} \]

and so the vectors \( W_{i/j} \) are (essentially) the columns of a Vandermonde matrix.
Equivalence

If $W$ is type II and $P_1, P_2$ are permutation matrices and $D_1, D_2$ are invertible diagonal matrices, then

$$P_1 D_1 W D_2 P_2$$

is type II. We say that it is equivalent to $W$. 
Equivalence

If $W$ is type II and $P_1, P_2$ are permutation matrices and $D_1, D_2$ are invertible diagonal matrices, then

$$P_1 D_1 W D_2 P_2$$

is type II. We say that it is equivalent to $W$.

**Theorem**

If $W$ and $W'$ are equivalent type-II matrices, there is a permutation matrix $P$ such that

$$\mathcal{N}_{W'} = P^T \mathcal{N}_W P.$$
Nontriviality

- \( I \in \mathcal{N}_W \)
Nontriviality

- $I \in \mathcal{N}_W$
- $J \in \mathcal{N}_W$ if and only if $W$ is a type-II matrix.
Spin Models

Definition

A type-II matrix is a spin model if $W \in \mathcal{N}_W$. 
Spin Models

**Definition**

A type-II matrix is a **spin model** if $W \in \mathcal{N}_W$.  

Each spin model determines a link invariant.
Examples

- The cyclic spin model.

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Examples

- The cyclic spin model.
- The Potts model.
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One Braid

\[ \sigma_1^2 \sigma_2 \sigma_3^{-1} \]
Two Braids

\[ \sigma_1 \sigma_2 \sigma_1 \quad \sigma_2 \sigma_1 \sigma_2 \]
**Generators and Relations**

**Definition**

The **braid group** $B_n$ on $n$ strands is generated by elements $\sigma_1, \ldots, \sigma_{n-1}$ and their inverses, subject to the relations:

- If $|i - j| > 1$, then $\sigma_i \sigma_j = \sigma_j \sigma_i$.
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$. 
Links

\[ \beta \]
Suppose $\alpha$ and $\beta$ are braids on $n$ strands. Then the following operations do not change the isotopy class of the closure of $\beta$:
Markov Moves

Suppose $\alpha$ and $\beta$ are braids on $n$ strands. Then the following operations do not change the isotopy class of the closure of $\beta$:

Markov I: $\beta \rightarrow \alpha^{-1} \beta \alpha$, 
Suppose $\alpha$ and $\beta$ are braids on $n$ strands. Then the following operations do not change the isotopy class of the closure of $\beta$:

Markov I: $\beta \rightarrow \alpha^{-1} \beta \alpha$,

Markov II: $\beta \rightarrow \beta \sigma_n$, 
Markov II

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Two braids give the same link if and only if they are Markov equivalent.
## Link Invariants

- Two braids give same link if and only if they are Markov equivalent.
- Given the first Markov move, we see that for a braid invariant to give us a link invariant, it must be constant on conjugacy classes in the braid group.
Let $V$ be a complex vector space of finite dimension and choose invertible elements $X$ and $Y$ in $\text{End}(V)$ such that $XYX = YXY$. Then we have a homomorphism, $\rho$ say, from $B_3$ into $\text{End}(V)$ such that
\[
\rho(\sigma_1) = X, \quad \rho(\sigma_2) = Y.
\]
A Markov Trace

If $\beta \in B_3$, then $\text{tr}(\rho(\beta))$ only depends on the conjugacy class of $\beta$. If we are lucky, this will be a link invariant.
Spin Models

Suppose $W$ is a spin model of order $v \times v$. Let $V = \text{Mat}_{d \times d}(\mathbb{C})$ and if $M \in V$, define

$$X(M) = \frac{1}{\sqrt{v}} WM, \quad Y(M) = W(-) \circ M.$$  

Then $XYX = YXY$, and we are lucky.
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A Transform

Suppose $W$ is a $v \times v$ type-II matrix. If $M \in \mathcal{N}_W$, let $\Theta(M)$ be the $v \times v$ matrix such that $\Theta(M)_{i,j}$ is the eigenvalue of $M$ on $W_{i,j}$. If $M, N \in \mathcal{N}_W$ then:
A Transform

Suppose $W$ is a $v \times v$ type-II matrix. If $M \in \mathcal{N}_W$, let $\Theta(M)$ be the $v \times v$ matrix such that $\Theta(M)_{i,j}$ is the eigenvalue of $M$ on $W_{i/j}$. If $M, N \in \mathcal{N}_W$ then:

- $\Theta(MN) = \Theta(M) \circ \Theta(N)$. 
A Transform

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- $\Theta(MN) = \Theta(M) \circ \Theta(N)$.
- $\Theta(M) \in \mathcal{N}_{WT}$. 
A Transform

Suppose $W$ is a $v \times v$ type-II matrix. If $M \in \mathcal{N}_W$, let $\Theta(M)$ be the $v \times v$ matrix such that $\Theta(M)_{i,j}$ is the eigenvalue of $M$ on $W_{i/j}$. If $M, N \in \mathcal{N}_W$ then:

- $\Theta(MN) = \Theta(M) \circ \Theta(N)$.
- $\Theta(M) \in \mathcal{N}_{W^T}$.
- $\Theta^2(M) = vM^T$. 
Schur Closure

Theorem (Jaeger, Nomura)

If $W$ is a type-II matrix then $\mathcal{N}_W$ is Schur-closed.
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Hence $\mathcal{N}_W$ has a basis of 01-matrices $\mathcal{A} = \{A_0, \ldots, A_d\}$ such that:
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- $A_0 = I$. 

Hence $\mathcal{N}_W$ has a basis of 01-matrices $\mathcal{A} = \{A_0, \ldots, A_d\}$ such that:

- $A_0 = I$.
- $\sum A_i = J$. 

**Axioms**
Hence $\mathcal{N}_W$ has a basis of 01-matrices $\mathcal{A} = \{A_0, \ldots, A_d\}$ such that:

- $A_0 = I$.
- $\sum A_i = J$.
- $A_i^T \in \mathcal{A}$, for all $i$. 

Axioms

Hence \( \mathcal{N}_W \) has a basis of 01-matrices \( \mathcal{A} = \{A_0, \ldots, A_d\} \) such that:

- \( A_0 = I \).
- \( \sum A_i = J \).
- \( A_i^T \in \mathcal{A} \), for all \( i \).
- \( A_i A_j \in \text{span}(\mathcal{A}) \).
Hence $\mathcal{N}_W$ has a basis of 01-matrices $\mathcal{A} = \{A_0, \ldots, A_d\}$ such that:

- $A_0 = I$.
- $\sum A_i = J$.
- $A_i^T \in \mathcal{A}$, for all $i$.
- $A_iA_j \in \text{span}(\mathcal{A})$.
- $A_iA_j = A_jA_i$, for all $i$ and $j$. 
Discreteness

There are only finitely many association schemes on \( v \) vertices.
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A Question

If $\mathcal{A}$ is an association scheme, can $\mathbb{C}[\mathcal{A}]$ contain infinitely many type-II matrices? (If the dimension of the scheme is three, then it contains at most six.)
Examples

We have the following classes of spin models:

- Cyclic models.
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- Higman-Sims—the Nomura algebra is the Bose-Mesner algebra of the Higman-Sims graph. (Found by Jaeger.)
Examples

We have the following classes of spin models:

- Cyclic models.
- Potts models.
- Higman-Sims—the Nomura algebra is the Bose-Mesner algebra of the Higman-Sims graph. (Found by Jaeger.)
- A class of examples with Nomura algebra equal to the Bose-Mesner algebra of distance-regular antipodal double cover of a complete bipartite graph. (Found by Nomura.)
Examples

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- Cyclic models.
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- A class of examples with Nomura algebra equal to the Bose-Mesner algebra of distance-regular antipodal double cover of a complete bipartite graph. (Found by Nomura.)
- Products of the above.
Problem

Find new non-trivial examples of type-II matrices $W$ such that $\dim(N_W) \geq 3$.

We do not have any examples that are not spin models!