Quantum Coloring Problems

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Fort Collins, November, 2011: In memory of Bob Liebler
Outline

1 Physics 101

2 The Unit Sphere
   - Coloring the Sphere
   - Projective Planes
   - Gleason
   - Equiangular Lines

3 The Unitary Group
   - Quantum Colorings
   - Mutually Unbiased Bases
   - Partitions
Cosmology

Quote

Hydrogen is a colorless, odorless gas which given sufficient time, turns into people. (Henry Hiebert)
Axioms

"The axioms of quantum physics are not as strict as those of mathematics"
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An orthogonality graph, and a problem

**Definition**

We define $\Omega(d)$ to be the graph with the unit vectors in $\mathbb{R}^d$ as its vertices, where two vertices are adjacent if and only if they are orthogonal.

**Problem**

What is $\chi(\Omega(d))$?
Clique in $\Omega(d)$

Since each orthonormal basis for $\mathbb{R}^d$ forms a clique in $\Omega(d)$, we have

$$\chi(\Omega(d)) \geq d.$$
A finite subgraph of $\Omega(d)$

**Definition**

Let $\Phi(d)$ denote the graphs with the $\pm 1$-vectors of length $d$ as vertices, where two vectors are adjacent if and only if they are orthogonal.
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2. If $d \equiv 2$ modulo four, $\Phi(d)$ is bipartite.
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4. $\omega(\Phi(d)) \leq d$, equality holds if and only if a Hadamard matrix exists.
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4. $\omega(\Phi(d)) \leq d$, equality holds if and only if a Hadamard matrix exists.
5. If $\chi(\Phi(d)) = d$ and there is a $d \times d$ Hadamard matrix, then $d$ is a power of two.
The chromatic number of $\Phi(d)$ increases exponentially

**Theorem (Frankl and Rödl)**

*There is a constant $c$ such that $0 < c < 2$ and if $4 \mid d$ and $d$ is large enough, then $\alpha(\Phi(d)) < c^d$.***
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Coloring planes

Definition

Let $\mathcal{P}(\mathbb{F})$ denote the projective plane over the $\mathbb{F}$. A proper coloring of $\mathcal{P}$ is a coloring of its points, such that each line gets exactly two colors.
Coloring planes

**Definition**

Let $\mathcal{P}(\mathbb{F})$ denote the projective plane over the $\mathbb{F}$. A **proper coloring** of $\mathcal{P}$ is a coloring of its points, such that each line gets exactly two colors.

**Theorem (Carter and Vogt, Hales and Straus)**

The proper colorings of $\mathcal{P}(\mathbb{F})$ correspond to the non-trivial non-Archimedean valuations of $\mathbb{F}$. 
Planes and spheres

Every coloring of $\Omega(3)$ gives a coloring of the projective plane, but the converse does not hold. But no coloring of the plane lists to a sphere coloring:

**Corollary**

$\chi(\Omega(3)) > 3$. 
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**Gleason’s theorem**

**Definition**

Let $\Omega(d)$ denote the graph whose vertices are the unit vectors in $\mathbb{R}^d$, where two unit vectors are adjacent if they are orthogonal. A frame function is a non-negative function on unit vectors that sums to 1 on each orthonormal basis.

**Theorem (Gleason, 1957)**

If $d \geq 3$ and $f$ is a frame function, then there is a positive semidefinite matrix $M$ such that $\text{tr}(M) = 1$ and $f(x) = x^T M x$ for all $x$. 
No $d$-colorings

**Corollary**

If $d \geq 3$ then $\chi(\Omega(d)) > d$. 
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Proof.

Suppose $\Omega(d)$ is $d$-colorable and let $S$ be a color class in a $d$-coloring. Then each orthonormal basis must contain a vertex in $S$, and therefore the characteristic vector of $S$ is a frame function.
No \( d \)-colorings

**Corollary**

If \( d \geq 3 \) then \( \chi(\Omega(d)) > d \).

**Proof.**

Suppose \( \Omega(d) \) is \( d \)-colorable and let \( S \) be a color class in a \( d \)-coloring. Then each orthonormal basis must contain a vertex in \( S \), and therefore the characteristic vector of \( S \) is a frame function. But this characteristic function is not continuous.
Applying compactness

Theorem (Kochen and Specker)
Assume $d \geq 3$. There is a finite subgraph of $\Omega(d)$ whose vertex set is a union of orthonormal bases, such that no coclique contains a vertex in each orthonormal basis.
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A line in $\mathbb{C}^d$ can be represented by a unit vector that spans it. If $x$ spans a line then

$$P = (x^* x)^{-1} xx^*$$

represents orthogonal projection onto the line spanned by $x$. 
The angle between the lines spanned by unit vectors $x$ and $y$ is determined by

$$|\langle x, y \rangle| = |x^* y|.$$ 

If $P$ and $Q$ are the projections $xx^*$ and $yy^*$, then

$$\text{tr}(PQ) = \text{tr}(xx^* yy^*) = \text{tr}(y^* xx^* y) = |\langle x, y \rangle|^2.$$
Linear combinations of projections

Suppose we have $m$ lines in $\mathbb{C}^d$ such that the angle between any pair of lines is the same. Let $P_1, \ldots, P_m$ be the corresponding projections. If

$$0 = \sum_r c_r P_r$$

then, if $\text{tr}(P_r P_s) = a^2$ when $r \neq s$,

$$0 = \sum_r c_r \text{tr}(P_k P_r) = c_k(1 - a^2) + a^2 \sum c_r.$$

Hence the coefficients $c_r$ are all equal and it follows they are all zero.
A bound on the size of a set of equiangular lines

**Lemma**

If $P_1, \ldots, P_m$ are the orthogonal projections onto a set of equiangular lines in $\mathbb{C}^d$, then they form a linearly independent subset of the vector space of $d \times d$ Hermitian matrices. Hence $m \leq d^2$. 
If equality holds, the angle is determined

**Theorem**

*If we have a set of $d^2$ equiangular lines in $\mathbb{C}^d$, then $a^2 = (d + 1)^{-1}$.***

**Proof.**

Suppose $\mathcal{L}$ is an equiangular set of $m$ lines in $\mathbb{C}^d$, with associated projections $P_1, \ldots, P_m$. If $m = d^2$ then there are scalars $c_i$ such that $I = \sum_r c_r P_r$ and therefore

$$1 = \text{tr}(P_k) = (1 - a^2) c_k + a^2 \sum_r c_r.$$  

So the scalars $c_r$ are all equal and, since $\text{tr}(I) = d$, we have $c_r = d/m$. Substituting this into the above equation yields the value stated for $a^2$. 

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Graph Spectra
An question about chromatic number

Let $X(d)$ be the graph on lines in $\mathbb{C}^d$, where lines given by projections $P$ and $Q$ are adjacent if $\text{tr}(PQ) = (d + 1)^{-1}$. Then $\omega(X(d)) \leq d^2$.

Problem

What is the chromatic number of $X(d)$?
What can we construct?

- Sets of $d^2$ lines that are equiangular to machine precision have been constructed up to $d = 67$ (Scott and Grassl 2009).
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- Equiangular sets with size $d^2$ exist when $d \in \{2, \ldots, 15, 19, 24, 35, 48\}$ (Scott and Grassl 2009).
What can we construct?

- Sets of $d^2$ lines that are equiangular to machine precision have been constructed up to $d = 67$ (Scott and Grassl 2009).
- Equiangular sets with size $d^2$ exist when $d \in \{2, \ldots, 15, 19, 24, 35, 48\}$ (Scott and Grassl 2009).
- In $\mathbb{R}^d$ we can get sets of size at most $\binom{d+1}{2}$ and, if $d > 3$, then $d$ is odd and $d + 2$ is a perfect square. Examples are known only for $d = 2, 3, 7, 23$. 
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A unitary Cayley graph

**Definition**

Let $cD$ denote the set of $d \times d$ unitary matrices for which all diagonal entries are zero. A graph $Y$ as a quantum $d$-coloring if there is a graph homomorphism from $Y$ into the Cayley graph $X(U(d), D)$. 

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$\Phi(d)$ has a quantum $d$-coloring if $d$ is a power of two.
Embedding the symmetric group

View the symmetric group $\text{Sym}(d)$ as a group of $d \times d$ permutation matrices.

- $\text{Sym}(d) \leq U(d, \mathbb{C})$ and two elements $\sigma$ and $\tau$ of $\text{Sym}(d)$ are adjacent in $X(U(d), D)$ if and only if $\tau \sigma^{-1}$ is a derangement.
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- A regular subgroup of $\text{Sym}(d)$ forms a clique of size $d$. 

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- A regular subgroup of $\text{Sym}(d)$ forms a clique of size $d$.
- The map that sends a permutation $\sigma$ to $1\sigma$ is a proper $d$-coloring of the image of $\text{Sym}(d)$.
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Unbiased bases

**Definition**

Two orthonormal bases $x_1, \ldots, x_d$ and $y_1, \ldots, y_d$ of $\mathbb{C}^d$ are unbiased if

$$|\langle x_r, y_s \rangle|$$

is independent of $r$ and $s$. (If it is, then it must be equal to $1/\sqrt{d}$.)
Unbiased bases

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Two orthonormal bases $x_1, \ldots, x_d$ and $y_1, \ldots, y_d$ of $\mathbb{C}^d$ are **unbiased** if

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is independent of $r$ and $s$. (If it is, then it must be equal to $1/\sqrt{d}$.)

If $U_1$ and $U_2$ are unitary matrices, their columns are unbiased if and only if all entries of $U_1^* U_2$ have the same absolute value, that is, if the matrix $U_1^* U_2$ is **flat**.
Let $\mathcal{F}$ denote the set of flat matrices in $U(d, \mathbb{C})$. Then a set of mutually unbiased bases for $\mathbb{C}^d$ is a clique in the Cayley graph $X(U(d), \mathcal{F})$. 
How large can a set of mutually unbiased bases be?

If \( U \) is a flat unitary matrix and \( D, E \) are diagonal matrices of order \( d \times d \), then

\[
\text{tr}(DU^{-1}EU) = \text{tr}(D) \text{tr}(E)
\]

If \( \mathcal{D} \) denotes the algebra of all diagonal matrices, it follows that

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\mathcal{D} \cap U^{-1}DU = \{cI : c \in \mathbb{C}\}.
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**Corollary**

*The columns of the unitary matrices \( U_1, \ldots, U_m \) are mutually unbiased if and only if for all \( r \) and \( s \) (with \( r \neq s \))

\[
U_r^{-1}\mathcal{D}U_r \cap U_s^{-1}\mathcal{D}U_s = \{ cI : c \in \mathbb{C} \}.
\]

*Hence we can have at most \( d + 1 \) mutually unbiased matrices in \( \mathbb{C}^d \).*
The basic question?

**Question**

For which values of $d$ can we construct a mutually unbiased set of $d + 1$ orthonormal bases of $\mathbb{C}^d$?
Some partial answers

- There are mutually unbiased bases of size $d + 1$ if $d$ is a prime power.
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- There is always a set of size three.
Some partial answers

- There are mutually unbiased bases of size $d + 1$ if $d$ is a prime power.
- There is always a set of size three.
- If $d = 2d_0$ where $d_0$ is odd, we do not know how to do better than three.
Constructions from projective planes

All known examples of sets of $d + 1$ mutually unbiased bases in $\mathbb{C}^d$ can be constructed from either:

- A $(d, d, d, 1)$-relative difference set in an abelian group of order $d^2$. 
Constructions from projective planes

All known examples of sets of \( d + 1 \) mutually unbiased bases in \( \mathbb{C}^d \) can be constructed from either:

- A \((d, d, d, 1)\)-relative difference set in an abelian group of order \( d^2 \).

- A symplectic spread in a vector space of even dimension: a set of \( q^d \) symmetric \( d \times d \) matrices such that the difference of any two distinct matrices is invertible.
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Graph Spectra
A graph on set partitions

**Definition**

Let $V$ be a set of size $d^2$. Define $\mathcal{P}(d)$ to be the graph whose vertices are the partitions of $V$ into $d$ cells of size $d$, where two such partitions are adjacent if each cell of the first partition contains a point from each of the $d$ cells of the second partition.
A graph on set partitions

**Definition**

Let $V$ be a set of size $d^2$. Define $\mathcal{P}(d)$ to be the graph whose vertices are the partitions of $V$ into $d$ cells of size $d$, where two such partitions are adjacent if each cell of the first partition contains a point from each of the $d$ cells of the second partition.

We can represent each partition $\pi$ by a $d^2 \times d$ matrix $M(\pi)$ whose columns are the characteristic vectors of its cells. Then $\pi \sim \rho$ in $\mathcal{P}(d)$ if and only if

$$M(\pi)^T M(\rho) = J_d.$$
Example

Assume $d = 3$. Then $\mathcal{P}(3)$ has 280 vertices and is regular with valency 36. There are 70 partitions which have 1 and 2 in the same cell, these form a coclique of maximal size (and all cocliques of size 70 are equivalent to this).
Coloring partitions

Meagher and Stevens:

\[ \chi(\mathcal{P}(d)) \leq \binom{d + 1}{2}. \]
Coloring partitions

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\[ \chi(\mathcal{P}(d)) \leq \binom{d+1}{2}. \]

(The upper bound is tight if \( d = 3 \). Nothing more is known.)
Clique in $\mathcal{P}(k)$

**Lemma**

The cliques of size $k$ in $\mathcal{P}(d)$ correspond to orthogonal arrays with $k$ rows and entries from $\{1, \ldots, d\}$. 

Corollary

$\omega(\mathcal{P}(d)) = d + 1$ if and only if there is an affine plane of order $d$. 

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The End(s)