Are Almost All Graphs Cospectral?

Chris Godsil

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Outline

1  Cospectral Graphs
   ■ Polynomials and Walks
   ■ Constructing Cospectral Graphs
   ■ Switching

2  1-Full Graphs
   ■ A Cyclic Subspace
   ■ Generating All Matrices
   ■ Cospectral 1-Full Graphs
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   - A Cyclic Subspace
   - Generating All Matrices
   - Cospectral 1-Full Graphs
The Characteristic Polynomial

**Definition**

Let $G$ be a graph with adjacency matrix $A$. The characteristic polynomial $\phi(G, t)$ of $G$ is the characteristic polynomial of $A$:

$$\phi(G, t) := \det(tI - A).$$
Examples

The characteristic polynomials of $K_1$, $K_2$ and $P_3$ are respectively:

$$t, \quad t^2 - 1, \quad t^3 - 2t.$$
WALKS

Lemma

If $A$ is the adjacency matrix of $G$, then $(A^r)_{i,j}$ is the number of walks in $G$ from vertex $i$ to vertex $j$ with length $r$. 

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Closed Walks

A walk in $G$ is **closed** if its first and last vertices are equal. The number of closed walks in $G$ with length $r$ is

$$\sum_{i \in V(G)} (A^r)_{i,i} = \text{tr}(A^r).$$
Short Closed Walks

If $G$ has $n$ vertices, $e$ edges and contains exactly $t$ triangles, then

$$\text{tr}(A^0) = n$$
$$\text{tr}(A^1) = 0$$
$$\text{tr}(A^2) = 2e$$
$$\text{tr}(A^3) = 6t.$$
Short Closed Walks

If $G$ has $n$ vertices, $e$ edges and contains exactly $t$ triangles, then

\[
\begin{align*}
\text{tr}(A^0) &= n \\
\text{tr}(A^1) &= 0 \\
\text{tr}(A^2) &= 2e \\
\text{tr}(A^3) &= 6t.
\end{align*}
\]

(And then it gets messy!)
A Generating Function

The generating function for the closed walks in $G$, counted by length, is

$$
\sum_{r \geq 0} \text{tr}(A^r)t^r.
$$

It is a rational function:

$$
\sum_{r \geq 0} \text{tr}(A^r)t^r = \frac{t^{-1}\phi'(G,t^{-1})}{\phi(G,t^{-1})}.
$$
A Characterisation

Corollary

Two graphs $G$ and $H$ are cospectral if and only if their generating functions for closed walks are equal.
The Smallest Cospectral Graphs

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The Smallest Connected Cospectral Graphs
Cospectral, Cospectral Complements?

The graphs $\overline{C_4 \cup K_1}$ and $\overline{K_{1,4}}$ have two and four triangles respectively—they are not cospectral.
Another Generating Function

The number of walks of length $r$ in $G$ is equal to

$$\text{tr}(A^r J) = 1^T A^r 1$$

and thus

$$\sum_{r \geq 0} \text{tr}(A^r J) t^r$$

is the generating function for all walks in $G$. 

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Complements and Walks

Theorem

Suppose $G$ and $H$ are cospectral graphs with respective adjacency matrices $A$ and $B$. Then $\overline{G}$ and $\overline{H}$ are cospectral if and only if the generating functions for all walks in $G$ and in $H$ are equal.
Regular Graphs

If $G$ is a $k$-regular graph on $n$ vertices then its walk generating function is

$$\frac{n}{1 - kt}.$$
Regular Graphs

If $G$ is a $k$-regular graph on $n$ vertices then its walk generating function is

$$\frac{n}{1 - kt}.$$
Two Irregular Graphs

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0-Sums

The 0-sum of two graphs $G$ and $H$ is got by identifying a vertex in $G$ with a vertex in $H$:

![Diagram of 0-sum of graphs G and H with identified vertex v]
Spectrum of a 0-Sum

If we create the 0-sum $F$ by merging $v$ in $G$ with $v$ in $H$, then

$$\phi(F) = \phi(G)\phi(H \setminus v) + \phi(G \setminus v)\phi(H) - t\phi(G \setminus v)\phi(H \setminus v).$$
Example

If $G = K_2$ and $K = K_2$ then their 0-sum $F$ is $P_3$, whence

$$\phi(P_3, t) = (t^2 - 1)t + t(t^2 - 1) - t(t^2) = t^3 - 2t.$$
Corollary

If we hold \( G \) and its vertex \( v \) fixed, then the characteristic polynomial of the 0-sum of \( G \) and \( H \) is determined by the characteristic polynomials of \( H \) and \( H \setminus v \).
Constructing Cospectral Graphs

Deleting Vertices

If $H$ is the graph

then $H \setminus u$ and $H \setminus v$ are isomorphic...
A Cospectral Pair

...and thus we obtain a pair of cospectral graphs:
Another Pair

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Constructing Cospectral Graphs

A Hint

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A Theorem

Theorem (Schwenk... Godsil & McKay)

*Almost all trees are cospectral...*
A Theorem

**Theorem (Schwenk... Godsil & McKay)**

*Almost all trees are cospectral... with cospectral complements.*
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Yet Another Construction

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If

\[ K := \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \]

then \( K1 = 1 \) and \( K^2 = I \), whence \( K \) is orthogonal, and

\[ KM = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = J_4 - M. \]
Switching

\[
\begin{pmatrix}
K & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
0 & M & 0 \\
M^T & A_1 & B_1 \\
0 & B_1^T & A_2
\end{pmatrix}
\begin{pmatrix}
K & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & (J - M)^T \\
(J - M)^T & A_1 & B_1 \\
0 & B_1^T & A_2
\end{pmatrix}
\]
Therefore...

**Theorem**

*Switching related graphs are cospectral, with cospectral complements.*
Is it true that almost all graphs are determined by their spectrum?
A Related Example

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Let $G$ be a graph on $n$ vertices with adjacency matrix $A$. Define $U$ to be the subspace of $\mathbb{R}^n$ spanned by the vectors $A^r 1$, for all non-negative integers $r$. 
Automorphisms

**Theorem**

*If the permutation matrix $P$ is in $\text{Aut}(G)$, then $Pu = u$ for all $u$ in $U$.***
Automorphisms

**Theorem**

*If the permutation matrix $P$ is in $\text{Aut}(G)$, then $Pu = u$ for all $u$ in $U$.***

**Proof.**

If $P$ is a permutation matrix, $P1 = 1$. If $P \in \text{Aut}(G)$, then $PA = AP$ and so, for all $r$

$$PA^r1 = A^rP1 = A^r1.$$
1-Rank

**Definition**

The 1-rank of $G$ is the dimension of $U$. 
1-Rank

Definition

The \textbf{1-rank} of $G$ is the dimension of $U$.

Lemma

\textit{The 1-rank of $G$ is less than or equal to the number of orbits of $\text{Aut}(G)$ on the vertices of $G$.}
**Definition**

A graph $G$ on $n$ vertices is 1-full if its 1-rank is $n$. 

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1-Full Graphs

Definition

A graph $G$ on $n$ vertices is 1-full if its 1-rank is $n$.

Corollary

A 1-full graph is asymmetric.
A Cyclic Subspace

Also...

Theorem (Godsil & McKay)

A 1-full graph is vertex reconstructible.
A Cyclic Subspace

A Question

Is it true that almost all graphs are 1-full?
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A Basis of Matrices

**Theorem**

Let $G$ be a graph on $n$ vertices. If $G$ is 1-full, the matrices

$$A^i J A^j,$$

$0 \leq i, j < n$

form a basis for $\text{Mat}_{n\times n}(\mathbb{R})$. 

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Are Almost All Graphs Cospectral?
The Proof

Proof.

For \( i = 0, \ldots, n - 1 \), set \( u_i = A^i \mathbf{1} \). Then \( A^i J A^j = u_i u_j^T \). The vectors \( u_0, \ldots, u_{n-1} \) are linearly independent, and so any non-zero linear combination of the matrices can be written as

\[
    u_0 v_0^T + \cdots + u_{n-1} v_{n-1}^T
\]

where none of the vectors \( v_0, \ldots, v_{n-1} \) are zero. Since the \( u_i \)'s are linearly independent, this sum cannot be zero. \( \Box \)
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Walk Equivalent

Definition

Two graphs $G$ and $H$ are walk equivalent if their generating functions for walks are equal.
Walk Equivalent

Definition

Two graphs $G$ and $H$ are walk equivalent if their generating functions for walks are equal.

(Thus any two $k$-regular graphs on the same number of vertices are walk equivalent.)
Walk-Equivalent 1-Full Graphs

Theorem

If $G$ and $H$ are walk equivalent graphs and $G$ is 1-full, then $G$ and $H$ are cospectral with cospectral complements.
An Endomorphism

Assume \( A \) and \( B \) are the adjacency matrices of \( G \) and \( H \) respectively. Since the matrices \( A^i J A^j \) (where \( 0 \leq i, j < n \)) form a basis for \( \mathcal{M} = \text{Mat}_{n \times n}(\mathbb{R}) \), there is a unique linear map \( \Phi : \mathcal{M} \to \mathcal{M} \) such that

\[
\Phi(A^i J A^j) = B^i J B^j.
\]
Let $w_r$ denote the number of walks of length $r$ in $G$. Then

$$A^i J A^j A^k J A^\ell = w_{j+k} A^i J A^\ell$$

and consequently

$$\Phi(A^i J A^j A^k J A^\ell) = w_{j+k} \Phi(A^i J A^\ell)$$

$$= w_{j+k} B^i J B^\ell$$

$$= B^i J B^j B^k J B^\ell$$
It follows that $\Phi$ is a homomorphism (and not just a linear map). Since $\mathcal{M}$ is a simple algebra, $\Phi$ is an isomorphism. By the Noether-Skolem theorem it follows that there is an invertible matrix $L$ such that

$$\Phi(M) = L^{-1}ML$$

for all matrices $M$. 
Conclusion

So we have

$$L^{-1} AL = B,$$

whence $G$ and $H$ are cospectral.
Conclusion

So we have

\[ L^{-1}AL = B, \]

whence \( G \) and \( H \) are cospectral.

Since \( \Phi(J) = J \), we have \( L^{-1}JL = J \), whence \( \overline{G} \) and \( \overline{H} \) are cospectral.