1. The *odd girth* of a graph is the length of the shortest odd cycle in it. Show that two cospectral graphs have the same odd girth.

2. Determine the eigenvalues and their multiplicities for $K_{m,n}$.

3. Prove that if $X$ has the same spectrum as $K_n$, then $X \cong K_n$. [discuss]

4. Prove or disprove: if $X$ has the same spectrum as $K_{1,n}$, then $X \cong K_{1,n}$.

5. Let $X$ be a graph on $n$ vertices with $n$ different eigenvalues. The object of this exercise is to prove that an automorphism of $X$ has order one or two. Proceed as follows:
   - (a) Suppose $z$ is an eigenvector for $A$. Show that if $PA = AP$, then $z$ is an eigenvector for $P$.
   - (b) If $P$ is a permutation matrix and $z$ is a real eigenvector for $P$, then $Pz = \pm z$.
   - (c) If $P$ is the permutation matrix corresponding to an automorphism of $X$, then $P^2 = I$.

6. If $X$ is a graph on $n$ vertices with $n$ distinct eigenvalues, show that Aut($X$) is abelian.

7. A *Hadamard matrix* is a square $\pm 1$-matrix of order $v \times v$ such that $H^TH = vI$. A Hadamard matrix is regular if its rows all have the same sum. Show how to construct a strongly regular graph from a regular symmetric Hadamard matrix with constant diagonal. [Do not attempt to prove this by induction on the number of constraints on the word 'matrix' :-)]

8. Observe that $2I_4 - J_4$ is a regular symmetric Hadamard matrix with constant diagonal. Look up 'Kronecker product', and construct a regular symmetric Hadamard matrix with constant diagonal of order $16 \times 16$. Identify the corresponding strongly regular graph with one already named.

9. A *quad* in a graph is a clique of size four such that each vertex not in the clique is adjacent to an even number of vertices in it. Assume
   
   $$ M := \frac{1}{2}J_4 - I_4 $$

   and
   
   $$ M_1 = \begin{pmatrix} M & 0 \\ 0 & I_{v-4} \end{pmatrix}. $$

   If $X$ is a graph whose first four vertices form a quad, prove that
   
   $$ M_1 A(X) M_1 $$

   is the adjacency matrix of a graph $Y$ cospectral to $X$. Show that the complements of $X$ and $Y$ are also cospectral.

10. Determine the strongly regular graphs with zero as an eigenvalue. [discuss]
11. Let $N$ be the incidence matrix of a Steiner triple system on $v$ points. Show that the columns of
\[
\begin{pmatrix}
N^T - \frac{3}{v} J
\end{pmatrix}
\]
are eigenvectors for the block graph $X$ of the system, and determine the eigenvalue. For a bonus, prove that these eigenvectors span the eigenspace of $X$ that they belong to. (Remark: the matrix $J$ above is not square in general.)

12. Let $X$ and $Y$ be the Latin square graphs corresponding to the groups $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$. Prove that they are not isomorphic.

13. Let $X$ be the graph with the $2 \times 2$ matrices over $\text{GF}(q)$ as vertices; where two matrices $A$ and $B$ are adjacent if and only $\text{rk}(A - B) = 1$. Prove that $X$ is a Cayley graph. Prove that $X$ is strongly regular and determine its parameters and its matrix of eigenvalues.

14. Let $V$ be the set of all triples from the set $S$, and declare two triples to be adjacent if they have exactly one element in common. Show that if $|S| = 7$ or $10$, then $X$ is strongly regular and determine its matrix of eigenvalues.

15. If $\psi$ is a monic polynomial, show that there are only finitely many connected graphs $X$ such that $\psi$ is the minimal polynomial of $A(X)$.

16. If $A$ is a symmetric matrix, then $n^+(A)$ denotes the number of positive eigenvalues of $A$, and $n^-(A)$ denotes the number of negative eigenvalues. Use interlacing to prove that if $n = |V(X)|$ and $A = A(X)$, then
\[
\alpha(X) \leq \min\{n - n^-, n - n^+\}.
\]

[Discuss]

17. Let $X$ be a strongly regular graph with parameters $(v, k; a, c)$ and set $\delta = a - c$. If $M$ is a matrix then $M_{D,D}$ denotes the submatrix of $M$ with rows and columns indexed by the entries of $D$.

(a) Prove that
\[
(t^2 - \delta t - k + c)(tI - A)^{-1} = A + (t - \delta) + \frac{c}{t - k} J.
\]

(b) If $D \subseteq V(X)$ we have the following identity (trust me):
\[
\det((tI - A)^{-1}_{D,D}) = \frac{\phi(X \setminus D, t)}{\phi(X, t)}.
\]

(c) Suppose $D$ is a coclique in $X$. Using the previous two results, derive a formula for $\phi(X \setminus D, t)$ involving only $t$, the eigenvalues of $X$, and their multiplicities.

[discuss]
18. Let $X$ be a putative strongly regular graph with parameters $(28, 9; 0, 4)$. Compute the eigenvalues of $X$ and their multiplicities (use whatever formulas you can find). Now use the results of the previous exercise to show that there is no strongly regular graph with parameters $(28, 9; 0, 4)$.

19. The Higman-Sims graph is strongly regular with parameters $(100, 22; 0, 6)$. Use the results of Exercise 17 to show that the vertices at distance two from a given vertex in the Higman-Sims graph induce a strongly regular graph on 77 vertices.

20. Use interlacing arguments to derive bounds on:
   
   (a) The value of $\omega(X)$, the maximum number of vertices in a clique.
   
   (b) The maximum number of vertices in an induced matching.

21. (We err, you) construct a Steiner triple system on $3m$ points when $m$ is odd.) Let $m$ be an odd integer and let $V$ be the set $\mathbb{Z}_m \times \mathbb{Z}_3$. If $a, b \in \mathbb{Z}_m$, let $a \star b$ denote the unique element $c$ in $\mathbb{Z}_m$ such that $2c = a + b$. Construct a Steiner triple system on $V$ that contains all triples
   
   $$\{(a, i), (b, i), (a \star b, i + 1)\}$$
   
   where $a \neq b$. [Discuss]

22. Derive an expression for the number of triangles in $X$, in terms of the eigenvalues of $X$.

23. The eigenvalue support of the vertex $u$ in $X$ the number of spectral idempotents $E$ such that $(E)_{u,u} \neq 0$. The radius of $u$ is the maximum distance in $X$ of a vertex from $u$. Prove that if $u$ has radius $r$, and $\sigma$ is the size of the eigenvalue support of $u$, then $r + 1 \leq \sigma$.

24. Prove that a connected regular graph with four eigenvalues is walk regular.

25. Prove that direct and Cartesian products of walk-regular graphs are walk regular.

26. Prove or disprove: the complement of a walk-regular graph is walk regular.

27. Let $X$ be a regular graph on $n = 2m$ vertices, and let $S$ be a subset of $V(X)$ with size $m$. Let $Y$ be the graph we get by adding a new vertex adjacent to each vertex in $S$. Let $Z$ be the graph we get by adding a new vertex adjacent to each vertex in $V(X) \setminus S$. Prove that $Y$ and $Z$ are cospectral.

28. (We continue with the previous question.) Show that if $\text{Aut}(X)$ does not contain an element of order two, then $Y$ and $Z$ are not isomorphic.

29. Let $X$ be a strongly regular graph and let $E$ and $F$ denote the two non-trivial idempotents in the spectral decomposition of $A(X)$.
   
   (a) Show that $E^3 = E \circ E \circ E$ is positive semidefinite.
   
   (b) Express the sum of the entries of $E^3$ in terms of $k$, $\theta$ and $\tau$. 

(c) If \( X \) exists, then the sum in (b) is non-negative. Using this show that there is no strongly regular graph with parameters \((64,21;0,10)\).

30. If \( X \) is a primitive strongly regular graph and \( u \in V(X) \), prove that the subgraph induced by the vertices at distance two from \( u \) is connected.

31. Let \( X \) be a graph on \( n \) vertices with least eigenvalue \( \tau \) and let \( \gamma \) be the greatest real number such that \( M = A - \tau I - \gamma J \) is positive semidefinite. Derive an upper bound for \( \alpha(X) \) in terms of \( n, \tau \) and \( \gamma \).

32. Let \( X \) be a \( k \)-regular graph on \( n \) vertices with least eigenvalue \( \tau \). Let \( S \) be a coclique whose size meets the ratio bound, and let \( x \) be its characteristic vector. Use the fact that
\[
z := x - \frac{|S|}{n} 1
\]
is an eigenvector of \( A = A(X) \) to prove that each vertex not in \( S \) has the same number of neighbours in \( S \).