CO444/644
Assignment 2: Homomorphisms

Due: Monday, March 4

1. If $X$ is bipartite, show that any cycle of length equal to the girth of $X$ is a retract. [discuss]

2. If $S \subseteq V(X)$ and $\chi(X \setminus S) < \chi(X)$, show that any retract of $X$ contains a vertex from $S$.

3. Show that the core of a generously transitive graph is generously transitive. [discuss]

4. If $X$ and $Y$ are graphs and $X \times X \cong Y \times Y$, prove that $X \cong Y$.

5. Assume $r \geq 2$. Show that $X \times K_r \cong Y \times K_r$ if and only if $X \times K_2 \cong Y \times K_2$.

6. Show that the subgraph obtained from the Petersen graph by deleting one vertex is a core.

7. Determine the core of $L(K_n)$ for each $n$.

8. If $X$ is a graph, let $\mathcal{N}(X)$ denote the multiset of neighborhoods of $X$. (Here a neighborhood is just a set of vertices.) If $X$ and $Y$ are graphs with the same vertex set, show that $X \times K_2 \cong Y \times K_2$ if and only if $\mathcal{N}(X) \cong \mathcal{N}(Y)$.

9. If $X$ is a Cayley graph for an abelian group, prove that $X \square X \rightarrow X$.

10. Let $P'_n$ be the graph formed from the path with vertex set $\{0, \ldots, n-1\}$ by putting a loop on 0. Construct the graph $\Delta_n(X)$ by adding a new vertex to $P'_n \times X$ and joining it to each vertex in the set $\{(n-1, u) : u \in V(X)\}$.

Show that there is a proper $r$-coloring of $\Delta_n(X)$ if and only if there is a walk of length $n$ in $K^X_r$ whose first vertex is a homomorphism and whose last vertex is a constant function. [discuss]

11. Prove or disprove: each proper homomorphomorph image of the Petersen graph contains a triangle.

12. Show that $X \rightarrow X \times Y$ if and only if $X \rightarrow Y$.

13. Let $r$ be fixed, and suppose that for each pair of distinct vertices $u$ and $v$ in $X$, there is an $r$-coloring of $X$ where $u$ and $v$ get different colors. Show that $X$ is a subgraph of a product of some number of copies of $K_r$, i.e., a subgraph of $K^m_r$ or some $m$.

14. Show that the Cayley graph $X(\mathbb{Z}_{13}, \{\pm 1, \pm 5\})$ has chromatic number four, but each proper induced subgraph is 3-colorable. [What about proper subgraphs.]

15. Show that if $X$ is cubelike and contains a coclique of size three, then $\chi(X) \leq |V(X)|/4$.

16. Prove that if the endomorphism $f$ of a connected graph is locally injective, it is an isomorphism.

17. The distance-two graph $X_2$ of $X$ is a graph with the same vertex set as $X$; two vertices are adjacent in $X_2$ if and only if they are at distance two in $X$. When $X$ is connected and bipartite, $X_2$ has exactly two components and these are called the halved graphs of $X$. (They need not be isomorphic in general.) Show that the halved graphs of the product of $K_2$ with the Kneser graph $K_{2k-1; k-1}$ are isomorphic to the Johnson graphs $J(2k-1, k-1, k-2)$.
18. Let $\mathcal{S}(d)$ denote the graph with the unit vectors in $\mathbb{C}^d$ as its vertices, with two unit vectors adjacent if they are orthogonal. Let $\mathcal{P}(d)$ be the graph with matrices $xx^*$ as vertices, where $x$ is a unit vector in $\mathbb{C}^d$; two matrices $P$ and $Q$ are adjacent if $\text{tr}(PQ) = 0$. Show that $\mathcal{P}(d)$ is a retract of $\mathcal{S}(d)$. [discuss]

19. Given a graph $X$, we define $P_3(X)$ to be the graph with the same vertex set of $X$, where two vertices are adjacent in $P_3(X)$ if the are joined in $X$ by a walk of length three. (Note that $X$ is a spanning subgraph of $P_3(X)$, and that if $X$ contains a triangle, then $P_3(X)$ has a loop.) A strong $n$-colouring of a graph is proper $n$-colouring such that the neighbourhood of each colour class is a coclique. Prove that $X$ has a strong $n$-colouring if and only if $P_3(X) \rightarrow K_n$. [discuss]

20. Prove that if $X \rightarrow Y$, then $P_3(X) \rightarrow P_3(Y)$. Prove that $P_3(X \times Y) \cong P_3(X) \times P_3(Y)$.

21. Define the graph $P_3^{-1}(X)$ as follows. Its vertices are the ordered pairs $(u, S)$, where $u \in V(X)$ and $S$ is a non-empty subset of the neighbours of $u$. The pairs $(u, S)$ and $(u, T)$ are adjacent if:
   (a) $u \in T$,
   (b) $v \in S$,
   (c) each vertex in $S$ is adjacent to each vertex in $T$.

   Prove that $P_3(P_3^{-1}(X))$ and $X$ are homomorphically equivalent. [I’m not sure whether $P_3^{-1}(P_3(X))$ is also homomorphically equivalent to $X$.]

22. Prove that if $X$ is connected, cubic and vertex transitive, then $X^*$ is $K_2$, an odd cycle, or $X$ itself.

23. Show that if $X$ is a quartic vertex-transitive graph on an odd number of vertices, then $X^*$ is either an odd cycle, a complete graph, or $X$ itself. [What if $|V(X)|$ is a power of two?]

24. If $X$ is connected, show that the constant functions from $V(X)$ to $V(F)$ induce a copy of $F$ in $F^X$.

25. If $X$ is connected and not bipartite, prove that $K_2^X$ is the disjoint union of $K_2$ with some number of isolated vertices. [discuss]

26. By constructing an explicit colouring, prove that for any graph $X$, the product $X \times K_n^X$ is $n$-colourable.

27. Prove the previous question by a counting argument.

28. Prove or disprove: $\chi(X \square Y) = \max\{\chi(X), \chi(Y)\}$.

29. [withdrawn]
30. Prove that if $X$ and $Y$ are connected graphs, then $K_n$ is a retract of $X \times Y$ only if it is a retract of $X$ or $Y$. [Remark: if we could prove this without the assumption that $X$ and $Y$ are connected, we would have established Hedetniemi's theorem.]

31. If $X = \mathbb{Z}_3(K_4)$, prove that $\alpha(X) \geq 4$ and $\omega(X) \geq 4$. (Gordon Royle has computed that equality holds in both cases, and that $\chi(X) = 7$.)

32. Prove that if $X \rightarrow Y$, then $\mathbb{Z}_n(X) \rightarrow \mathbb{Z}_n(Y)$.

33. [withdrawn]

34. If $C$ is an odd cycle, prove that $\chi(\Delta_3(C)) = 4$. (It may be easier if you do not use exponential graphs.)

35. Prove that if $n$ is odd, the folded $n$-cube contains an induced subgraph isomorphic to $\Delta_{(n-1)/2}(C_n)$. [The folded $n$-cube is the graph we get from the $n$-cube by identifying the pairs of vertices at distance $n$, it is cubelike.]

36. If $X$ is connected and $\chi(X) = r$, prove that $X$ folds onto $K_r$.

37. Assume $X$ is connected. Show that the subgraph of $C_{2k+1}^X$ induced by the loops is connected if and only if $X$ is bipartite and does not fold onto $C_{2k+1} \times K_2 = C_{4k+2}$. [Discuss]

38. Show that the subgraph of $C_{2k}^X$ induced by the loops has exactly two components if and only if $X$ does not fold onto $C_{2k}$.