Unique Factorisation of Additive Induced-Hereditary Properties

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Abstract

An additive hereditary graph property is a set of graphs, closed under isomorphism and under taking subgraphs and disjoint unions. Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be additive hereditary graph properties. A graph G has property $(\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n)$ if there is a partition $(V_1, \ldots, V_n)$ of $V(G)$ into n sets such that, for all i, the induced subgraph $G[V_i]$ is in $\mathcal{P}_i$. A property $\mathcal{P}$ is reducible if there are properties $\mathcal{Q}$, $\mathcal{R}$ such that $\mathcal{P} = \mathcal{Q} \circ \mathcal{R}$; otherwise it is irreducible. Mihók, Semanišin and Vasky [J. Graph Theory **33** (2000), 44–53] gave a factorisation for any additive hereditary property $\mathcal{P}$ into a given number $dc(\mathcal{P})$ of irreducible additive hereditary factors. Mihók [Discuss. Math. Graph Theory **20** (2000), 143–153] gave a similar factorisation for properties that are additive and induced-hereditary (closed under taking induced-subgraphs and

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disjoint unions). Their results left open the possiblity of different factorisations, maybe even with a different number of factors; we prove here that the given factorisations are, in fact, unique.

1 Introduction

A *graph property* is an isomorphism-closed set of graphs. A graph G *has property* $\mathcal{P}$ if $G \in \mathcal{P}$. The *universal property* $\mathcal{U}$ is the set of all (finite, unlabelled, simple) graphs. A property $\mathcal{P}$ is *non-trivial* if $\emptyset \neq \mathcal{P} \neq \mathcal{U}$.

A property is *hereditary*, *induced-hereditary* or *additive* if it is closed under taking subgraphs, induced-subgraphs or disjoint unions, respectively. If $\mathcal{P}$ is additive, and every component of a graph X is in $\mathcal{P}$, then X is also in $\mathcal{P}$.

Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be graph properties. A $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$*-partition* of a graph G is a partition $(V_1, \ldots, V_n)$ of $V(G)$ into n (possibly empty) sets such that, for all i, the induced subgraph $G[V_i]$ is in $\mathcal{P}_i$. The property $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$ is the set of all graphs having a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition. The $\mathcal{P}_i$'s are *factors* or *divisors* of $\mathcal{P}$, while $\mathcal{P}$ is the *product* of the $\mathcal{P}_i$'s. It is easy to see that the product of additive (or hereditary or induced-hereditary) properties is also additive (or hereditary or induced-hereditary).

In this article, an additive (induced-)hereditary property is *reducible* if it is the product of at least two non-trivial *additive (induced-)hereditary* properties; otherwise it is *irreducible*. We show in [4] that if an additive (induced-)hereditary property is the product of *any* two non-trivial properties, then it is also the product of two additive (induced-)hereditary non-trivial properties. So the concept of reducibility used here turns out to be the same as a more natural concept of reducibility; we point out, however, that the proofs in [4] depend on this article.

Mihók, Semanišin and Vasky [8] gave a factorisation of an additive hereditary property $\mathcal{P}$ into a given number $dc(\mathcal{P})$ of irreducible additive hereditary factors. This factorisation was shown to be well-defined, but it was also claimed to be unique. The argument was that if $\mathcal{P} = \mathcal{Q} \circ \mathcal{R}$, where by induction $\mathcal{Q}$ and $\mathcal{R}$ each have a unique factorisation, then $\mathcal{P}$ also has a unique factorisation. However, there is still the possibility that $\mathcal{P}$ factors as $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_r$, where no subset of the $\mathcal{P}_i$'s has either $\mathcal{Q}$ or $\mathcal{R}$ as a product.

As an analogy, consider the ring $\{x + y\sqrt{5} \mid x, y \in \mathbb{Z}\}$. In this integral domain 2, $1 + \sqrt{5}$ and $1 - \sqrt{5}$ are all irreducible — they have no factorisation into two non-unit factors. In particular, 2 has a unique factorisation, but $4 = 2^2$ does not, because we have $4 = (1+\sqrt{5})(-1+\sqrt{5})$.

We show in Theorem 3.2 that similar anomalies do not occur with additive hereditary properties if the two factorisations have *exactly* $dc(\mathcal{P})$ factors. Could there then be factorisations with different numbers of factors? This is not an idle question, as Mihók et al. showed in Example 4.2 of the same paper that a certain hereditary (but not additive) property $\mathcal{P}_1 \circ \mathcal{P}_2$ has another factorisation $\mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \mathcal{Q}_3$ where even the number of irreducible hereditary factors is different. One of the main contributions of this paper is Theorem 3.1, where we prove that in any factorisation into irreducible additive hereditary factors, the number of factors of $\mathcal{P}$ must be *exactly* $dc(\mathcal{P})$.

In [7], Mihók gave a remarkably general construction of uniquely partitionable graphs, and used this to produce a factorisation for the wider class of properties that are additive and induced-hereditary. This was claimed to be unique using the same argument as in [8]. We generalise his construction, and our own results (Theorems 4.9, 5.2 and 5.3, respectively) to prove that this factorisation is in fact unique.

We note that unique factorisation was settled completely in [6] for a significant class of additive hereditary properties, the proof depending on the structure of those properties (and in the spirit of the proof we give here). It is possible to use the structure of the factorisation presented in [8] to show that any factorisation with exactly $dc(\mathcal{P})$ additive hereditary factors must be the one constructed in that article (a similar proof is possible for the factorisation of [7]); the appeal of the proofs of uniqueness given here is that they are independent of the structure of the factors of $\mathcal{P}$. Thus, they depend only on the more elementary aspects of [8] and [7].

In the next section we reproduce the essential concepts, definitions and results adapted from [8]; stating those results in a stronger fashion here, and sometimes omitting their simple proofs. Our own techniques and proofs are presented in Section 3. Sections 4 and 5 are the induced-hereditary analogues of these two sections. We end with some corollaries of unique factorisation and a list of open questions.

A second paper [4] contains related results on uniquely partitionable graphs, a characterisation of induced-hereditary properties uniquely factorisable into arbitrary properties (not necessarily induced-hereditary). A technical report [5] contains the results of both papers, and generalises them. More recently, the first author [3] used the results in [7] and in this paper to show that it is NP-hard to recognise reducible additive induced-hereditary properties, with the exception of the set of bipartite graphs.

2 Definitions and results from [8]

In this section and the next we will actually prove unique factorisation for a class of properties strictly larger than the additive hereditary class. We use $G \subseteq H$ to denote that G is a subgraph of H. A *hereditary compositive* property is a hereditary property $\mathcal{P}$ where, for any two graphs $G_1, G_2 \in \mathcal{P}$, there is a graph $H \in \mathcal{P}$ such that $G_i \subseteq H, i = 1, 2$. It turns out that the proof of unique factorisation for additive hereditary properties carries over to the hereditary compositive case without any change. For our purposes, a hereditary compositive property is reducible if it is the product of two non-trivial hereditary compositive properties; otherwise it is irreducible.

The unique factorisation result for additive induced-hereditary properties includes as a special case the result for additive hereditary properties (Prop. 6.4), but not the one for hereditary compositive properties. An additive hereditary property is both additive induced-hereditary, and hereditary compositive. However, for a fixed finite graph S, properties of the form $\mathcal{P}_S := \{G \mid G \subseteq S\}$ are hereditary compositive but not additive. In [5] we prove unique factorisation for a class that strictly includes additive induced-hereditary properties, but still does not contain properties of the form $\mathcal{P}_S$.

In addition, the structures of the proofs of Theorems 3.5 and 5.6 are similar. Having in mind the simpler proof for Theorem 3.5 before attempting the more difficult proof of Theorem 5.6 is very helpful.

The smallest hereditary property that contains a set $\mathcal{G}$ is denoted by $[\mathcal{G}]$. This is the hereditary property *generated by* $\mathcal{G}$, or that $\mathcal{G}$ *generates*. $\mathcal{G}$ *is a generating set for* $\mathcal{P}$ if $[\mathcal{G}] = \mathcal{P}$. It is easily seen that

$$[\mathcal{G}] \quad = \quad \{G \mid \exists\, H \in \mathcal{G},\ G \subseteq H\}.$$

The *completeness* $c(\mathcal{P})$ of a hereditary property $\mathcal{P} \neq \mathcal{U}$ is $\max\{k : K_k \in \mathcal{P}\}$[1], where K_k is the complete graph on k vertices; clearly, $c(\mathcal{Q} \circ \mathcal{R}) = c(\mathcal{Q}) + c(\mathcal{R})$. Thus, any factorisation of a hereditary property $\mathcal{P}$ has at most $c(\mathcal{P})$ non-trivial factors.

The *join* $G_1 + \cdots + G_n$ of n graphs $G_1, \ldots, G_n$ consists of disjoint copies of the G_i's, and all edges between $V(G_i)$ and $V(G_j)$, for $i \neq j$. A graph G is

[1]In some of the literature, the convention is to define $c(\mathcal{P}) := \max\{k : K_{k+1} \in \mathcal{P}\}$.

decomposable if it is the join of two graphs; otherwise, G is *indecomposable*. It is easy to see that G is decomposable if and only if its complement $\overline{G}$ is disconnected; G is the join of the complements of the components of $\overline{G}$, so every decomposable graph can be expressed uniquely as the join of indecomposable subgraphs, the *ind-parts* of G. The number of ind-parts of G is the *decomposability number* $dc(G)$ of G.

For a hereditary property $\mathcal{P}$, a graph G is $\mathcal{P}$*-strict* if $G \in \mathcal{P}$ but $G + K_1 \notin \mathcal{P}$. The set $\mathcal{M}(\mathcal{P})$ of $\mathcal{P}$-maximal graphs is defined as:

$$\begin{aligned}
\mathcal{M}(n,\mathcal{P}) &:= \{G \in \mathcal{P} \mid |V(G)| = n \text{ and for all } e \in E(\overline{G}), G + e \notin \mathcal{P}\};\\
\mathcal{M}(\mathcal{P}) &:= \bigcup_{n=1}^{\infty} \mathcal{M}(n,\mathcal{P})
\end{aligned}$$

Since, for $1 \leq n \leq c(\mathcal{P})$, $M(n,\mathcal{P}) = \{K_n\}$, it is also useful to define

$$\mathcal{M}^*(\mathcal{P}) := \bigcup_{n=c(\mathcal{P})}^{\infty} \mathcal{M}(n,\mathcal{P}).$$

2.1. Lemma [8]. *Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be hereditary properties of graphs, and denote $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$ by $\mathcal{P}$. A graph G belongs to $\mathcal{M}(\mathcal{P})$ if and only if, for every $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-partition $(V_1, \ldots, V_m)$ of $V(G)$, the following holds: $G[V_i] \in \mathcal{M}(\mathcal{P}_i)$ for $i = 1, \ldots, n$, and $G = G[V_1] + \cdots + G[V_m]$. Moreover, if $G \in \mathcal{M}^*(\mathcal{P})$, then it is $\mathcal{P}$-strict, each $G[V_i]$ is $\mathcal{P}_i$-strict, and is in $\mathcal{M}^*(\mathcal{P}_i)$; in particular, each $G[V_i]$ is non-empty.* $\square$

It follows that if $\mathcal{P}$ is reducible, then every graph in $\mathcal{M}^*(\mathcal{P})$ is decomposable. We note that the join of a $\mathcal{Q}$-maximal graph G and an $\mathcal{R}$-maximal graph H need not be $(\mathcal{Q} \circ \mathcal{R})$-maximal; for example, take G to be complete, $|V(G)| \leq c(\mathcal{Q}) - 2$, and H not complete.

Clearly $[\mathcal{M}^*(\mathcal{P})] = \mathcal{P}$, but if $\mathcal{P}$ is additive it is not the unique generating set. If $\mathcal{G}$ is a generating set for the hereditary property $\mathcal{P}$, its *decomposability number* $dc(\mathcal{G})$ is $\min\{dc(G) \mid G \in \mathcal{G}\}$; the *decomposability number* of $\mathcal{P}$ is $dc(\mathcal{P}) := dc(\mathcal{M}^*(\mathcal{P}))$. A property with $dc(\mathcal{P}) = 1$ is *indecomposable*; by Lemma 2.1 such a property must be irreducible, and we shall see that for hereditary *compositive* properties the converse is also true. The converse is not true for hereditary properties in general, as shown in [8].

2.2. Lemma [8]. *Let $\mathcal{P}$ be a hereditary property and let $G \in \mathcal{M}^*(\mathcal{P})$, $H \in \mathcal{P}$. If $G \subseteq H$ then $dc(H) \le dc(G)$. If we have equality, with $G = G_1 + \cdots + G_n$ and $H = H_1 + \cdots + H_n$ being the respective expressions as joins of ind-parts, then we can relabel the ind-parts of H so that each G_i is an induced subgraph of H_i.* $\square$

2.3. Lemma [8]. *If $\mathcal{G}$ generates the hereditary property $\mathcal{P}$, then $dc(\mathcal{G}) \le dc(\mathcal{M}^*(\mathcal{P}))$, with equality if $\mathcal{G} \subseteq \mathcal{M}^*(\mathcal{P})$.* $\square$

For $\mathcal{G} \subseteq \mathcal{P}$ and $H \in \mathcal{P}$, let $\mathcal{G}[H] := \{G \in \mathcal{G} \mid H \subseteq G\}$.

2.4. Lemma [8]. *Let $\mathcal{G}$ generate the hereditary compositive property $\mathcal{P}$, and let H be an arbitrary graph in $\mathcal{P}$. Then $\mathcal{G}[H]$ also generates $\mathcal{P}$.* $\square$

For a generating set $\mathcal{G} \subseteq \mathcal{M}^*(\mathcal{P})$, let $\mathcal{G}^{\downarrow} := \{G \in \mathcal{G} \mid dc(G) = dc(\mathcal{P})\}$.

2.5. Lemma [8]. *If $\mathcal{G} \subseteq \mathcal{M}^*(\mathcal{P})$ generates the hereditary compositive property $\mathcal{P}$, then so does $\mathcal{G}^{\downarrow}$.* $\square$

3 Unique factorisation for hereditary compositive properties

Our interpretation of [8] is that Mihók et al. proved that every hereditary compositive property $\mathcal{P}$ has a factorisation into $dc(\mathcal{P})$ indecomposable factors. Therefore, reducibility and decomposability are the same thing. Our purpose here is to show that every hereditary compositive property has at most one factorisation into indecomposable hereditary compositive factors. We do so in the following two results.

3.1. Theorem. *Let $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$ be a factorisation of the hereditary compositive property $\mathcal{P}$ into indecomposable hereditary compositive properties. Then $m = dc(\mathcal{P})$.*

3.2. Theorem. *A hereditary compositive property $\mathcal{P}$ can have only one factorisation with exactly $dc(\mathcal{P})$ indecomposable hereditary compositive factors.*

The following result from [8] shows there is at least one factorisation.

3.3. Theorem [8]. *A hereditary compositive property has a factorisation into $dc(\mathcal{P})$ (necessarily indecomposable) hereditary compositive factors.* $\square$

This in turn implies the following.

3.4. Corollary [8]. *A hereditary compositive property is irreducible if and only if it is indecomposable.* $\square$

Putting this all together, we conclude:

3.5. Hereditary Compositive Unique Factorisation Theorem. *A hereditary compositive property has a unique factorisation into irreducible hereditary compositive factors, and the number of factors is exactly $dc(\mathcal{P})$.* $\square$

Our proofs depend heavily on the following construction of a generating set for $\mathcal{P}$. Suppose $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$ is a factorisation of $\mathcal{P}$ into indecomposable hereditary compositive factors, and, for each i, we are given a generating set $\mathcal{G}_i \subseteq \mathcal{M}^*(\mathcal{P}_i)$ and a graph $H_i \in \mathcal{P}_i$. By Lemmas 2.4 and 2.5, the set $\mathcal{G}_i^{\downarrow}[H_i] := \{G \in \mathcal{G}_i \mid H_i \subseteq G,\ dc(G) = 1\}$ is also a generating set for $\mathcal{P}_i$.

We set[2] $\mathcal{G}_1^{\downarrow}[H_1] + \cdots + \mathcal{G}_m^{\downarrow}[H_m] := \{G_1 + \cdots + G_m \mid \forall\, i\ G_i \in \mathcal{G}_i^{\downarrow}[H_i]\}$. This is clearly a generating set for $\mathcal{P}$, but need not consist of $\mathcal{P}$-maximal graphs (even if $m = dc(\mathcal{P})$). However, we can add edges to each graph $G_1 + \cdots + G_m$ until we get (in all possible ways) a $\mathcal{P}$-maximal graph G'. Using $G \subset\!\!\cdot\; H$ to mean that G is a spanning subgraph of H, we can now describe the generating set we want:

$$\begin{aligned}(\mathcal{G}_1[H_1] + \cdots + \mathcal{G}_m[H_m])^{\downarrow} \ :=\ & \{G' \in \mathcal{M}^*(\mathcal{P}) \mid dc(G') = dc(\mathcal{P}),\ \text{and} \\ & \exists\, G \in \mathcal{G}_1^{\downarrow}[H_1] + \cdots + \mathcal{G}_m^{\downarrow}[H_m]\,,\ G \subset\!\!\cdot\; G'\}.\end{aligned}$$

The following is immediate from the definition, and from Lemma 2.5.

3.6. Lemma. *Let $\mathcal{G} = (\mathcal{G}_1[H_1] + \cdots + \mathcal{G}_m[H_m])^{\downarrow}$. Then:*

1. *$\mathcal{G}$ is a generating set for $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$;*

[2] Our notation extends easily to the join of any m sets: $\mathcal{G}_1 + \cdots + \mathcal{G}_m$; and to generating sets that contain several specified subgraphs: $\mathcal{G}[H_1, \ldots, H_r]$.

2. *if $G \in \mathcal{G}$, then $dc(G) = dc(\mathcal{P})$; and*

3. *every $G \in \mathcal{G}$ is spanned by a join of m indecomposable graphs, each of which contains a different one of $H_1, \ldots, H_m$.* $\square$

Because we take $G' = G'_1 + \cdots + G'_{dc(\mathcal{P})} \in \mathcal{G}$ to be a *spanned* supergraph of $G = G_1 + \cdots + G_m \in \mathcal{G}_1^{\downarrow}[H_1] + \cdots + \mathcal{G}_m^{\downarrow}[H_m]$, we must have, for each i, $V(G_i) = V(\sum_{j \in J_i} G'_j)$, where $(J_1, J_2, \ldots, J_m)$ is some partition of $\{1, 2, \ldots, n\}$. That is, each of the m ind-parts of G is a spanning subgraph of a join of ind-parts from G'. We note that although $G_i \in \mathcal{P}_i$, none of the G'_j, $j \in J_i$, need be in $\mathcal{P}_i$. In particular, the crucial observation that Theorem 3.1 rests on is that, if $|J_i| > 1$, then $G_i \subsetneqq \sum_{j \in J_i} G'_j$, and, since G_i was $\mathcal{P}_i$-maximal, $\sum_{j \in J_i} G'_j$ is not in $\mathcal{P}_i$.

We present first the proof of Theorem 3.2, since it is simpler.

Proof of Theorem 3.2: Let $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n = \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_n$ be two factorisations of $\mathcal{P}$ into $n = dc(\mathcal{P})$ indecomposable hereditary compositive factors.

Label the $\mathcal{P}_i$'s inductively, beginning with $i = n$, so that, for each i, $\mathcal{P}_i$ is inclusion-wise maximal among $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_i$. For each i, j such that $i > j$, if $\mathcal{P}_i \setminus \mathcal{P}_j \neq \emptyset$, then let $X_{i,j} \in \mathcal{P}_i \setminus \mathcal{P}_j$; if $\mathcal{P}_i \setminus \mathcal{P}_j = \emptyset$, then $\mathcal{P}_i = \mathcal{P}_j$ and we set $X_{i,j}$ to be the null graph. For each i, by compositivity there is an $H_{i,0} \in \mathcal{P}_i$ that contains all the $X_{i,j}$'s as subgraphs. The important point is that if $\{L_1, L_2, \ldots, L_n\}$ is an unordered $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of some graph G such that, for each $i = 1, 2, \ldots, n$, $H_{i,0} \subseteq G[L_i]$, then, by reverse induction on i starting at n, $G[L_i] \in \mathcal{P}_i$.

For each i, let $\mathcal{G}_i = \{G_{i,0}, G_{i,1}, G_{i,2}, \ldots\}$ be a generating set for $\mathcal{P}_i$. When graphs have a double subscript, we will use the second number to denote which step of our construction we are in. We start with $H_0 = H_{1,0} + \cdots + H_{n,0}$.

For each $s \geq 0$, let $H_{s+1} \in (\mathcal{G}_1[H_{1,s}, G_{1,s}] + \cdots + \mathcal{G}_n[H_{n,s}, G_{n,s}])^{\downarrow}$. Then H_{s+1} has an ind-part from each $\mathcal{G}_i[H_{i,s}, G_{i,s}]$; we label the ind-parts as $H_{1,s+1}$, $\ldots$, $H_{n,s+1}$, so that, for each i, $H_{i,1} \subseteq H_{i,2} \subseteq H_{i,3} \subseteq \cdots$

For $\mathcal{G}_i[H_{i,s}, G_{i,s}]$ to be non-empty, we must have $H_{i,s} \in \mathcal{P}_i$. We know that the $H_{i,s+1}$'s give an unordered $\{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$-partition of H_{s+1}. From the earlier remark, for $i = 1, 2, \ldots, n$, $H_{i,s+1} \in \mathcal{P}_i$.

By Lemma 2.1, the ind-parts of H_s form its $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_n\}$-partition, so there is some permutation φ_s of $\{1, 2, \ldots, n\}$ such that, for each i, $H_{i,s} \in \mathcal{Q}_{\varphi_s(i)}$. Since there are only finitely many permutations of $\{1, 2, \ldots, n\}$, there must be some permutation φ that appears infinitely often. Now whenever $\varphi_t = \varphi$, we have $H_{i,1} \subseteq H_{i,2} \subseteq \cdots \subseteq H_{i,t} \in \mathcal{Q}_{\varphi(i)}$, so by heredity, for every

$s \le t$, $H_{i,s}$ is in $\mathcal{Q}_{\varphi(i)}$. Therefore, we can take $\varphi_s = \varphi$ for all s. By re-labelling the $\mathcal{Q}_i$'s, we can assume φ is the identity permutation, so that $H_{i,s} \in \mathcal{Q}_i$ for all i and s.

For each i and s, $G_{i,s-1} \subseteq H_{i,s}$, so that $\mathcal{H}_i := \{H_{i,1}, H_{i,2}, \ldots\}$ is a generating set for $\mathcal{P}_i$. But $\mathcal{H}_i \subseteq \mathcal{Q}_i$, so $\mathcal{P}_i = [\mathcal{H}_i] \subseteq \mathcal{Q}_i$.

By the same reasoning, there is a permutation τ such that $\mathcal{Q}_i \subseteq \mathcal{P}_{\tau(i)}$. We cannot relabel the $\mathcal{P}_i$'s as well, but if $\tau^k(i) = i$, then we have $\mathcal{P}_i \subseteq \mathcal{Q}_i \subseteq \mathcal{P}_{\tau(i)} \subseteq \mathcal{Q}_{\tau(i)} \subseteq \mathcal{P}_{\tau^2(i)} \subseteq \mathcal{Q}_{\tau^2(i)} \subseteq \cdots \subseteq \mathcal{P}_{\tau^k(i)} = \mathcal{P}_i$, so we must have equality throughout; in particular, $\mathcal{P}_i = \mathcal{Q}_i$ for each i. $\square$

Now for the proof of Theorem 3.1.

Proof of Theorem 3.1: Given any generating set $\mathcal{G}_i$ for $\mathcal{P}_i$, every graph in $\mathcal{G}_1 + \cdots + \mathcal{G}_m$ has decomposability m by construction. Then every graph in $(\mathcal{G}_1 + \cdots + \mathcal{G}_m)^{\downarrow} \subseteq \mathcal{M}^*(\mathcal{P})$ has decomposability at least m, so $dc(\mathcal{P}) \ge m$.

If $m < n := dc(\mathcal{P})$, and G is a $\mathcal{P}$-maximal graph with decomposability n, then in any $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-partition of G some $\mathcal{P}_i$-part is the join of two or more ind-parts. There is only a finite number of ways in which this can happen, and we will construct a sequence of generating sets so that each one excludes at least one of the possibilities until we reach a contradiction.

When graphs or sets have a double subscript, we will use the second number to denote which step of our construction we are in. For each i, we start with some generating set $\mathcal{G}_i$ consisting only of indecomposable $\mathcal{P}_i$-strict graphs.

Let $H_1 \in (\mathcal{G}_1 + \cdots + \mathcal{G}_m)^{\downarrow}$; then H_1 is a join $H_{1,1} + \cdots + H_{n,1}$ of n ind-parts. In general suppose we have graphs $H_1, H_2, \ldots, H_{k-1}$ such that, for each $s = 1, 2, \ldots, k-1$:

(a) H_s is $\mathcal{P}$-maximal;

(b) $dc(H_s) = n$, and $H_s = H_{1,s} + \cdots + H_{n,s}$;

(c) for $j = 1, \ldots, n$, $H_{j,1} \subseteq H_{j,2} \subseteq \cdots \subseteq H_{j,k-1}$; and

(d) there is a partition $(J_{1,s}, J_{2,s}, \ldots, J_{m,s})$ of $\{1, 2, \ldots, n\}$ such that

$$\sum_{j \in J_{i,s}} H_{j,s} \in \mathcal{P}_i.$$

Now let $H_k \in (\mathcal{G}_1[\sum_{j\in J_{1,(k-1)}} H_{j,(k-1)}] + \cdots + \mathcal{G}_m[\sum_{j\in J_{m,(k-1)}} H_{j,(k-1)}])^{\downarrow}$. As H_k contains H_{k-1}, by Lemma 2.2 we can label the ind-parts of $H_k = H_{1,k} + \cdots + H_{n,k}$ so that $H_{1,(k-1)} \subseteq H_{1,k}, \ldots, H_{n,(k-1)} \subseteq H_{n,k}$. It is important to note that the indecomposable graph from $\mathcal{G}_i[\sum_{j\in J_{i,(k-1)}} H_{j,(k-1)}]$ therefore spans $\sum_{j\in J_{i,(k-1)}} H_{j,k}$ (note the change in subscript). By Lemma 2.1 there is a partition $(J_{1,k}, \cdots, J_{m,k})$ of $\{1, 2, \ldots, n\}$ so that $\sum_{j\in J_{i,k}} H_{j,k} \in P_i$.

Since there is only a finite number of partitions of $\{1, 2, \ldots, n\}$, at some step B we must end up with a partition that occurred at some previous step $A < B$. Without loss of generality, suppose that $|J_{1,A}| = r \geq 2$. Then $\sum_{j\in J_{1,A}} H_{j,A} \in \mathcal{P}_1$; the indecomposable graph H_A from $\mathcal{G}_1[\sum_{j\in J_{1,A}} H_{j,A}]$ that is used in step $A+1$ spans $\sum_{j\in J_{1,A}} H_{j,(A+1)}$; this join properly contains the $\mathcal{P}_1$-maximal graph H_A and therefore is not in $\mathcal{P}_1$. But, for each j, $H_{j,(A+1)} \subseteq H_{j,(A+2)} \subseteq \cdots \subseteq H_{j,B}$, and so $\sum_{j\in J_{1,A}} H_{j,(A+1)} \subseteq \sum_{j\in J_{1,A}} H_{j,B}$. But $J_{1,A} = J_{1,B}$ and $\sum_{j\in J_{1,B}} H_{j,B} \in \mathcal{P}_1$, so $\sum_{j\in J_{1,A}} H_{j,(A+1)} \in \mathcal{P}_1$, a contradiction.

Thus we must have $|J_{i,A}| = 1$, for each $i = 1, 2, \ldots, m$, and so $m = n$. $\square$

4 Definitions and results from [7]

This section and the next are the induced-hereditary analogues of Sections 2 and 3, along with a highly important result (Theorem 4.9) adapted from [7].

In [7] Mihók generalised the results of [8] from additive hereditary properties to the wider class of additive *induced*-hereditary graph properties (we point out again, though, that hereditary *compositive* properties are not all additive); the concepts introduced in that article are presented here. We caution the reader that there are some significant differences between the old and new definitions of "generating set", "join", "decomposability", "$\mathcal{P}$-strict" and "ind-part"; these new definitions will apply throughout the rest of the paper, even for hereditary properties (that are *a fortiori* induced-hereditary).

We use $G \leq H$ to denote that G is an induced subgraph of H. The smallest induced-hereditary property that contains a set $\mathcal{G}$ is denoted by $\langle\mathcal{G}\rangle$. This is the induced-hereditary property *generated by* $\mathcal{G}$, or that $\mathcal{G}$ *generates*. We say that $\mathcal{G}$ *is a generating set for* $\mathcal{P}$ if $\langle\mathcal{G}\rangle = \mathcal{P}$. It is easy to see that:

$$\langle\mathcal{G}\rangle \;=\; \{G \mid \exists\, H \in \mathcal{G},\; G \leq H\}.$$

The $*$-*join* of n graphs $G_1, \ldots, G_n$ is the set

$$G_1 * \cdots * G_n := \{G \mid \bigcup_{i=1}^{n} G_i \subseteq G \subseteq \sum_{i=1}^{n} G_i\}$$

where $\bigcup$ and $\sum$ represent disjoint union and join, respectively. Given n sets of graphs, we define their $*$-join by

$$S_1 * \cdots * S_n := \bigcup (G_1 * \cdots * G_n) \ ,$$

the union being over all ways of the selecting the G_i so that $G_i \in S_i$ for all i. We note that this is just the same as $S_1 \circ \cdots \circ S_n$, but it is aesthetically pleasing to have the $*$ notation. If $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are additive properties, and $G_i \in \mathcal{P}_i$ for all i, then for all positive integers k we have

$$kG_1 * \cdots * kG_n \subseteq \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$$

where kG is the disjoint union of k copies of G.

A *$\mathcal{P}$-decomposition of G with n parts* is a partition $(V_1, \ldots, V_n)$ of $V(G)$ such that for all i $V_i \neq \emptyset$, and for all positive integers k we have $kG[V_1] * \cdots * kG[V_n] \subseteq \mathcal{P}$. The *$\mathcal{P}$-decomposability number $dec_{\mathcal{P}}(G)$* of G is the maximum number of parts in a $\mathcal{P}$-decomposition of G; if $G \notin \mathcal{P}$, then we put $dec_{\mathcal{P}}(G) = 0$. If $G \in \mathcal{P}$, then, for all positive integers k, $kG \in \mathcal{P}$; therefore $G \in \mathcal{P}$ if and only if $dec_{\mathcal{P}}(G) \geq 1$. Also, G is *$\mathcal{P}$-decomposable* if $dec_{\mathcal{P}}(G) > 1$. If $\mathcal{P}$ is the product of two additive induced-hereditary properties, then *every* graph in $\mathcal{P}$ with at least two vertices is $\mathcal{P}$-decomposable.

4.1. Lemma. *Let $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$, where the $\mathcal{P}_i$'s are additive properties of graphs. Then any $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-partition of a graph G is a $\mathcal{P}$-decomposition of G. If the $\mathcal{P}_i$'s are induced-hereditary, then every graph in $\mathcal{P}$ with at least m vertices has a partition with all m parts non-empty.* □

A graph G is *$\mathcal{P}$-strict* if $G \in \mathcal{P}$ but $G * K_1 \not\subseteq \mathcal{P}$; we denote the set of $\mathcal{P}$-strict graphs by $\mathbf{S}(\mathcal{P})$. If $f(\mathcal{P}) = \min\{|V(F)| \mid F \notin \mathcal{P}\}$, then $G * K_1 * \cdots * K_1 \not\subseteq \mathcal{P}$, where the $*$ operation is repeated $f(\mathcal{P})$ times. Thus, every $G \in \mathcal{P}$ is an induced-subgraph of some $\mathcal{P}$-strict graph (with fewer than $|V(G)| + f(\mathcal{P})$ vertices), and so $\langle \mathbf{S}(\mathcal{P}) \rangle = \mathcal{P}$. Similarly, $dec_{\mathcal{P}}(G) < f(\mathcal{P})$.

The *decomposability number $dec(\mathcal{G})$* of a generating set $\mathcal{G}$ of $\mathcal{P}$ is

$$\min\{dec_{\mathcal{P}}(G) \mid G \in \mathcal{G}\};$$

the *decomposability number* $dec(\mathcal{P})$ of $\mathcal{P}$ is $dec(\mathbf{S}(\mathcal{P}))$. A property with $dec(\mathcal{P}) = 1$ is *indecomposable*. An indecomposable property is also irreducible and it will turn out that the converse is also true.

4.2. Lemma. *Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be induced-hereditary properties of graphs, and let G be a $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$-strict graph. Then, for every $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-partition $(V_1, \ldots, V_m)$ of $V(G)$, $G[V_i]$ is $\mathcal{P}_i$-strict (and in particular non-empty).* $\square$

It follows that $dec(\mathcal{A} \circ \mathcal{B}) \geq dec(\mathcal{A}) + dec(\mathcal{B})$, and thus any factorisation of an additive induced-hereditary property $\mathcal{P}$ has at most $dec(\mathcal{P})$ irreducible additive induced-hereditary factors.

4.3. Lemma [7]. *Let G be a $\mathcal{P}$-strict graph, and let $G' \in \mathcal{P}$ be an induced supergraph of G, i.e., $G \leq G'$. Then G' is $\mathcal{P}$-strict, and $dec_{\mathcal{P}}(G) \geq dec_{\mathcal{P}}(G')$.*

Proof: Every graph in $G * K_1$ is an induced subgraph of a graph in $G' * K_1$, so G' must be $\mathcal{P}$-strict. If $(V_1, \ldots, V_n)$ is a $\mathcal{P}$-decomposition of G' with n parts, then $(V_1 \cap V(G), \ldots, V_n \cap V(G))$ is a $\mathcal{P}$-decomposition of G; moreover, it has n parts unless, for some i, $V_i \cap V(G) = \emptyset$, which is impossible because G is $\mathcal{P}$-strict. $\square$

4.4. Lemma [7]. *If $\mathcal{G}$ generates the induced-hereditary property $\mathcal{P}$, then $dec(\mathcal{G}) \leq dec(\mathbf{S}(\mathcal{P}))$, with equality if $\mathcal{G} \subseteq \mathbf{S}(\mathcal{P})$.* $\square$

For $\mathcal{G} \subseteq \mathcal{P}$, and $H \in \mathcal{P}$, let $\mathcal{G}[H] := \{G \in \mathcal{G} \mid H \leq G\}$.

4.5. Lemma [7]. *Let $\mathcal{G}$ generate the additive induced-hereditary property $\mathcal{P}$, and let H be an arbitrary graph in $\mathcal{P}$. Then $\mathcal{G}[H]$ also generates $\mathcal{P}$.* $\square$

For a generating set $\mathcal{G}$, let $\mathcal{G}^{\downarrow} := \{G \in \mathcal{G} \mid G \in \mathbf{S}(\mathcal{P}),\ dec_{\mathcal{P}}(G) = dec(\mathbf{S}(\mathcal{P}))\}$. The following is a simple consequence of Lemmas 4.3 and 4.5.

4.6. Lemma [7]. *If $\mathcal{G}$ generates the additive induced-hereditary property $\mathcal{P}$, then so does $\mathcal{G}^{\downarrow}$.* $\square$

A graph $G \in \mathcal{P}$ is *uniquely $\mathcal{P}$-decomposable* if there is only one $\mathcal{P}$-decomposition of G with $dec_{\mathcal{P}}(G)$ parts. If $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$, then by Lemma 4.1

a uniquely $\mathcal{P}$-decomposable graph G with $dec_{\mathcal{P}}(G) = n$ must be uniquely $\{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$-partitionable (every $\{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$-partition gives the same unordered partition of $V(G)$). If $(V_1, \ldots, V_n)$ is the unique $\mathcal{P}$-decomposition of G, we call the graphs $G[V_1], \ldots, G[V_n]$ its *ind-parts* (although they are themselves usually $\mathcal{P}$-decomposable).

4.7. Lemma. *Let G be a graph in $\mathbf{S}(\mathcal{P})$ with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$, and suppose that G has a unique $\mathcal{P}$-decomposition $(V_1, \ldots, V_{dec(\mathcal{P})})$ with $dec(\mathcal{P})$ parts. If $G \leq H$, then $H \in \mathbf{S}(\mathcal{P})$, $dec_{\mathcal{P}}(H) = dec(\mathcal{P})$, and, for any $\mathcal{P}$-decomposition $(W_1, \ldots, W_{dec(\mathcal{P})})$ of H, we can relabel the W_i's so that, for all i, $W_i \cap V(G) = V_i$.* □

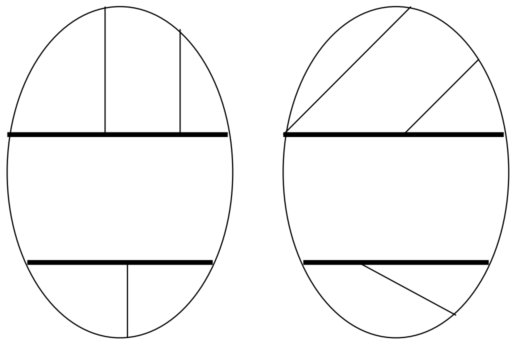
In the hereditary case it was very important that if $G = G_1 + \cdots + G_m$, each G_i is the join of ind-parts (the partition into ind-parts "respected" the partition into G_i's); in the induced-hereditary case we can prove analogous results (e.g., Corollaries 4.10–4.13) for $\mathcal{P}$-strict, uniquely $\mathcal{P}$-decomposable graphs with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$, which allows us to generalise Theorem 3.1.

4.8. Definitions. Let $d_0 = (U_1, U_2, \ldots, U_m)$ be a $\mathcal{P}$-decomposition of a graph G. A $\mathcal{P}$-decomposition $d_1 = (V_1, V_2, \ldots, V_n)$ of G *respects* d_0 if no V_i intersects two or more U_j's; that is, each V_i is contained in some U_j, and so each U_j is a union of V_i's.

If G is a graph, then $s\circledast G$ denotes the set $G * G * \cdots * G$, where there are s copies of G. For $G^* \in s\circledast G$, denote the copies of G by $G^1, \ldots, G^s$. Then G^* *respects* d_0 if $G^* \in sG[U_1] * \cdots * sG[U_m]$; that is, two vertices in different G^i's are joined by an edge only if they are also contained in different U_j's. A $\mathcal{P}$-decomposition $d = (V_1, \ldots, V_n)$ of G^* *respects* d_0 *uniformly* if, for each V_i, there is a U_j such that, for every G^k, $V_i \cap V(G^k) \subseteq U_j$. The decomposition of G^k induced by d is denoted $d|G^k$.

If G is uniquely $\mathcal{P}$-decomposable, its ind-parts *respect* d_0 if its unique $\mathcal{P}$-decomposition with $dec_{\mathcal{P}}(G)$ parts respects d_0. If G^* is uniquely $\mathcal{P}$-decomposable, its ind-parts *respect* d_0 *uniformly* if: (a) for some s, $G^* \in s\circledast G$; (b) G^* respects d_0; and (c) G^*'s unique $\mathcal{P}$-decomposition with $dec_{\mathcal{P}}(G^*)$ parts respects d_0 uniformly.

The *extension of* d_0 to G^* is the decomposition obtained by repeating d_0 on each copy of G. If G^* respects d_0, or if it has a $\mathcal{P}$-decomposition that



G^i G^j

Figure 1: d (vertical lines) respects d_0 (horizontal lines) uniformly

respects d_0 uniformly, then the extension of d_0 is also a $\mathcal{P}$-decomposition of G^*. In particular, G^* is a graph in $\mathcal{P}$.

We will sometimes write $G^i \cap U_x$ (or just U_x when it is clear we are referring to G^i) to mean the vertices of G^i that correspond to U_x, and $G^* \cap U_x$ (or just U_x, when it is clear from the context) to mean $\bigcup_i (G^i \cap U_x)$.

The required result is a corollary of the following theorem of Mihók; he actually proved it when $m = n$ (Corollary 4.11), but very little modification is needed to establish the general case, and we follow his proof and notation rather closely.

4.9. Theorem. *Let G be a $\mathcal{P}$-strict graph with $dec_{\mathcal{P}}(G) = n$, and let $d_0 = (U_1, U_2, \ldots, U_m)$ be a fixed $\mathcal{P}$-decomposition of G. Then there is a $\mathcal{P}$-strict graph $G^* \in s \circledast G$ (for some s) that respects d_0, and moreover any $\mathcal{P}$-decomposition of G^* with n parts respects d_0 uniformly.*

Proof: Let $d_i = (V_{i,1}, V_{i,2}, \ldots, V_{i,n})$, $i = 1, \ldots, r$, be the $\mathcal{P}$-decompositions of G with n parts which do not respect d_0. Since G is a finite graph, r is a nonnegative integer. If $r = 0$, take $G^* = G$; otherwise we will construct a graph $G^* = G^*(r) \in s \circledast G$ as above, denoting the s copies of G by $G^1, \ldots, G^s$.

If the resulting G^* has a $\mathcal{P}$-decomposition d with n parts, then, since G is $\mathcal{P}$-strict, $d|G^i$ will also have n parts. The aim of the construction is to add new edges $E^* = E^*(r)$ to sG to exclude the possibility that $d|G^i = d_j$,

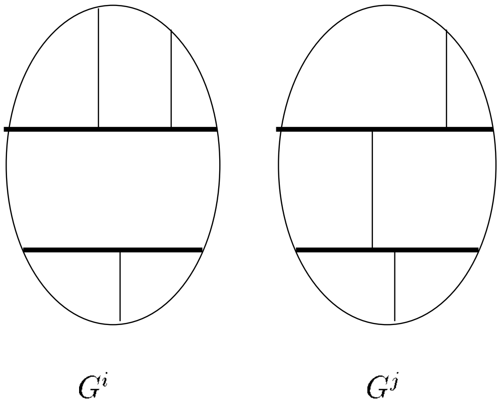


Figure 2: d (vertical lines) respects d_0 (horizontal lines) on both G^i and G^j, but not uniformly

for any $1 \leq i \leq s, 1 \leq j \leq r$. We will only add edges between $G^i \cap U_x$ and $G^j \cap U_y$, where $i \neq j$ and $x \neq y$, so that G^* will respect d_0.

We shall use two types of constructions.

Construction 1. $G^i \Rightarrow G^j$.

This is a graph in $2 \circledast G$ such that, if d is a $\mathcal{P}$-decomposition of $G^i \Rightarrow G^j$ and $d|G^i$ respects d_0, then $d|G^j$ respects d_0; moreover, d respects d_0 uniformly on $G^i \Rightarrow G^j$. (We comment that this corrects a minor error in [7]. The author of [7] was independently aware of both the error and its correction.)

Since G is $\mathcal{P}$-strict, there is a graph $F \in (G * K_1) \setminus \mathcal{P}$. Let $N_F(z)$ be the neighbours in G of $z \in V(K_1)$. For $y = 1, 2, \ldots, m$, let Z_y denote $U_y \cap N_F(z)$. Let $G^i, G^j, i \neq j$ be disjoint copies of G; join every vertex of U_x in G^j to every vertex of Z_y, $x \neq y$, in G^i. Note that $G^i \Rightarrow G^j \in 2G[U_1]*2G[U_2]*\cdots*2G[U_m]$. Since d_0 is a $\mathcal{P}$-decomposition of G, this implies that $(G^i \Rightarrow G^j) \in \mathcal{P}$.

Let $d = (V_1, V_2, \ldots, V_\ell)$ be a $\mathcal{P}$-decomposition of $H = (G^i \Rightarrow G^j)$ such that $d|G^i$ respects d_0, but $d|G^j$ does not respect d_0 (or at least, not in the same manner, i.e., d does not respect d_0 uniformly). Then there is a k such that $V_k \cap G^i \subseteq U_y$, but $v \in V_k \cap G^j$ belongs to U_x, $x \neq y$. We claim F is an induced subgraph of a graph in $H[V_1] * H[V_2] * \cdots * H[V_\ell]$, which implies $F \in \mathcal{P}$, a contradiction.

To see this, consider the vertex v and G^i. We have edges from v to every vertex in Z_w, for all $w \neq x$. We are only missing the edges from v to

$Z_x \cap G^i$. But $d|G^i$ respects d_0, so $U_x \cap G^i$ is the union of, say, $V_{t_1} \cap G^i, V_{t_2} \cap G^i, \ldots, V_{t_r} \cap G^i$. Since $V_k \cap G^i \subseteq U_y$ and $y \neq x$, $k \notin \{t_1, t_2, \ldots, t_r\}$. Since d is a $\mathcal{P}$-decomposition of H, we may freely add edges between V_i's and remain in $\mathcal{P}$; in particular, one graph in $H[V_1] * H[V_2] * \cdots * H[V_\ell]$ is obtained by adding precisely the edges between v and $Z_x \cap G^i$. Clearly F is the subgraph of this graph induced by G^i and v, as claimed.

Construction 2. $m \bullet k_t G$.

For a $\mathcal{P}$-decomposition $d_t = (V_{t,1}, V_{t,2}, \ldots, V_{t,dec_{\mathcal{P}}(G)})$ of G that does not respect d_0, $m \bullet k_t G$ is a graph in $(mk_t) \circledast G$ having no $\mathcal{P}$-decomposition $d = (W_1, W_2, \ldots, W_{dec_{\mathcal{P}}(G)})$ such that, for all of the mk_t induced copies G^i of G, $d|G^i = d_t$.

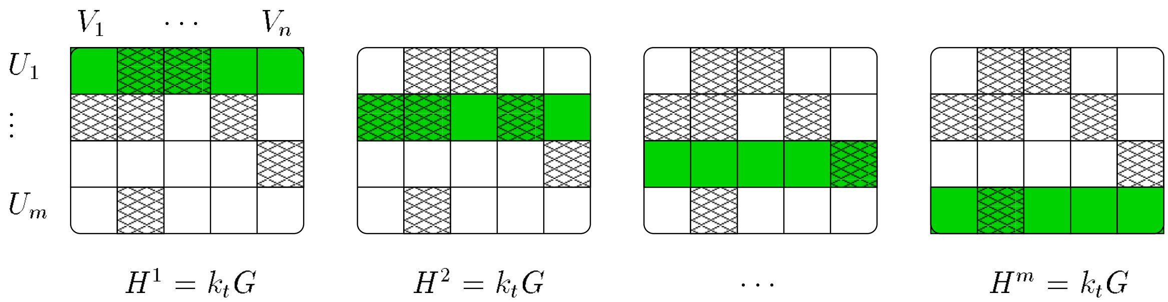


Figure 3: $m \bullet k_t G$ - we only put edges between the m shaded parts

Let $n = dec_{\mathcal{P}}(G)$ and let $A_{i,j}(t)$ denote $U_i \cap V_{t,j}, 1 \leq i \leq m, 1 \leq j \leq n$. Since d_t does not respect d_0, at least $n+1$ sets $A_{i,j}(t)$ are nonempty. Because $dec_{\mathcal{P}}(G) = n$, there exists a positive integer k_t such that $k_t G[A_{1,1}(t)] * k_t G[A_{1,2}(t)] * \cdots * k_t G[A_{m,n}(t)] \not\subset \mathcal{P}$. Fix a graph $F_t \in (k_t G[A_{1,1}(t)] * k_t G[A_{1,2}(t)] * \cdots * k_t G[A_{m,n}(t)]) \setminus \mathcal{P}$.

Suppose that in $H = k_t G$ we replace the edges between $H \cap U_x$ and $H \cap U_y$ by the edges that there are between $F_t \cap U_x$ and $F_t \cap U_y$, for all $x \neq y$; the U_x's still form a $\mathcal{P}$-decomposition of the resulting graph, $\tilde{H}$, so it is in $\mathcal{P}$. If the extension of d_t were also a $\mathcal{P}$-decomposition of $\tilde{H}$, we could obtain F_t immediately by replacing the edges between $\tilde{H} \cap V_{t,i}$ and $\tilde{H} \cap V_{t,j}$, by those between $F_t \cap V_{t,i}$ and $F_t \cap V_{t,j}$, for all $i \neq j$. The only problem is that $\tilde{H}$ does not contain k_t disjoint copies of G, as we altered edges *inside* the copies of G.

So instead we construct $m \bullet k_t G$ from m disjoint copies of $H = k_t G$, denoted by $H^j, j = 1, 2, \ldots, m$ (see Figure 3). We add edges between $H^x \cap U_x$ and $H^y \cap U_y$, corresponding to the edges that there are between $F_t \cap U_x$ and $F_t \cap U_y$, for all $x \neq y$. Because $d_0 = (U_1, U_2, \ldots, U_m)$ is a $\mathcal{P}$-decomposition, H is in $\mathcal{P}$.

Now $H^1 \cap U_1, \ldots, H^m \cap U_m$ form a copy of $\tilde{H}$ in $m \bullet k_t G$. Suppose $H' = m \bullet k_t G$ has a $\mathcal{P}$-decomposition $d = (W_1, W_2, \ldots, W_n)$ such that, for every one of the mk_t induced copies G^i of G, $d|G^i = d_t$; then we can obtain F_t as an induced subgraph of a graph in $H'[W_1] * H'[W_2] * \cdots * H'[W_n]$ (by changing edges in the copy of $\tilde{H}$ as explained above).

We now construct G^* as follows. First let $G(1) := m \bullet k_1 G$. For $1 < \ell \leq r$, construct $G(\ell)$ by taking mk_ℓ disjoint copies $G(\ell-1)^1, \ldots, G(\ell-1)^{mk_\ell}$ of $G(\ell - 1)$. For each copy G^i of G in $G(\ell-1)^i$ and each copy G^j of G in $G(\ell-1)^j$, we add the edges between them that are between the i^{th} and j^{th} copies of G in $m \bullet k_\ell G$. (See Figure 4.)

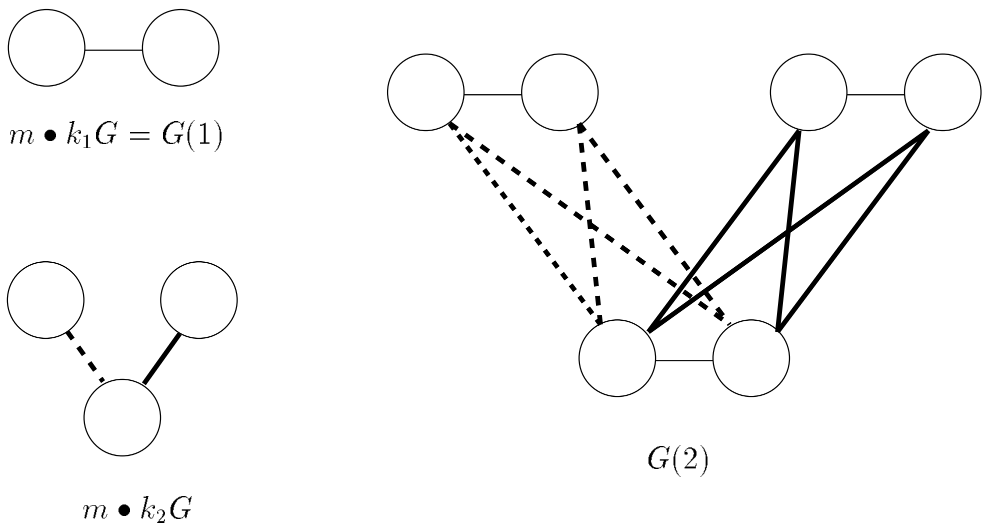


Figure 4: Constructing $G(2)$ from $G(1)$ and $m \bullet k_2 G$

Finally, from $G(r)$, which is in, say, $s \circledast G$, consisting of copies G^1, G^2, $\ldots$, G^s of G, we create G^* by adding two more copies G^0 and G^{s+1} of G.

For each $i \in \{1, 2, \ldots, s\}$, we add the edges between G^0 and G^i to create the graph $G^0 \Rightarrow G^i$, we add the edges between G^i and G^{s+1} to create the graph $G^i \Rightarrow G^{s+1}$, and we add the edges between G^{s+1} and G^0 to create the graph $G^{s+1} \Rightarrow G^0$.

Let d be a $\mathcal{P}$-decomposition of G^* with n parts (it might be that none exists, in which case we are done). For $1 \le \ell \le r$, if every copy of $G(\ell - 1)$ in $G(\ell)$ contains a copy of G for which $d|G = d_\ell$, then we would have mk_ℓ such copies of G inducing a copy of $m \bullet k_\ell G$, which we know is impossible. So by induction from r to 1, there is a copy G^p of G for which $d|G^p$ is none of $d_1, d_2, \ldots, d_r$. Thus, $d|G^p$ respects d_0. But $G^p \Rightarrow G^{s+1}$ is an induced subgraph of G^*, so $d|G^{s+1} = d_0$ (and in fact d respects d_0 uniformly on these two copies of G). Similarly, $d|G^0$ respects d_0 and, again in the same way, d respects d_0 uniformly, as required. $\square$

4.10. Corollary. *Let G be a $\mathcal{P}$-strict graph with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$, and let $d_0 = (U_1, U_2, \ldots, U_m)$ be a fixed $\mathcal{P}$-decomposition of G. Then there is a $\mathcal{P}$-decomposition of G with exactly $dec(\mathcal{P})$ parts that respects d_0.*

Proof: In Theorem 4.9, since $G^* \ge G$ we know G^* is $\mathcal{P}$-strict, and so $dec(\mathcal{P}) \le dec_{\mathcal{P}}(G^*) \le dec_{\mathcal{P}}(G) = dec(\mathcal{P})$. Thus G^* has at least one $\mathcal{P}$-decomposition d with $dec(\mathcal{P})$ parts; $d|G$ also has $dec(\mathcal{P})$ parts (since G is $\mathcal{P}$-strict) and respects d_0. $\square$

4.11. Corollary [7]. *Let G be a $\mathcal{P}$-strict graph with $dec_{\mathcal{P}}(G) = n$, and let $d_0 = (U_1, U_2, \ldots, U_n)$ be a fixed $\mathcal{P}$-decomposition of G with n parts. Then there is a $\mathcal{P}$-strict graph $G^* \in s \circledast G$ (for some s) which has a unique $\mathcal{P}$-decomposition d with n parts, and $d|G^j = d_0$ for all j.*

Proof: The only $\mathcal{P}$-decomposition of G with n parts that respects d_0 is d_0 itself (since here d_0 has exactly n parts). Thus in Theorem 4.9, the only possible decomposition of G^* with n parts is the extension of d_0, which is a $\mathcal{P}$-decomposition of G^* by construction. $\square$

The next result tells us that under certain conditions, given a factorisation $\mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_m$ of $\mathcal{P}$ into additive induced-hereditary properties, and a $\mathcal{P}$-decomposition d_0 of G, we can group the parts of d_0 to get a $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition of G. Of course, d_0 does not respect *all* $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partitions; in fact, if $m = dec(\mathcal{P})$, d_0 can only respect one partition, namely d_0 itself (note that none of the parts of a partition can be empty, because G is $\mathcal{P}$-strict). We will see later (Theorem 5.3) that when we factor the $\mathcal{Q}_i$'s as far as possible we do get exactly $dec(\mathcal{P})$ irreducible factors, say $\mathcal{P}_1, \ldots, \mathcal{P}_{dec(\mathcal{P})}$, and applying the corollary we get that d_0 *is* a $(\mathcal{P}_1, \ldots, \mathcal{P}_{dec(\mathcal{P})})$-partition.

4.12. Corollary. *Let G be a $\mathcal{P}$-strict graph with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$, and with some $\mathcal{P}$-decomposition $d_0 = (U_1, U_2, \ldots, U_{dec(\mathcal{P})})$. If $\mathcal{P} = \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_m$, then G has a $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition that d_0 respects.*

Proof: The graph G^* of Corollary 4.11 has some $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition d_1; this is also a $\mathcal{P}$-decomposition. By Corollary 4.10 the unique $\mathcal{P}$-decomposition d of G^* with $dec(\mathcal{P})$ parts must respect d_1; and the restriction of d to G is just d_0. $\square$

The set of $\mathcal{P}$-strict, uniquely $\mathcal{P}$-decomposable graphs with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$ is denoted $\mathbf{S}^{\Downarrow}(\mathcal{P})$, or just $\mathbf{S}^{\Downarrow}$. By Lemma 4.6 and Corollary 4.11 $\mathbf{S}^{\Downarrow}$ is a generating set for $\mathcal{P}$; in fact, for any $G \in \mathbf{S}^{\downarrow}$ and any specific $\mathcal{P}$-decomposition d of G, we can find an induced supergraph in $\mathbf{S}^{\Downarrow}$ whose ind-parts uniformly respect d.

4.13. Corollary. *Let G be a $\mathcal{P}$-strict graph with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$, and let $d_0 = (U_1, U_2, \ldots, U_m)$ be a fixed $\mathcal{P}$-decomposition of G. Then there is a uniquely $\mathcal{P}$-decomposable $\mathcal{P}$-strict graph $G^* \geq G$ whose ind-parts respect d_0 uniformly.* $\square$

4.14. Corollary. *Let $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{dec(\mathcal{P})}$. Let G be a $\mathcal{P}$-strict graph with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$. If $d_0 = (U_1, U_2, \ldots, U_m)$ is a $\mathcal{P}$-decomposition of G, then there is a factorization $\mathcal{P} = \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_m$ such that d_0 is a $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition of G.*

Proof: By Corollary 4.13 there is a a uniquely $\mathcal{P}$-decomposable graph $G^* \geq G$ whose ind-parts respect d_0 uniformly. Let $(V_1, V_2, \ldots, V_{dec(\mathcal{P})})$ be the unique $\mathcal{P}$-decomposition of G^*. By Lemma 4.1, the ind-parts of G^* must form its unique $(\mathcal{P}_1, \ldots, \mathcal{P}_{dec(\mathcal{P})})$-partition, so there is a partition $(J_1, J_2, \ldots, J_m)$ of $\{1, 2, \ldots, dec(\mathcal{P})\}$ such that, for each i, $U_i = \cup_{j \in J_i} V_j$ (when we restrict the V_j to a particular copy of G in G^*). It follows that $G[U_i] \in \prod_{j \in J_i} \mathcal{P}_j$, so we may set $\mathcal{Q}_i = \prod_{j \in J_i} \mathcal{P}_j$. $\square$

5 Unique factorisation for additive induced-hereditary properties

The strategy for proving the uniqueness of the factorisation of an additive induced-hereditary property into irreducible additive induced-hereditary properties is the same as for hereditary compositive properties. We shall first show that there is at most one into $dec(\mathcal{P})$ factors and then that any such factorisation must have $dec(\mathcal{P})$ factors.

The following construction of a generating set for $\mathcal{P}$ will be essential in proving unique factorisation. Suppose we are given a factorisation $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$ into indecomposable additive induced-hereditary factors, and, for each i, we are given a generating set $\mathcal{G}_i$ of $\mathcal{P}_i$ and a graph $H_i \in \mathcal{P}_i$. By Lemmas 4.5 and 4.6, the set $\mathcal{G}_i^{\downarrow}[H_i] := \{G \in (\mathcal{G}_i \cap \mathbf{S}(\mathcal{P}_i)) \mid H_i \leq G,\ dec_{\mathcal{P}_i}(G) = 1\}$ is also a generating set for $\mathcal{P}_i$.

The $*$-join of these m sets is then a generating set for $\mathcal{P}$, and we can once again pick out just those graphs that are strict and have minimum decomposability:

$$(\mathcal{G}_1[H_1] * \cdots * \mathcal{G}_m[H_m])^{\downarrow} := \{G' \in \mathbf{S}(\mathcal{P}) \mid dec_{\mathcal{P}}(G') = dec(\mathcal{P}),\ \text{and } \forall\, i,\ 1 \leq i \leq m,\ \exists\, G_i \in \mathcal{G}_i^{\downarrow}[H_i],\ G' \in G_1 * \cdots * G_m\}.$$

5.1. Lemma. *Let $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$. Then: $\mathcal{G} = (\mathcal{G}_1[H_1] * \cdots * \mathcal{G}_m[H_m])^{\downarrow} \subseteq \mathbf{S}(\mathcal{P})$ is a generating set for $\mathcal{P}$; every $G \in \mathcal{G}$ has $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$; and every $G \in \mathcal{G}$ is in the $*$-join of m $\mathcal{P}_i$-indecomposable graphs which contain $H_1, \ldots, H_m$ respectively.* $\square$

We are now ready to prove unique factorisation. As in the hereditary case, we first show that any two factorisations with exactly $dec(\mathcal{P})$ indecomposable factors must be the same, and then prove that any factorisation into indecomposable factors must have exactly $dec(\mathcal{P})$ terms.

5.2. Theorem. *An additive induced-hereditary property $\mathcal{P}$ can have only one factorisation with exactly $dec(\mathcal{P})$ indecomposable factors.*

Proof: Let $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n = \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_n$ be two factorisations of $\mathcal{P}$ into $n = dec(\mathcal{P})$ indecomposable factors. Label the $\mathcal{P}_i$'s inductively, beginning with $i = n$, so that, for each i, $\mathcal{P}_i$ is inclusion-wise maximal among $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_i$. For each i, j such that $i > j$, if $\mathcal{P}_i \setminus \mathcal{P}_j \neq \emptyset$, then let $X_{i,j} \in \mathcal{P}_i \setminus \mathcal{P}_j$; if $\mathcal{P}_i \setminus \mathcal{P}_j = \emptyset$, then $\mathcal{P}_i = \mathcal{P}_j$ and we set $X_{i,j}$ to be the null graph. For each i, set $H_{i,0} := \bigcup_{j<i} X_{i,j}$. Note $H_{i,0} \in \mathcal{P}_i$. The important point is that if $\{L_1, L_2, \ldots, L_n\}$ is an unordered $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of some graph G such that, for each $i = 1, 2, \ldots, n$, $H_{i,0} \leq G[L_i]$, then, by reverse induction on i starting at n, $G[L_i] \in \mathcal{P}_i$.

For each i, let $\mathcal{G}_i = \{G_{i,0}, G_{i,1}, G_{i,2}, \ldots\}$ be a generating set for $\mathcal{P}_i$. We will construct another generating set for each $\mathcal{P}_i$ that will turn out to be contained in some $\mathcal{Q}_j$; for graphs $G_{i,s}, H_{i,s}$, we will use the second subscript to denote which step of our construction we are in.

For each $s \geq 0$, choose a graph $H'_{s+1} \in (\mathcal{G}_1[H_{1,s}, G_{1,s}] * \cdots * \mathcal{G}_n[H_{n,s}, G_{n,s}])^{\downarrow}$, and find an induced supergraph H_{s+1} whose unique $\mathcal{P}$-decomposition with $dec(\mathcal{P})$ parts uniformly respects the obvious decomposition of H'_{s+1}. We label as $H_{i,s+1}$ the ind-part of H_{s+1} that contains the graph from $\mathcal{G}_i[H_{i,s}, G_{i,s}]$. Then, for each i, $H_{i,0} \leq H_{i,1} \leq H_{i,2} \leq \cdots$

For $\mathcal{G}_i[H_{i,s}, G_{i,s}]$ to be non-empty, we must have $H_{i,s} \in \mathcal{P}_i$. We know that the $H_{i,s+1}$'s give an unordered $\{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$-partition of H_{s+1}. From the earlier remark, for $i = 1, 2, \ldots, n$, $H_{i,s+1} \in \mathcal{P}_i$.

The ind-parts of H_s also form its unique $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_n\}$-partition. Thus, there is some permutation φ_s of $\{1, 2, \ldots, n\}$ such that, for each i, $H_{i,s} \in \mathcal{Q}_{\varphi_s(i)}$. Since there are only finitely many permutations of $\{1, 2, \ldots, n\}$, there must be some permutation φ that appears infinitely often. Now whenever $\varphi_t = \varphi$, we have $H_{i,1} \leq H_{i,2} \leq \cdots \leq H_{i,t} \in \mathcal{Q}_{\varphi(i)}$ so by induced-heredity, for every $s \leq t$, $H_{i,s}$ is in $\mathcal{Q}_{\varphi(i)}$. Therefore, we can take $\varphi_s = \varphi$, for all s. By re-labelling the $\mathcal{Q}_i$'s, we can assume φ is the identity permutation, so that $H_{i,s} \in \mathcal{Q}_i$ for all i and s.

Now for each i and s, $G_{i,s-1} \leq H_{i,s}$, so that $\mathcal{H}_i := \{H_{i,1}, H_{i,2}, \ldots\}$ is a generating set for $\mathcal{P}_i$. But $\mathcal{H}_i \subseteq \mathcal{Q}_i$, so $\mathcal{P}_i = \langle \mathcal{H}_i \rangle \subseteq \mathcal{Q}_i$.

By the same reasoning there is a permutation τ such that $\mathcal{Q}_i \subseteq \mathcal{P}_{\tau(i)}$. We cannot relabel the $\mathcal{P}_i$'s as well, but if $\tau^k(i) = i$, then we have $\mathcal{P}_i \subseteq \mathcal{Q}_i \subseteq \mathcal{P}_{\tau(i)} \subseteq \mathcal{Q}_{\tau(i)} \subseteq \mathcal{P}_{\tau^2(i)} \subseteq \mathcal{Q}_{\tau^2(i)} \subseteq \cdots \subseteq \mathcal{P}_{\tau^k(i)} = \mathcal{P}_i$, so we must have equality throughout; in particular, $\mathcal{P}_i = \mathcal{Q}_i$ for each i. $\square$

The second piece is analogous to Theorem 3.1, but the technical details are rather different.

5.3. Theorem. *Let $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$ be a factorisation of the additive induced-hereditary property $\mathcal{P}$ into indecomposable additive induced-hereditary properties. Then $m = dec(\mathcal{P})$.*

Proof: By Lemma 4.1 any $\mathcal{P}$-strict graph G has $dec_{\mathcal{P}}(G) \geq m$, so $dec(\mathcal{P}) \geq m$. To prove the reverse inequality, we suppose $m < n := dec(\mathcal{P})$ and then construct a sequence of graphs until we get a contradiction. When graphs or sets have a double subscript, we will use the second number to denote which step of our construction we are in. For each i, we start with some generating set $\mathcal{G}_i$ consisting only of $\mathcal{P}_i$-indecomposable $\mathcal{P}_i$-strict graphs.

Let $H_1' \in (\mathcal{G}_1 * \cdots * \mathcal{G}_m)^{\downarrow}$, with a corresponding $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-partition $d_1 = (G_{1,1}', \ldots, G_{m,1}')$, where each $G_{i,1}'$ is $\mathcal{P}_i$-strict and $\mathcal{P}_i$-indecomposable. By Corollary 4.13 there is a graph $H_1 \geq H_1'$ in $\mathbf{S}^{\Downarrow}$ whose ind-parts respect d_1 uniformly. That is, denoting the ind-parts by $H_{1,1}, \ldots, H_{n,1}$, there is a partition $(J_{1,1}, J_{2,1}, \ldots, J_{m,1})$ of $\{1, 2, \ldots, n\}$ such that $\bigcup_{j \in J_{i,1}} V(H_{j,1})$ induces $G_{i,1} = t_1 G_{i,1}'$. By additivity of $\mathcal{P}_i$, $t_1 G_{i,1}'$ is in $\mathcal{P}_i$, and by Lemma 4.3 it is $\mathcal{P}_i$-strict and $\mathcal{P}_i$-indecomposable.

In general suppose we have graphs $H_1, H_2, \ldots, H_{k-1}$ such that, for each $s = 1, 2, \ldots, k-1$:

(a) H_s is $\mathcal{P}$-strict and uniquely $\mathcal{P}$-decomposable;

(b) $dec_{\mathcal{P}}(H_s) = n$, with ind-parts $H_{1,s}, \ldots, H_{n,s}$;

(c) $H_1 \leq \cdots \leq H_{k-1}$, with the ind-parts labelled such that, for $j = 1, \ldots, n$, $H_{j,1} \leq H_{j,2} \leq \cdots \leq H_{j,k-1}$;

(d) there is a partition $(J_{1,s}, J_{2,s}, \ldots, J_{m,s})$ of $\{1, 2, \ldots, n\}$ such that the union $\bigcup_{j \in J_{i,s}} V(H_{j,s})$ induces a $\mathcal{P}_i$-indecomposable graph $G_{i,s}$; and

(e) for $p < q$, there is at least one i for which $\bigcup_{j \in J_{i,p}} V(H_{j,q})$ does not induce a graph in $\mathcal{P}_i$.

We will find two graphs H_k', H_k'' before constructing H_k itself. Because $m < n$, some $G_{i,(k-1)}$ contains more than one ind-part. Since $G_{i,(k-1)}$ is $\mathcal{P}_i$-indecomposable, for some t there is some $H_k' \in tH_{1,(k-1)} * \cdots * tH_{n,(k-1)}$ for which $\bigcup_{j \in J_{i,(k-1)}} tV(H_{j,(k-1)})$ does not induce a graph in $\mathcal{P}_i$. Now $H_{k-1} \cup H_k'$ is $\mathcal{P}$-strict with decomposability n (by Lemma 4.3, because it contains H_{k-1}) and has a $\mathcal{P}$-decomposition d_k' with n parts, each part being just $(t+1)H_{j,(k-1)}$. By Corollary 4.11 we find a graph $H_k'' \geq (H_{k-1} \cup H_k')$ in $\mathbf{S}^{\Downarrow}$ whose ind-parts are just the extension of d_k'.

The graph H_k'' has some $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-partition, and we can extend this to a partition d_k of $H_k'' \cup H_1'$ with $G_{i,1}'$ in the $\mathcal{P}_i$-part. We now find a graph $H_k \geq (H_k'' \cup H_1')$ in $\mathbf{S}^{\Downarrow}$ whose ind-parts respect this partition.

Properties (a) and (b) hold for H_k by virtue of being in $\mathbf{S}^{\Downarrow}$. Since $H_{k-1} \leq H_k'' \leq H_k$, and H_{k-1} is uniquely $\mathcal{P}$-decomposable, by Lemma 4.7 we can label the ind-parts of H_k to satisfy (c). Condition (e) then follows for any $p < k-1$, while for $p = k-1$ it holds because of the induced uniquely $\mathcal{P}$-decomposable subgraph H_k'' (which itself contains the "bad" subgraph H_k'). Finally, d_k determines a partition of H_k with the i^{th} part in $\mathcal{P}_i$ (because the ind-parts of H_k respect d_k) and $\mathcal{P}_i$-indecomposable (by Lemma 4.3, since the i^{th} part contains $G_{i,1}'$).

Since there is only a finite number of partitions of $\{1, 2, \ldots, n\}$, at some step B we must end up with a partition that occurred at some previous step $A < B$. But then (d) contradicts (e). □

5.4. Theorem [7]. *An additive induced-hereditary property has a factorisation into dec($\mathcal{P}$) (necessarily indecomposable) additive induced-hereditary factors.* □

5.5. Corollary [7]. *An additive induced-hereditary property is irreducible if and only if it is indecomposable.* □

5.6. Additive Induced-Hereditary Unique Factorisation Theorem. *An additive induced-hereditary property has a unique factorisation into irreducible additive induced-hereditary factors, and the number of factors is exactly dec($\mathcal{P}$).* □

6 Related results

An important consequence of Theorem 5.3 is that, for irreducible $\mathcal{P}_i$'s, there are uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graphs, given by Mihók's construction (Corollary 4.11). This was used by Broere and Bucko [1] to determine when such uniquely partitionable graphs exist if the $\mathcal{P}_i$'s are allowed to be reducible; and by the first author [3] to show that recognising reducible properties is NP-hard, with the exception of the set of bipartite graphs.

Before proving the uniqueness of the factorisations in [8] and [7], we tried without success to prove some related results. Their validity for induced-hereditary properties in general is still open. However, for *additive* induced-hereditary properties these results follow quite easily from Unique Factorisation, and we state them explicitly below. Note that Proposition 6.3 is equivalent to Theorem 5.3. We also show that unique factorisation for additive hereditary properties follows both from the result for hereditary compositive properties, and from the one for additive induced-hereditary properties.

6.1. Cancellation. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be additive induced-hereditary properties, $\mathcal{A} \neq \mathcal{U}$. If $\mathcal{A} \circ \mathcal{B} = \mathcal{A} \circ \mathcal{C}$, then $\mathcal{B} = \mathcal{C}$.* □

6.2. Corollary. *For additive induced-hereditary properties $\mathcal{A}', \mathcal{A}, \mathcal{B}', \mathcal{B}$, $\mathcal{A} \neq \mathcal{U} \neq \mathcal{B}$, if $\mathcal{A}' \circ \mathcal{B}' = \mathcal{A} \circ \mathcal{B}$, and $\mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{B}' \subseteq \mathcal{B}$, then $\mathcal{A}' = \mathcal{A}$, $\mathcal{B}' = \mathcal{B}$.* □

6.3. Proposition. *If $\mathcal{Q}$ and $\mathcal{R}$, are additive induced-hereditary properties, then $dec(\mathcal{Q} \circ \mathcal{R}) = dec(\mathcal{Q}) + dec(\mathcal{R})$.* □

A property is *strongly irreducible* if it has no factorisation into two non-trivial properties. We recall that an additive hereditary property is irreducible additive hereditary (respectively, irreducible additive induced-hereditary or irreducible hereditary compositive) if it has no factorisation into two non-trivial additive hereditary (respectively, additive induced-hereditary, or hereditary compositive) properties.

6.4. Proposition. *Let $\mathcal{P}$ be an additive hereditary property. Then:*

A. $\mathcal{P}$ is irreducible additive hereditary iff it is strongly irreducible

B. $\mathcal{P}$ has a unique factorisation into irreducible additive hereditary factors, *and the number of factors is exactly $dc(\mathcal{P}) = dec(\mathcal{P})$;*

C. *if $\mathcal{P} = \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_r$, and the $\mathcal{Q}_j$'s are all additive induced-hereditary (or all hereditary compositive), then they are all additive hereditary.*

Proof: A. If $\mathcal{P} = \mathcal{S} \circ \mathcal{T}$, where $\mathcal{S}$ and $\mathcal{T}$ are any two properties, then $\mathcal{S} + \mathcal{T} := \{G + H \mid G \in \mathcal{S}, H \in \mathcal{T}\}$ is a generating set for $\mathcal{P}$, with $dc(\mathcal{S} + \mathcal{T}) \geq 2$. By Lemma 2.3, $dc(\mathcal{P}) \geq 2$, and by Theorem 1.1 of [8] $\mathcal{P}$ has a factorisation into $dc(\mathcal{P})$ additive hereditary properties.

B. Let $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$, where the $\mathcal{P}_i$'s are irreducible additive hereditary properties. Then by A, this must be its unique factorisation into $dc(\mathcal{P})$ irreducible hereditary compositive properties, and also its unique factorisation into $dec(\mathcal{P})$ irreducible additive induced-hereditary properties.

C. If we factor each $\mathcal{Q}_j$ into its irreducible additive induced-hereditary factors, then by B these irreducible factors are all additive hereditary, so each $\mathcal{Q}_j$ is the product of additive hereditary factors. $\square$

An irreducible additive hereditary property is thus *strongly uniquely factorizable* — it has exactly one factorisation even when we allow factors that are not additive or hereditary. Szigeti and Tuza [10, Problem 4, p. 144] asked whether this was true for all additive hereditary properties. Semanišin [9] gave a class of examples of additive hereditary properties with non-hereditary factors. We show in [4] that the only reducible additive hereditary property that is strongly uniquely factorisable is the set of bipartite graphs, which is contained in any reducible additive hereditary property.

In [8], however, it is claimed that if the factors of an additive hereditary property are all hereditary then they must in fact all be additive hereditary. The argument assumes that the factorisation of Theorem 3.3 is unique when factoring into any hereditary properties; we do not believe that this has been proved — our proofs of uniqueness depend heavily on the additivity of the factors — so we leave this as an open question:

If $\mathcal{P} = \mathcal{Q} \circ \mathcal{R}$, with $\mathcal{P}$ additive and induced-hereditary, and $\mathcal{Q}$ and $\mathcal{R}$ induced-hereditary, must $\mathcal{Q}$ and $\mathcal{R}$ be additive too? cf. [10, Problem 4]

After this paper was first submitted, we discussed this work with Mihók, who now agrees with our interpretation of the results of [7] and [8]. He has also provided a different, perhaps simpler proof of Theorems 3.1 and 5.3. We expect this proof to appear in some other publication.

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