Two highly representative maps on one surface

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Abstract: We show that, for each orientable surface Σ, there is a constant c_Σ so that, if G_1 and G_2 are embedded simultaneously in the orientable surface Σ, with representativities r_1 and r_2, respectively, then the minimum number $\mathrm{cr}(G_1, G_2)$ of crossings between the two maps satisfies

$$\mathrm{cr}(G_1, G_2) \le \frac{c_\Sigma}{r_1 r_2} |E(G_1)||E(G_2)| \ .$$

This refines earlier estimates by Negami. Furthermore, we provide a counterexample to a conjecture of Archdeacon and Bonnington by exhibiting, for each k, embeddings G_1 and G_2 in the double torus so that, if we force all the vertices of G_1 to be in the same face of G_2, then the number of crossings between G_1 and G_2 is at least $k \cdot \mathrm{cr}(G_1, G_2)$.

1 Introduction

Let G_1, G_2 be graphs that embed in the surface Σ. Negami [2] introduced two different notions of how many crossings G_1 and G_2 have. One is $\mathrm{cr}(G_1, G_2)$, which is the minimum, over all embeddings of G_1 and G_2 in Σ, of the number of crossings between the two graphs, while the other, $\mathrm{cr}(\phi_1(G_1), \phi(G_2))$ is the minimum, over fixed homeomorphism classes $\phi(G_1)$ and $\phi(G_2)$ of embeddings for each of G_1 and G_2, of the number of crossings between the two maps. He proved:

Theorem 1 *[2] If Σ is the orientable surface with g handles, then*

$$cr(\phi(G_1), \phi(G_2)) \le 4g|E(G_1)||E(G_2)| \ .$$

Notice that this implies the obvious fact that if Σ is the sphere, then

$$\mathrm{cr}(G_1, G_2) = \mathrm{cr}(\phi(G_1), \phi(G_2)) = 0 \ .$$

Following up on Negami's work, Archdeacon and Bonnington [1] considered the cases of the torus and the projective plane. In both cases, they were able to get quite accurate estimates for $\mathrm{cr}(G_1, G_2)$. Essentially, they reduced the problem to computing $\mathrm{cr}_{\mathrm{sf}}(\phi_1(G_1), \phi_2(G_2))$, in which the only permitted homeomorphs $\phi_1'(G_1)$ and $\phi_2'(G_2)$ of $\phi_1(G_1)$ and $\phi_2(G_2)$ are those for which, for $\{i,j\} = \{1,2\}$, all the vertices of G_i are in the same face of $\phi_j'(G_j)$. Motivated by their results, they put forward the following conjecture.

Conjecture 1 *[1] For each surface Σ, there is a constant c so that*

$$cr(\phi_1(G_1), \phi_2(G_2)) \le c \cdot \mathrm{cr}_{\mathrm{sf}}(\phi_1(G_1), \phi_2(G_2)) \ .$$

In this work we improve Negami's result as follows. The *representativity* of an embedded graph $\phi(G)$ in a surface Σ is the smallest number of intersections of a non-contractible curve in Σ with $\phi(G)$. Note that representativity is not defined for embeddings in the sphere.

Theorem 2 *If Σ is an orientable surface of positive genus, then there is a constant c', so that if, for $i = 1, 2$, $\phi_i(G_i)$ has representativity r_i, then*

$$cr(\phi(G_1), \phi(G_2)) \le \frac{c'}{r_1 r_2} |E(G_1)||E(G_2)| \ .$$

Negami had already observed that there is a constant $N(\Sigma, r)$ so that if $r_1 \ge r$ and $r_2 \ge r$, then

$$\mathrm{cr}(\phi(G_1), \phi(G_2)) \le N(\Sigma, r)|E(G_1)||E(G_2)| \ ,$$

and, furthermore, if r is sufficiently large, then $N(\Sigma, r) \le 1$.

We remark that Negami's proof depends on the deep result of Robertson and Seymour [4] that if H is a fixed embedded graph in Σ, then there is a constant $\rho(H)$ such that every graph embedded in Σ with representativity at least $\rho(H)$ contains H as a minor. Our arguments rely on more elementary facts and give explicit bounds.

We also give a counterexample in the double torus to the Archdeacon and Bonnington conjecture.

At the heart of the proof of Theorem 2 is a result that guarantees, for a graph G embedded on an orientable surface Σ, the existence of a closed disc $D \subseteq \Sigma$, that contains all the vertices of G, and "most" of the edges.

Theorem 3 *For each orientable surface Σ there are constants $F(\Sigma), R(\Sigma)$ with the following property. Suppose that G is embedded on Σ with representativity $r > R(\Sigma)$. Then there is a closed disc $D \subseteq \Sigma$ that satisfies the following conditions.*

(i) All the vertices of G are contained in D.

(ii) At most $F(\Sigma)|E(G)|/r$ edges of G intersect $\Sigma \setminus D$.

(iii) For each edge e, $(\Sigma \setminus D) \cap e$ is either empty or has one component.

The proof of this statement is at the end of Section 3.

This theorem implies the intuitively clear result that, as $r \to \infty$, the proportion of the edges of G contained in the disc tends to 1.

2 The Main Theorem

The version of the main theorem we prove in this paper is the following.

Theorem 4 *For any orientable surface Σ, not the sphere, there exist constants $R(\Sigma), c(\Sigma)$, and $c'(\Sigma)$ with the following properties. For $i = 1, 2$, let $\phi_i : G_i \to \Sigma$ be an embedding with representativity r_1. If $r_1 \geq R(\Sigma)$, then*

$$cr(\phi_1(G_1), \phi_2(G_2)) \leq \frac{c(\Sigma)}{r_1} |E(G_1)||E(G_2)|.$$

Moreover, if also $r_2 \geq R(\Sigma)$, then

$$cr(\phi_1(G_1), \phi_2(G_2)) \leq \frac{c'(\Sigma)}{r_1 r_2} |E(G_1)||E(G_2)|.$$

Proof. Let $F(\Sigma), R(\Sigma)$ be as in the statement of Theorem 3 and let g be the genus of Σ. Suppose that the representativity r_1 of G_1 is at least $R(\Sigma)$. (We also use G_i as a shorthand for $\phi_i(G_i)$.) Apply Theorem 3 to G_1, and let D_1 denote the disc thereby obtained.

Negami [2] proved that (any two embedded graphs) G_1, G_2 can be jointly embedded so that every edge of G_1 intersects each edge of G_2 at most $4g$ times. It is clear that we can arrange the joint embedding of G_1 and G_2 in such a way that this condition is satisfied, and so that the disc D_1 is contained in a single face of G_2. Since such a joint embedding has at most $4g(F(\Sigma)|E(G_1)|/r_1)|E(G_2)|$ crossings, it follows that the first part of Theorem 4 holds with $c(\Sigma) = 4gF(\Sigma)$.

Finally suppose that r_2 is also at least $R(\Sigma)$, and let D_2 be a closed disc that results from the application of Theorem 3 to G_2. We can now arrange the joint embedding of G_1 and G_2 in such a way that the only crossings occur between edges in $\Sigma \setminus D_1$ and edges in $\Sigma \setminus D_2$, and, as before, so that no two edges cross each other more than $4g$ times. Since such a joint embedding has at most $4g(F(\Sigma)|E(G_1)|/r_1)(F(\Sigma)|E(G_2)|/r_2)$ crossings, it follows that the second part of Theorem 4 holds with $c'(\Sigma) = 4g(F(\Sigma))^2$. ∎

3 A decomposition theorem for densely embedded graphs

Our aim in this section is to prove Theorem 3.

However, in order to establish Theorem 3 (by induction on the genus of the surface), it is convenient to actually prove a slightly stronger statement, namely that the disc D can be asked to contain any fixed (sufficiently small, compared to the representativity of G) set of faces.

For a face F of a graph G embedded in a surface Σ, $\mathrm{bo}(F)$ denotes the set of vertices of G incident with F.

Theorem 5 *For each orientable surface Σ and nonnegative integer k, there are constants $r(\Sigma, k)$ and $f(\Sigma, k)$ with the following properties. Suppose that G is embedded in Σ with representativity $r > r(\Sigma, k)$, and let $\mathcal{F}$ be a collection of k faces in G. Then there is a closed disc D in Σ such that:*

(i) every vertex of G is contained in D;

(ii) at most $f(\Sigma, k)|E(G)|/r$ edges intersect $\Sigma \setminus D$;

(iii) if $F \in \mathcal{F}$, then $\overline{F}$ is contained in the interior of D; moreover, no edge that intersects $\Sigma \setminus D$ is incident with a vertex of $bo(F)$; and

(iv) for each edge e, $(\Sigma \setminus D) \cap e$ is either empty or has one component, and, if it has one component, then $D \cup e$ contains an essential arc.

Remark If D is a disc that satisfies Conditions (i)–(iv) in Theorem 5, then it is straightforward to modify D (if necessary) to obtain a closed disc that also satisfies (i)–(iv), and whose boundary intersects G only in vertices.

Proof of Theorem 5. The proof is by induction on the genus of Σ. First we prove the theorem when Σ has genus 1 (i.e., Σ is the torus), but for arbitrary k.

Base Case: Σ is the torus. We show that Theorem 5 holds with the values $r(\Sigma, k) := \max\{9k, 9\}$ and $f(\Sigma, k) := 6$.

Let G be a graph embedded in the torus with representativity $r > \max\{9k, 9\}$, and let $\mathcal{F}$ be a (possibly empty) collection of at most k faces of G. We shall show the existence of a disc D satisfying Conditions (i)–(iv) in the statement of Theorem 5, with $f(\Sigma, k) = 6$.

Step 1 *G contains a minimal subgraph K that contracts to the toroidal grid $C_{\lfloor 3r/4 \rfloor} \times C_{\lfloor 3r/4 \rfloor}$.*

Since $r > 5$, G has a minor isomorphic to $T_{\lfloor 3r/4 \rfloor} = C_{\lfloor 3r/4 \rfloor} \times C_{\lfloor 3r/4 \rfloor}$ (see [5]). Let K be a minimal subgraph of G that has $T_{\lfloor 3r/4 \rfloor}$ as a minor. It is easy to see that each vertex of $T_{\lfloor 3r/4 \rfloor}$ corresponds to a tree (having at most four leaves) in K and that each edge of $T_{\lfloor 3r/4 \rfloor}$ corresponds to an edge of K. Contracting the edges in the trees turns K into $T_{\lfloor 3r/4 \rfloor}$.

It follows that K has two sets $\mathcal{R}$ and $\mathcal{B}$ of $\lfloor 3r/4 \rfloor$ pairwise disjoint cycles, so that any cycles $R \in \mathcal{R}$ and $B \in \mathcal{B}$ intersect in a path consisting of at least one vertex and that, in the embedding of K induced by G, the cycles R and B cross at this intersection.

Moreover, since $3r/4 \geq 5$, the cycles in $\mathcal{R}$ occur in the embedding in a natural cyclic order $R_1, R_2, \ldots, R_{\lfloor 3r/4 \rfloor}$ so that, for $i = 1, 2, \ldots, \lfloor 3r/4 \rfloor$, R_i and R_{i+1} bound a cylinder $C_R(i)$ that is disjoint from all the other R_j (indices are read modulo $\lfloor 3r/4 \rfloor$). Likewise we can cyclically order as $B_1, B_2, \ldots, B_{\lfloor 3r/4 \rfloor}$ the cycles in $\mathcal{B}$ so that, for each $i = 1, 2, \ldots, \lfloor 3r/4 \rfloor$, B_i and B_{i+1} bound a cylinder $C_B(i)$ that is disjoint from all the other B_j.

Step 2 *Pick "nice" cylinders $C_R(i)$ and $C_B(j)$.*

Let $\kappa(\mathcal{F})$ denote the set of vertices of G incident with a face in $\mathcal{F}$, i.e., $\kappa(\mathcal{F}) = \cup_{F \in \mathcal{F}} \mathrm{bo}(F)$.

We claim that there are at most $3k$ B–cylinders that contain a vertex in $\kappa(\mathcal{F})$. For it is apparent that each face F of G is contained in a face of K, which is necessarily in some $C_B(i)$. The boundary of F may, therefore, intersect B_i and B_{i+1}, but is otherwise disjoint from every other $C_B(j)$ and so the boundary of F has vertices in at most $C_B(i-1)$, $C_B(i)$, and $C_B(i+1)$, as required. Since $|\mathcal{F}| = k$, the claim follows.

Similarly, at most $3k$ R–cylinders contain a vertex in $\kappa(\mathcal{F})$.

Therefore, there are at least $\lfloor 3r/4 \rfloor - 3k > r/3$ (here we are using the assumptions $r > 9k$ and $r > 9$) B–cylinders that do not contain any vertex in $\kappa(\mathcal{F})$. Similarly, there are more than $r/3$ R–cylinders that do not contain any vertex in $\kappa(\mathcal{F})$.

Let $C_B(i_0)$ (respectively $C_R(j_0)$) be a cylinder whose interior has the fewest edges among all B–cylinders (respectively R–cylinders) that do not contain any vertex in $\kappa(\mathcal{F})$. Thus, there are fewer than $|E(G)|/(r/3)$ edges in the interior of $C_B(i_0)$ (respectively $C_R(j_0)$).

Step 3 *Construction of D.*

Among all the cycles X of G such that $V(X) \subseteq V(B_{i_0})$ and X is freely homotopic to B_{i_0}, choose B'_{i_0} to minimize the number of edges in the subcylinder $C'_B(i_0)$ of $C_B(i_0)$ bounded by B'_{i_0} and B_{i_0+1}. The point here is to ensure that every edge of G in $C'_B(i_0)$ having both ends in B'_{i_0} is actually in B'_{i_0}.

Similarly, select R'_{j_0} and $C'_R(j_0)$ relative to R_{j_0} and $C_R(j_0)$.

Now let β_0 be a simple closed curve in $C'_B(i_0)$ near to and freely homotopic to B'_{i_0}, so that β_0 crosses only edges of G and, in fact, crosses just once each edge of G both incident with a vertex of B'_{i_0} and in the interior of $C'_B(i_0)$. Then β_0 crosses at most $|E(G)|/(r/3)$ edges of G.

Likewise, following R'_{j_0} from one side of β_0 to the other in the interior of $C'_R(j_0)$, let ρ_0 be a simple arc that only crosses, and just once, each edge of G incident with a vertex of R'_{j_0} and in the interior of $C'_R(j_0)$. Note that ρ_0 crosses at most $|E(G)|/(r/3)$ edges of G.

Taking U to be an appropriate small open neighbourhood of $\beta_0 \cup \rho_0$, the set $D = \Sigma \setminus U$ is a closed disc, whose boundary ∂D intersects G only in edges, and each edge intersected by ∂D is crossed precisely twice. Notice that U has only pieces of edges of G, there are at most $(6/r)|E(G)|$ such edges, and each has precisely one component in U.

It is straightforward to check that D satisfies Conditions (i)–(iv) in the statement of Theorem 5, with $f(\Sigma, k) = 6$.

Inductive Step: Σ has genus $g > 1$. Suppose that Theorem 5 holds for every orientable surface Σ_j with positive genus j smaller than g, and, in all cases, for every nonnegative integer k. We complete the proof by showing that it follows that Theorem 5 also holds for Σ, for every nonnegative integer k.

Set $R(g,k) := \max_{j<g,\ t\le k+1}\{r(\Sigma_j, t)\}$ and $s(g,k) := \max_{j<g,\ t\le k+1}\{f(\Sigma_j, t)\}$. Now define $r(\Sigma, k)$ and $f(\Sigma, k)$ as:

$$r(\Sigma, k) := \max\{48, 48k, 24(R(g,k)+1)\}$$

and

$$f(\Sigma, k) := 96s(g,k) + 2\ .$$

Let G be embedded in Σ with representativity $r > r(\Sigma, k)$, and let $\mathcal{F}$ be a collection of at most k faces of G.

Step 1 *Find a noncontractible separating cycle C_{i_0} such that (i) C_{i_0} divides G into two graphs G_h, G_ℓ of sufficiently high representativity; and (ii) C_{i_0} is disjoint from the boundary of each face in $\mathcal{F}$.*

First we note that it follows from [3] that G contains $\lfloor (r-1)/8 \rfloor$ pairwise disjoint, pairwise homotopic, noncontractible separating cycles $C_1, \ldots, C_{\lfloor (r-1)/8 \rfloor}$.

We claim that there is an integer i_0, $\lfloor r/24 \rfloor \le i_0 \le \lfloor r/12 \rfloor$, such that C_{i_0} is disjoint from $\mathrm{bo}(F)$ for every $F \in \mathcal{F}$. To see this, note that there are $\lfloor r/12 \rfloor - \lfloor r/24 \rfloor + 1 > r/24$ cycles C_i such that $\lfloor r/24 \rfloor \le i \le \lfloor r/12 \rfloor$. On the other hand, it is easy to see that, for each face F in $\mathcal{F}$, $\mathrm{bo}(F)$ intersects at most two cycles C_i. Therefore at most $2k$ cycles C_i intersect $\mathrm{bo}(F)$ for some $F \in \mathcal{F}$. Since $2k < r/24$, the claim follows.

The cycle C_{i_0} separates Σ into two (one–holed) surfaces Σ_h and Σ_ℓ of (positive) genera h, ℓ, such that $h + l = g$. Moreover, G (as embedded in Σ) can be viewed as the join (along C_{i_0}) of two graphs G_h, G_ℓ (embedded in Σ_h and Σ_ℓ, respectively), in each of which the cycle C_{i_0} is the boundary of a face. That is, in order to obtain G from G_h and G_ℓ, we identify G_h and G_ℓ along C_{i_0}.

Let F_h (respectively F_ℓ) denote the face in G_h (respectively G_ℓ) bounded by C_{i_0}.

Step 2 *Defining $\mathcal{F}_h$ and $\mathcal{F}_\ell$.*

Each face in G_h and G_ℓ, with the exception of F_h and F_ℓ, corresponds naturally to a face in G. Let $\mathcal{F}_h$ (respectively $\mathcal{F}_\ell$) denote the collection that consists of those faces in G_h (respectively G_ℓ) whose corresponding face in G belongs to $\mathcal{F}$, plus the face F_h (respectively F_ℓ).

Step 3 *Apply Theorem 5 (using the inductive hypothesis) to G_h (and $\mathcal{F}_h$), and to G_ℓ (and $\mathcal{F}_\ell$), and let D_h, D_ℓ denote the discs thereby obtained.*

Let $r(G_h)$ (respectively $r(G_\ell)$) denote the representativity of G_h (respectively G_ℓ). It is easy to see (since $\lfloor r/24 \rfloor \le i_0 \le \lfloor r/12 \rfloor$) that both $r(G_h)$ and $r(G_\ell)$ are at least $\lfloor r/24 \rfloor$. Since $h < g$, we can apply Theorem 5 (by the inductive hypothesis) to G_h and its collection of faces $\mathcal{F}_h$. Indeed, the inductive hypothesis applies, since $r(G_h)$ is at least $\lfloor r/24 \rfloor$, which is in turn greater than $r(\Sigma_h, |\mathcal{F}_h|)$, by the definition of $r(\Sigma, k)$.

Thus, by applying Theorem 5 to G_h (and $\mathcal{F}_h$), we obtain a closed disc D_h in Σ_h that satisfies Conditions (i)–(iv). In particular, at most

$$f(\Sigma_h, |\mathcal{F}_h|)|E(G_h)|/r(G_h)$$

edges of G_h intersect $\Sigma_h \setminus D_h$. Moreover (see Remark after Theorem 5), we choose D_h so that its boundary intersects G_h only in vertices.

Similarly, by applying Theorem 5 to G_ℓ (and $\mathcal{F}_\ell$), we obtain a closed disc D_ℓ in Σ_ℓ that satisfies Conditions (i)–(iv). In particular, at most

$f(\Sigma_\ell, |\mathcal{F}_\ell|)|E(G_\ell)|/r(G_\ell)$ edges of G_ℓ intersect $\Sigma_\ell \setminus D_\ell$. As with D_h, we choose D_ℓ so that its boundary intersects G_ℓ only in vertices.

Step 4 *Glue back G_h and G_ℓ along C_{i_0}. The set T_0 that results is an essential cylinder in Σ, that contains all the vertices of G, whose boundary components intersect G only in vertices, and that has at least r vertices on each boundary component.*

The disc D_h (respectively D_ℓ) contains the cycle C_{i_0} and the face F_h (respectively F_ℓ) in its interior, since F_h (respectively F_ℓ) belongs to $\mathcal{F}_h$ (respectively $\mathcal{F}_\ell$). Thus we can identify G_h and G_ℓ along their respective copies of C_{i_0}, to get back G (embedded in Σ), and in the process obtain a compact set T_0 that results from the union of $D_h \setminus F_h$ and $D_\ell \setminus F_\ell$. We note that T_0 is homeomorphic to a cylinder.

The following observations are readily verified from the construction of T_0, and the properties of the discs D_h and D_ℓ:

(1) every vertex of G is contained in T_0;

(2) at most $f(\Sigma_h, |\mathcal{F}_h|)|E(G_h)|/r(G_h) + f(\Sigma_\ell, |\mathcal{F}_\ell|)|E(G_\ell)|/r(G_\ell)$ edges intersect (moreover, are contained in) $\Sigma \setminus T_0$;

(3) if $F \in \mathcal{F}$, then $\overline{F}$ is contained in the interior of T_0; moreover, no edge that intersects $\Sigma \setminus T_0$ is incident with a vertex of $\mathrm{bo}(F)$; and

(4) for each edge e, $(\Sigma \setminus T_0) \cap e$ is either empty or has one component, and, if it has one component, then $T_0 \cup e$ contains an essential arc.

Moreover, noting that (a) each of $f(\Sigma_h, |\mathcal{F}_h|)$ and $f(\Sigma_\ell, |\mathcal{F}_\ell|)$ is at most $s(g,k)$; (b) each of $|E(G_h)|$ and $|E(G_\ell)|$ is less than $|E(G)|$; and (c) each of $r(G_h)$ and $r(G_\ell)$ is at least $\lfloor r/24 \rfloor > r/48$, Condition (2) above implies:

(2′) at most $96s(g,k)|E(G)|/r$ edges intersect (moreover, are contained in) $\Sigma \setminus T_0$.

The similarities between the properties of T_0 and the properties of the disc we require to complete the proof are, of course, not incidental. Indeed, we will show that T_0 contains a closed disc with the required properties.

We also note that ∂T_0 intersects G only in vertices. Since both boundary components of T_0 are noncontractible, separating curves in Σ, it follows that each boundary component of T_0 contains at least r vertices of G.

Step 5 *Find two arcs γ, γ' across the cylinder T_0, very close to each other, that intersect as few edges as possible.*

An application of Menger's Theorem, plus the fact that G is r–representative in Σ, yields that there are at least r pairwise disjoint paths $P_0, \ldots, P_{r-1}$ from the vertices in one boundary component of T_0 to the vertices in the other boundary component of T_0.

Moreover, these paths can be labeled so that, for each i, P_i and P_{i+1} bound, together with two disjoint pieces of ∂T_0, a closed disc Q_i that does not intersect $P_{i+2} \cup \cdots \cup P_{i-1}$ (indices are read modulo r).

As before, every face F in $\mathcal{F}$ is contained in the interior of T_0, so $\mathrm{bo}(F)$ is contained in one Q_i. Consequently $\mathrm{bo}(F)$ intersects at most most three Q_i's Since $r > 6k$, and $|\mathcal{F}| = k$, it follows that there exist at least $r/2$ Q_i's that are disjoint from $\mathrm{bo}(F)$ for every F in $\mathcal{F}$. Let Q_{j_0} be such a disc with the least number of edges in its interior. Thus, $E(Q_{j_0}) \leq |E(G)|/(r/2)$.

There exists a simple open arc γ with the following properties: (i) one point of $\overline{\gamma} \setminus \gamma$ is in one boundary component of T_0, and the other point is in the other boundary component; (ii) neither of the two points in $\overline{\gamma} \setminus \gamma$ is in G; (iii) γ is contained in the interior of Q_{j_0}; (iv) γ intersects G only in edges, and γ does not intersect any edge more than once.

Let γ' be a simple open arc, drawn sufficiently close to γ (but with $\overline{\gamma} \cap \overline{\gamma'} = \emptyset$), so that γ' also satisfies (i), (ii), (iii), and (iv), and γ and γ' intersect exactly the same edges of G, and do so in exactly the same order.

Step 6 *The closed disc D that results by removing from T_0 the small strip bounded by $\overline{\gamma}$ and $\overline{\gamma'}$ is the required disc.*

The set $T_0 \setminus (\overline{\gamma} \cup \overline{\gamma'})$ consists of two components, one of which (say Y) is contained in Q_{j_0}. It is immediately checked that $D := T_0 \setminus Y$ is homeomorphic to a closed disc.

It is straightforward to verify that D satisfies Conditions (i), (iii), and (iv) in Theorem 5.

We finally proceed to bound the number of edges that intersect $\Sigma \setminus D$.

First we note that the number of edges that intersect $\Sigma \setminus D$ equals the number of edges that intersect $\Sigma \setminus T_0$ plus the number of edges that intersect γ. As we observed above, at most $96s(g,k)|E(G)|/r$ edges intersect $\Sigma \setminus T_0$. On the other hand, since every edge that intersects γ is contained in the interior of Q_{j_0}, and the interior of Q_{j_0} has at most $|E(G)|/(r/2)$ $= 2|E(G)|/r$ edges, it follows that at most $2|E(G)|/r$ edges intersect γ.

We therefore conclude that at most $(96s(g,k)+2)|E(G)|/r$ edges of G intersect $\Sigma \setminus D$. This proves that Condition (ii) holds, as claimed, with $f(\Sigma,k) = 96s(g,k)+2$. ∎

Proof of Theorem 3. Let $f(\Sigma,0)$ and $R(\Sigma)$ be as in the statement of Theorem 5. It is immediately checked that Theorem 3 holds with $F(\Sigma) := f(\Sigma,0)$ and with the same value of $R(\Sigma)$. ∎

4 A counterexample to a conjecture by Archdeacon and Bonnington

In this section we give constructions of joint embeddings of graphs that disprove Conjecture 1.

Let $\mathrm{cr}_{\mathrm{sf}}(G_1,G_2)$ denote the minimum number of pairwise crossings of edges in a joint embedding of G_1 and G_2 in which all vertices of G_1 lie in a single face of G_2 and vice versa.

Using this definition, the conjecture can be restated in the following equivalent form.

Conjecture 1 (Archdeacon and Bonnington [1]) *For each surface Σ, there is a constant $d(\Sigma)$ such that, for any two graphs G_1, G_2 jointly embedded in Σ,*

$$\frac{\mathrm{cr}_{\mathrm{sf}}(G_1,G_2)}{\mathrm{cr}(G_1,G_2)} \le d(\Sigma).$$

Consider the embedded graph $\phi(G)$ that consists of three disjoint loops, embedded in the double torus as in Figure 1.

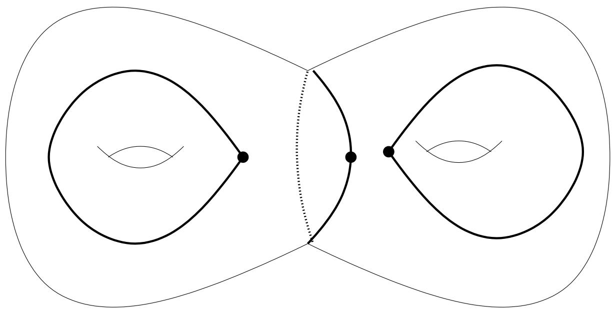

FIGURE 1. A doubly–toroidal embedding of the graph G that consists of three disjoint loops.

Now let $G_1 = G_2 = G$. Then clearly $\mathrm{cr}(\phi(G_1),\phi(G_2)) = 0$, since G_1 and G_2 can be jointly embedded with no crossings of edges. On the other hand, every joint embedding of G_1 and G_2 in which all the vertices of G_1 lie in a single face of G_2 has at least two crossings. This observation shows that the joint embedding (G_1, G_2) disproves the conjecture above, since $\mathrm{cr}_{\mathrm{sf}}(G_1, G_2) \geq 2$ and $\mathrm{cr}(G_1, G_2) = 0$.

One might object that, in this example, the graphs G_1 and G_2 are not connected. With this potential objection in mind, one could slightly modify this example as follows.

Let t be any positive integer. For each loop ℓ of G, draw an additional $t-1$ loops homotopic to ℓ, and based on the same vertex. Finally, add two edges (links) to the resulting graph, so as to make it connected. The embedded graph G' that results has a total of $3t$ loops, plus 2 links. Now let $G_1' = G_2' = G'$. It is easily verified that G_1 and G_2 can be jointly embedded with at most $4t$ crossings of edges. Thus $\mathrm{cr}(G_1', G_2') \leq 4t$. It is also easy to check that any joint embedding of G_1 and G_2 in which all the vertices of G_1 lie on a single face of G_2 must have more than t^2 crossings. Therefore $\mathrm{cr}_{\mathrm{sf}}(G_1', G_2') \geq t^2$. Hence, $\mathrm{cr}_{\mathrm{sf}}(G_1', G_2')/\mathrm{cr}(G_1', G_2') \geq t/4$. Therefore, no constant $d(\Sigma)$ as in the conjecture can possibly exist, even for joint embeddings of connected graphs.

5 Concluding Remarks

In [2], Negami included the following conjecture.

Conjecture (Negami [2]) There is a constant C with the following property. If G_1 and G_2 are embedded on a surface, then

$$\mathrm{cr}(G_1, G_2) \leq C \cdot |E(G_1)||E(G_2)|.$$

On the one hand, this seems eminently reasonable: why should two edges be forced to cross more than once? On the other hand, if one considers two sets of bouquets, both having a maximum number of loops so as to have no two loops homotopic, then, given the homeomorphism classes of the embeddings, some pair of loops must cross many times. (It is possible to prove that they need never cross more than g times, which improves Negami's $4g$ coefficient to g. The argument is not very complicated, but it does not seem worthwhile at the moment.)

The conjecture by Archdeacon and Bonnington that we disproved in Section 4 was inspired from the fact that the statement is true for toroidal or projective embeddings. The examples above have shown that it is already not true for doubly–toroidal embeddings. In our examples, the joint crossing number can be achieved (up to a constant factor) by a joint embedding in which all the faces of each graph lie on *two* faces of the other graph. With this observation in mind, we suggest the following revised version of the Archdeacon and Bonnington conjecture.

Conjecture *For each orientable surface Σ, there is a constant $t(\Sigma)$ with the following property. Let G_1 and G_2 be two graphs jointly embedded in Σ. Then there exists a joint embedding achieving (up to a constant factor) the joint crossing number such that all the vertices of G_1 lie in at most $t(\Sigma)$ faces of G_2 and vice versa.*

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