Embedding Grids in Surfaces

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Abstract

We show that if a very large grid is embedded in a surface, then a large subgrid is embedded in a disc in the surface. This readily implies that: (a) a minor-minimal graph that does not embed in a given surface has no very large grid; and (b) a minor-minimal k-representative embedding in the surface has no very large grid. Similar arguments show (c) that if G is minimal with respect to crossing number, then G has no very large grid. This work is a refinement of C. Thomassen, *A simpler proof of the excluded minor theorem for higher surfaces*, J. Combin. Theory, Ser. B **70** (1996) 306-311.

1 Introduction

As part of their "Graph Minors" project, Robertson and Seymour [6] proved the following result.

Theorem 1 *For any surface Σ, there are only finitely many graphs that do not embed in Σ and that are minor–minimal with this property.*

The proof by Robertson and Seymour is long and difficult. However, there is now a remarkably accessible proof based on their original ideas. This proof is summarized in the following three results.

(1) *Let b be an integer and let $G_1, G_2, \ldots$ be an infinite sequence of graphs each with branch–width at most b. Then there exist $i < j$ such that G_i is a minor of G_j.*

(2) *For any positive integer k there is an integer $f(k)$ such that if G is a graph with branch-width at least $f(k)$, then G contains the k by k grid as a minor.*

(3) *For any surface Σ there exists an integer k such that, if G is a graph that has a k by k grid as a minor, and G does not embed in Σ, then there exists an edge e of G such that $G - e$ does not embed in Σ.*

A graph has small branch–width (or, similarly, tree–width) if it can be decomposed across noncrossing separations into small pieces. (As we do not need these notions here we shall not give precise definitions.) The *k by k grid* is the graph on k^2 vertices $v_{i,j}$, $1 \leq i, j \leq k$, such that $v_{i,j}$ is adjacent to $v_{i\pm1,j}$ and $v_{i,j\pm1}$, whenever the subscripts are between 1 and k.

Robertson and Seymour's proof of (1), in [7], was stated in terms of tree–width and relied on a result of Thomas [9]. These results have simpler proofs for branch–width; see Geelen, Gerards, and Whittle [3]. A marvellously elementary proof of (2) is given by Diestel, Jensen, Gorbunov and Thomassen [1]. (Actually, their proof is for tree–width, rather than branch–width. These two statements are equivalent, and their proof becomes slightly easier in the branch–width version.) Thomassen [10] gives an elegant short proof of (3), the final link in the chain, and we provide another short proof in the present paper. The main distinctions between [10] and the current work are: (a) we highlight more directly the embeddings of grids in surfaces; and (b) we demonstrate that essentially the same arguments work for crossing number rather than genus.

For this work, a *surface* is a compact connected 2-manifold without boundary. The *Euler genus* $\varepsilon(\Sigma)$ of a surface Σ is $k + 2h$, where Σ is obtained from the sphere by the addition of k crosscaps and h handles. Thus, for example, the torus and Klein bottle both have Euler genus 2.

The main purpose of this article is to prove that, when a large grid is embedded on a fixed surface, most of the grid is embedded in a planar way.

Lemma 1 *Let G be a grid embedded in a surface Σ. Then, the number of noncontractible 4–cycles in the grid is at most $9\varepsilon(\Sigma)$.*

Consider a very large grid embedded on a fixed surface Σ. By Lemma 1, most of the 4–cycles in the grid are contractible. Contractible cycles bound discs, and in Σ these small discs are glued together along edges of the grid. Thus, we see that much of the grid is embedded in a planar way on Σ. In particular, some large subgrid is embedded in a disc. These observations lead to an easy proof of (3). Moreover, we can also deduce that a minor-minimal r-representative embedding in a surface has bounded grid size.

This article is an amalgamation of joint research of the first two authors with independent work of the third author. The methods in Section 2 are those of the first two authors; the third author proved analogues of Lemmas 4 and 5 with different techniques. The material on crossing numbers is essentially due to the third author; in particular, Theorem 4 and the "bounded path-width" conjecture are originally due to him.

2 Grids in Surfaces

In this section, we prove Lemma 1. The main topological result we require is the following.

Lemma 2 *Let Σ be a surface and let $\mathcal{C}$ be a set of simple closed noncontractible curves in Σ such that some component of $\Sigma \setminus (\cup_{C \in \mathcal{C}} C)$ has all the curves in $\mathcal{C}$ in its boundary. Then $|\mathcal{C}| \leq \varepsilon(\Sigma)$.*

Proof: Let Γ be a component of $\Sigma \setminus (\cup_{C \in \mathcal{C}} C)$ such that all the curves of $\mathcal{C}$ are contained in the boundary of Γ. Suppose that $\mathcal{C}$ contains a separating curve C, and let Σ_1 and Σ_2 be the components of $\Sigma \setminus C$, where Γ is contained in Σ_1. Since C is noncontractible, Σ_2 is not a disc, and, hence, contains a nonseparating closed curve C'. Replacing C with C' in $\mathcal{C}$ reduces the number of separating curves in $\mathcal{C}$. Thus, we may assume that $\mathcal{C}$ contains only nonseparating curves.

Let n be the number of components of $\Sigma \setminus (\cup_{C \in \mathcal{C}} C)$, let h be the number of 2–sided curves in $\mathcal{C}$, let k be the number of 1–sided curves in $\mathcal{C}$, and let Γ be the component of $\Sigma \setminus (\cup_{C \in \mathcal{C}} C)$ incident with every member of $\mathcal{C}$. Some collection of $h-(n-1)$ 2–sided curves in $\mathcal{C}$ do not separate Σ. Thus, $\varepsilon(\Sigma) \geq 2(h-(n-1))+k$. If $n=1$, then $\varepsilon(\Sigma) \geq 2h+k \geq |\mathcal{C}|$, as required. Thus, we may assume that $n>1$. Now, since $\mathcal{C}$ does not contain a separating curve, every component of $\Sigma \setminus (\cup_{C \in \mathcal{C}} C)$ other than Γ has at least two curves in its boundary, so $h \geq 2(n-1)$. Therefore, $\varepsilon(\Sigma) \geq 2(h-(n-1))+k \geq h+k = |\mathcal{C}|$, as required. **Q.E.D.**

We obtain Lemma 1 as a consequence of our next result. A subgraph H of a graph G *separates* G if there exist proper subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$ and $H = G_1 \cap G_2$. The point is, if H is a nonseparating subgraph of G and G is embedded in a surface Σ, then, relative to the induced embedding of H in Σ, the rest of G is contained in (the closure of) one face of H.

Lemma 3 *Let G be a connected graph embedded in a surface Σ and let $\mathcal{C}$ be a set of pairwise disjoint cycles in G such that $\cup_{C \in \mathcal{C}} C$ does not separate G. If every cycle in $\mathcal{C}$ is noncontractible in Σ, then $|\mathcal{C}| \leq \varepsilon(\Sigma)$.*

Proof: Let $H = \cup_{C \in \mathcal{C}} C$. Since H does not separate G, there is one component of $\Sigma \setminus H$ in which the rest of G is embedded. Since G is connected, each cycle in H is in the boundary of this component. By Lemma 2 $|\mathcal{C}| \leq \varepsilon(\Sigma)$. **Q.E.D.**

Now for the proof of Lemma 1. Let G be a k by k grid embedded in a surface Σ. We can give the 4–cycles coordinates (i, j), where $1 \leq i, j \leq k-1$, in the natural way. Considering the coordinates modulo 3 gives a partitioning of the 4–cycles of G into 9 sets. Note that the union of cycles in any one of these sets is nonseparating in G. Therefore, by Lemma 3, each of these sets contains at most $\varepsilon(\Sigma)$ noncontractible cycles. Hence, there are at most $9\varepsilon(\Sigma)$ noncontractible 4–cycles in G. (We believe that the correct upper bound is actually $2\varepsilon(\Sigma) + 8$.)

We now derive two easy consequences of Lemma 3. The first shows that, if a very large grid is embedded in a surface, then a large subgrid is embedded in a disc, while the second says that if a very large grid is embedded in a surface, then some ring of 4-cycles surrounding the centre of the grid is embedded in a cylinder. (These results follow more easily from Lemma 1, but Lemma 3 gives sharper bounds.)

The *boundary cycle* of a k by k grid is the cycle that bounds the infinite face in the usual planar embedding of the grid (that is, consists of the subgraph induced by the $v_{i,j}$ such that $\{i, j\} \cap \{1, k\} \neq \emptyset$).

Lemma 4 *Let t, k, n be positive integers such that $n \geq t(k+2)$ and let G be an n by n grid. If G is embedded in a surface Σ of Euler genus at most $t^2 - 1$, then some k by k subgrid of G is embedded in a closed disc in Σ such that the boundary cycle of the k by k grid is the boundary of the disc.*

Proof: Clearly G contains t^2 k by k subgrids that are pairwise disjoint and no two vertices from distinct subgrids are adjacent in the grid. By Lemma 3, not all of these subgrids can have noncontractible 4-cycles, so one has only contractible 4-cycles. Each of these bounds a closed disc and the union of these closed discs is the required closed disc. **Q.E.D.**

We comment that Lemma 4 is at least nearly best possible. It is straightforward to embed the tk by tk grid in an orientable surface of genus at most $t^2 - 1$ such that no k by k subgrid is embedded in a closed disc. Lemma 4 shows this is not true of the $t(k+2)$ by $t(k+2)$ grid.

For the second use, let G be a k by k grid. For $1 \leq i < k/2 - 2$, the i^{th}-*collar* is the subgraph C_i induced by the vertices at distance either $i-1$ or i from a nearest vertex on the boundary cycle of the grid. The *exterior cycle* and *interior cycle* of C_i are the cycles induced by the vertices at distance $i-1$ and distance i, respectively. We note that the exterior cycle of C_i is the boundary cycle of a $(k-2i+2)$ by $(k-2i+2)$ subgrid.

Lemma 5 *Let G be a $(6\varepsilon+6)$ by $(6\varepsilon+6)$ grid embedded in a surface Σ of Euler genus ε. Then, for some $i \in \{1, 2, \ldots, 3\varepsilon+1\}$, the i^{th}-collar is embedded in a cylinder in Σ.*

Proof: By Lemma 3, one of the $\varepsilon+1$ i^{th}-collars, $i \in \{1, 4, 7, \ldots, 3(\varepsilon+1)-2\}$, contains only contractible 4-cycles. For one in which all 4-cycles are contractible, the 4-cycles bound discs and the union of these discs is a cylinder. **Q.E.D.**

3 Applications

Our first application is to prove the following form of (3).

Theorem 2 *Let Σ be a surface of Euler genus ε. Let $n \geq \left\lceil \sqrt{\varepsilon+3} \right\rceil (6\varepsilon+7)$ and let G be a graph containing an n by n grid as a minor. If, for every $e \in E(G)$, $G - e$ embeds in Σ, then G embeds in Σ.*

In the proof, we shall need the concept of a bridge. Let G be a graph and let H be a subgraph of G. A *bridge* of H is either an edge of G, not in H, together with its ends, if both ends are in H, or a component of $G - V(H)$, together with any edge of G incident with a vertex in that component, and the ends from H of such edges. We note that a subgraph H of G is separating if and only if it has at least two bridges.

An edge e of a k by k grid is *central* if either (1) $k = 2m - 1$ is odd and e is incident with $v_{m,m}$ or (2) $k = 2m$ is even and e is in the 4-cycle induced by $v_{m,m}$, $v_{m,m+1}$, $v_{m+1,m}$, and $v_{m+1,m+1}$.

Proof: Let e be any edge of G. An embedding of $G - e$ in Σ can be used to obtain an embedding of G in Σ', obtained from Σ by adding a handle.

Let M be a minimal subgraph of G that contracts to the n by n grid. There is an injection between the cycles of the grid and the cycles of M; a *quadrangle* is a cycle of M that contracts to one of the 4-cycles of the grid.

Since $\varepsilon(\Sigma') = \varepsilon(\Sigma) + 2 = \varepsilon + 2$, Lemma 4 implies that some subgraph K of M that contracts to a $(6\varepsilon + 7)$ by $(6\varepsilon + 7)$ grid is embedded in a disc in the embedding of G in Σ'. Let K be a minimal subgraph with this property.

A vertex of K is an *interior vertex* if it is not in the boundary cycle of K. A bridge of K is an *interior bridge* if it contains an interior vertex of K.

Every interior bridge of K must be embedded, in Σ', in a face of K bounded by a quadrangle. Furthermore, every nonquadrangular cycle of K bounds a disc in Σ, and separates both Σ and G.

Let e be a central edge of K. Let K' be a minimal subgraph of $K - e$ that contracts to a $(6\varepsilon + 6)$ by $(6\varepsilon + 6)$ grid.

Embed $G - e$ in Σ. By Lemma 5, there is a collar J of K' that is embedded in a cylinder. Let I be the interior cycle of J and let E be the exterior cycle of J.

Let G_1 be the subgraph of G contained in the closed disc Δ_1 in Σ' bounded by I and not containing E. Let G_2 be the bridge of G_1 in G containing E and let $\mathcal{B}$ denote the set of all the other bridges of G_1. Note all attachments of G_2 are in I and any bridge in $\mathcal{B}$ has only attachments in the I-side of some quadrangle in J and is embedded, in Σ', in the disc bounded by such a quadrangle.

We obtain an embedding of G in Σ as follows. Let Δ_2 be a small closed disc in Δ_1, disjoint from G_1. Then G_1 is embedded in a cylinder Γ obtained by deleting the interior of Δ_2 from Δ_1.

Consider the induced embedding of $G_2 \cup I$ in Σ. Some face F has I in its boundary. There is a cylinder in F with I as one of its boundary curves. We can identify Γ with this cylinder, being sure to identify the two copies of I.

Furthermore, E and I bound a cylinder in Σ. Any bridge in $\mathcal{B}$ not already embedded in $\Sigma \setminus F$ can be embedded in this cylinder, using its embedding from the embedding of G in Σ'. This completes the embedding of G in Σ. **Q.E.D.**

An embedding of a graph G in a surface Σ is *r-representative* if every noncontractible closed curve in Σ intersects the graph at least r times. The embedding is *minor-minimal r-representative* if it is r-representative and the deletion or contraction (in the surface) of any edge produces an embedding which is not r-representative. (An introduction to representativity is given by Robertson and Vitray [8].) We note the following result.

Theorem 3 *Let G be a minor-minimal r-representative embedding in a surface Σ of Euler genus ε. Let $n \geq \left\lceil \sqrt{\varepsilon+1} \right\rceil (r+3)$. Then G has no n by n grid minor.*

Proof: If G has an n by n minor, then by Lemma 4 G has a subgraph M that contracts to an $r+1$ by $r+1$ grid such that M is embedded in a disc with the outer boundary of M being the boundary of the disc. Delete any central edge e of such a subgraph. Suppose there is a noncontractible curve γ having fewer than r intersections with $G-e$. Then γ must intersect e and, therefore, must come into and leave the disc containing M. But then γ must cross the remaining r by r grid at least r times, a contradiction. **Q.E.D.**

We note that (using (1) and (2)) Theorem 2 implies Theorem 1. It is not clear that Theorem 3 implies the number of minor-minimal r-representative embeddings in a fixed surface is finite. This fact is, however, proved by other means in [5, 2, 4].

In a slightly different direction, let $cr(G)$ denote the crossing number of G, i.e., the minimum number of pairwise crossings of edges in a drawing of G in the plane.

Theorem 4 *Let $k \geq 1$ be an integer. Let G be a graph such that $cr(G) \leq k$ and, for every edge e of G, $cr(G-e) < k$. Let $n \geq \left\lceil \sqrt{2k+1} \right\rceil (12k-5)$. If G contains an n by n grid as a minor, then $cr(G) < k$.*

Proof: Let D be a drawing of G in the plane with k crossings. Let M be a minimal subgraph of G that contracts to an n by n grid. Since M contains $2k+1$ pairwise disjoint $12k-5$ by $12k-5$ subgrids, one of these subgrids K is such that no edge of the subgrid is crossed in D and none of its internal bridges has an edge crossed in D.

Let e be any central edge of K. Let B be a drawing of $G-e$ in the plane with fewer than k crossings. Since $K-e$ contains a $12k-6$ by $12k-6$ subgrid, one of the $(2k-1)$ i^{th}-collars, for $i \in \{1,4,7,\ldots,3(2k-1)-2\}$, in $K-e$ has no crossings of any of its edges in the drawing B. Let J be such a collar and let I and E be the interior and exterior cycles of J.

Just as we obtained an embedding of G in Σ in the proof of Theorem 2, we can get a drawing of G with fewer than k crossings. Let G_1 denote the subgraph of G drawn by D on or inside I, let G_2 denote the bridge of G_1 containing E, and let $\mathcal{B}$ denote the set all the other bridges of G_1.

We get our drawing of G by removing everything inside I in B (i.e., the side of I not containing E), plugging in the drawing of G_1 from D and using the uncrossed collar J to embed all the bridges from $\mathcal{B}$, using individually their drawings from D. **Q.E.D.**

There is obviously a version of this last result that also applies to the crossing number of a graph drawn on some surface, not just the plane. This does not seem to be of independent interest at the moment.

Theorem 4 does not imply a useful finiteness result, since the property of having crossing number $\leq k$ is not closed under contraction of an edge. Even for $k=2$, the number of graphs that have crossing number 2 and all proper subgraphs have crossing number at most 1 is infinite.

One example of an infinite class is obtained by taking three paths with common ends, but otherwise disjoint, each of length at least 2, doubling their edges and adding a new vertex adjacent to exactly one internal vertex of each of the paths. It is easy to see that this graph has crossing number 2 and that the deletion of any edge reduces the crossing number to 1. One of these graphs gives another as a minor by deleting one of two parallel edges (reducing the crossing number to 1) and contracting the second of the two parallel edges (raising the crossing number back to 2). As all known examples of infinite crossing-critical families have this "repetitive structure", we put forward the following.

Conjecture 1 *Let k be a positive integer. Then there is an integer $f(k)$*

such that if G is a graph for which $cr(G) = k$ and $cr(G-e) < k$ for all edges e of G, then the path-width of G is at most $f(k)$.

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References

[1] R. Diestel, T. Jensen, K. Gorbunov and C. Thomassen, *Highly connected sets and the excluded grid theorem*, J. Combin. Theory Ser. B, **75**, (1999), 61-73.

[2] Z. Gao, P.D. Seymour and R.B. Richter, *Irreducible triangulations of surfaces*, J. Combin. Theory Ser. B, **68**, (1996), 206–217.

[3] J.F. Geelen, A.M.H. Gerards and G. Whittle, *Branch–width and well–quasi–ordering for matroids and graphs*, in preparation.

[4] M. Juvan, A. Malnič and B. Mohar, *Systems of curves on surfaces*, J. Combin. Theory Ser. B, **68**, (1996), 7–22.

[5] A. Malnič and R. Nedela, *k-minimal triangulations of surfaces*, Acta Math. Univ. Comenian (N.S.) **64**, (1995), 57-76.

[6] N. Robertson and P.D. Seymour, *Graph Minors VIII: A Kuratowski theorem for general surfaces*, J. Combin. Theory Ser. B, **48**, (1990), 255-288.

[7] N. Robertson and P.D. Seymour, *Graph Minors IV: Tree-width and well-quasi-ordering*, J. Combin. Theory Ser. B, **48**, (1990), 227-254.

[8] N. Robertson and R.P. Vitray, *Representativity of surface embeddings*, in Paths, flows and VLSI-layout, Algorithms Combin., **9**, (1990), 293-328.

[9] R. Thomas, *A Menger-like property of tree-width: the finite case*, J. Combin. Theory Ser. B, **48**, (1990), 67-76.

[10] C. Thomassen, *A simpler proof of the excluded minor theorem for higher surfaces*, J. Combin. Theory Ser. B, **70**, (1997), 306–311.