Fold-2-covering triangular embeddings

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Abstract

For a graph G and a positive integer m, $G_{(m)}$ is the graph obtained from G by replacing every vertex by an independent set of size m and every edge by m^2 edges joining all possible new pairs of ends. If G triangulates a surface, then it is easy to see from Euler's formula that $G_{(m)}$ can, in principle, triangulate a surface. For m prime and at least 7, it has previously been shown that in fact $G_{(m)}$ does triangulate a surface, and in fact does so as a "covering with folds" of the original triangulation. For $m = 5$, this would be a consequence of Tutte's 5-Flow Conjecture. In this work, we investigate the case $m = 2$ and describe simple classes of triangulations G for which $G_{(2)}$ does have a triangulation that covers G "with folds", as well as providing a simple infinite class of triangulations G of the sphere for which $G_{(2)}$ does not triangulate any surface.

1 Introduction

Let G be a (simple) graph that triangulates a surface Σ and let m be a positive integer. Then $G_{(m)}$ is the graph obtained by replacing each vertex v of G by an independent set $v_0, v_1, \ldots, v_{m-1}$ of m vertices and replacing each edge uv of G by m^2 edges u_iv_j, $i, j = 0, 1, 2, \ldots, m-1$. Clearly $|V(G_{(m)})| = m|V(G)|$ and $|E(G_{(m)})| = m^2|E(G)|$. For each triangle $[uvw]$ of the triangulation, there are m^3 triangles $[u_iv_jw_k]$ in $G_{(m)}$. If G triangulates a surface with Euler Characteristic χ, a simple calculation shows that if $G_{(m)}$ triangulates a surface, then that surface has Euler Characteristic $(1-m)m|V(G)| + m^2\chi$.

In fact, by requiring the triangles in the triangulation of $G_{(m)}$ to cover triangles in the triangulation of G (i.e. if $[u_iv_jw_k]$ is a face of $G_{(m)}$, then $[uvw]$

is a face of G), previous work has shown that such a triangulation of $G_{(m)}$ exists if

1. $m \geq 7$ is prime [3] or
2. every vertex of G has even degree [2, 4] or
3. G is 4-chromatic and distinct from K_4 [1].

Since $G_{(mn)} = \left(G_{(m)}\right)_{(n)}$, to prove that $G_{(m)}$ always has a triangular embedding, it suffices to prove it for prime values of m. The techniques of [3] turn the embedding problem into a problem on the dual about flows. By Seymour's 6-Flow Theorem [7], every 2-connected graph has a 6-flow. This implies 1 above. Tutte's 5-Flow Conjecture (c.f., [5]) implies that $G_{(5)}$ has a triangular embedding.

So it is natural now to consider the cases $m = 2$ and $m = 3$ and in this work we treat the case $m = 2$ and show that there are many examples of G for which $G_{(2)}$ has a triangular embedding, while for many G that triangulate the sphere, $G_{(2)}$ does not triangulate any surface. The only previous negative result was that of Jungerman [6], who showed that for $G = K_4$, which triangulates the sphere, $G_{(3)}$ does not triangulate any orientable surface.

One of the features of the study of these covering triangulations that we find intriguing is the natural occurrence of classical graph theoretic parameters in the context. As mentioned above, flows occur in a dual sense to give a triangulation of $G_{(m)}$. We shall see in this paper that if G has a perfect matching, then $G_{(2)}$ has a triangulation that covers G with folds.

2 Basics of Coverings with Folds

Let G be a graph that triangulates a surface Σ. We need to distinguish between *triangles* and 3-cycles in the graph, the former being the 3-cycles that bound faces in the triangulation. It is sometimes advantageous to think of the triangulation as a 2-dimensional simplicial complex, whose 1-skeleton is G. For each vertex v of G, the *link* $lk(v)$ of v is the subgraph consisting of the edges uw for which $[uvw]$ is a triangle, together with their incident vertices. We note that in a triangulation, for every vertex v, $lk(v)$ is a cycle.

Consider the graph $G_{(2)}$ and suppose it triangulates a surface Σ' in such a way that if $[u_iv_jw_k]$ is a triangle of $G_{(2)}$, then $[uvw]$ is a triangle of G. Such a triangulation is a *covering with folds* (or just *covering*) — the reason for the terminology will become clear shortly. For a vertex v, edge $e = uv$ and triangle $t = [uvw]$ of G, respectively, the vertex $\tilde{v} = v_i$, edge $\tilde{e} = u_i v_j$ and triangle $\tilde{t} = [u_i v_j w_k]$ of $G_{(2)}$, respectively, *cover*, or *are above*, v, e and t, respectively. The edge $\tilde{e}$ is *regular* if the two triangles of $G_{(2)}$ containing $\tilde{e}$ cover distinct triangles of G. Otherwise, $\tilde{e}$ is a *fold*. See Figure 1.

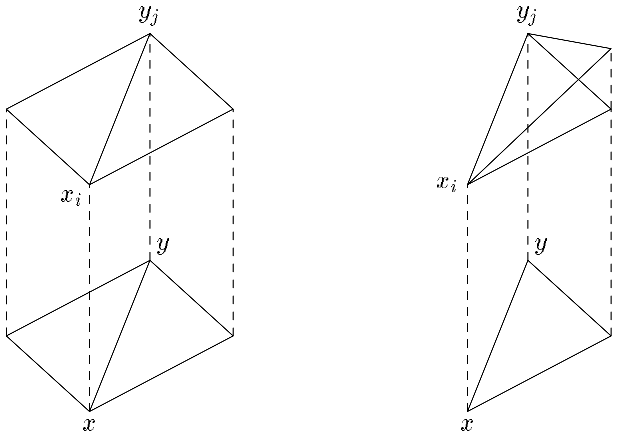

Figure 1: A regular edge and a fold

Let $\tilde{u}$ be a vertex of $G_{(2)}$. If $G_{(2)}$ is a covering with folds, then $lk(\tilde{u})$ is a cycle in $G_{(2)}$. We create a graph $H_{\tilde{u}}$, allowing parallel edges, as follows. The vertices are the neighbors of u and, for each edge $v_i w_j$ of $lk(\tilde{u})$, we put an edge joining v and w. Thus, the number of edges in $H_{\tilde{u}}$ joining v and w is the number of times there is an edge $v_i w_j$ in $lk(\tilde{u})$. Since there are two vertices v_0 and v_1 above the vertex $v \in V(lk(u))$ and each of these vertices is incident to two edges in $lk(\tilde{u})$, we see that $H_{\tilde{u}}$ is regular of degree 4. Thus, if e and f are successive edges of $lk(u)$, and there are α and β edges above e and f, respectively, in $lk(\tilde{u})$, then $\alpha + \beta = 4$. Note that neither of these numbers can be 0, since then $lk(\tilde{u})$ would not be connected. We can think of $H_{\tilde{u}}$ as being $lk(u)$ with some of the edges replicated to make $H_{\tilde{u}}$ 4-regular.

There are only two solutions to this equation, namely $\{\alpha, \beta\} = \{1, 3\}$ and $\{\alpha, \beta\} = \{2, 2\}$. We are going to associate with each $\tilde{u}$ an index, which is essentially the winding number of the projection of $lk(\tilde{u})$ onto $lk(u)$, around u.

Suppose first that $\{\alpha, \beta\} = \{1, 3\}$. Then the multiplicities 1 and 3 alternate as we go around the cycle $lk(u)$. It follows that u has even degree.

If $e = vw$ has multiplicity 3, then $lk(\tilde{u})$ contains three of the 4 edges $v_i w_j$. Since the labels i and j are arbitrary in $\{0, 1\}$, we can assume $lk(\tilde{u})$ contains v_0w_0, v_0w_1, v_1w_0. In this case, the edges $\tilde{u}v_0$ and $\tilde{u}w_0$ are folds, while the edges $\tilde{u}v_1$ and $\tilde{u}w_1$ are regular. It follows that $lk(\tilde{u}) = (\ldots, w_1, v_0, w_0, v_1, \ldots)$ and that this pattern repeats as $(\ldots, y_1, x_0, y_0, x_1, w_1, v_0, w_0, v_1, \ldots)$, being completed by going once around $lk(u)$. In this case, we define the *index* of $\tilde{u}$ to be 1.

In the event that $\{\alpha, \beta\} = \{2, 2\}$, there are two possibilities. In the first case, every edge incident with $\tilde{u}$ is regular. Suppose the vertices, in cyclic order, in $lk(u)$ are $(v^1, v^2, \ldots, v^r)$. Then every edge $u_i v_k^j$, $i, k \in \{0, 1\}$, $j = 1, 2, \ldots, r$, is regular. Therefore, the link of $\tilde{u}$ above u has substrings of the form $v_{k_{j-1}}^{j-1} v_{k_j}^j v_{k_{j+1}}^{j+1}$, where the k_s are in $\{0, 1\}$. It follows that $lk(\tilde{u}) = (v_{k_1}^1, v_{k_2}^2, \ldots, v_{k_r}^r, v_{1-k_1}^1, v_{1-k_2}^2, \ldots, v_{1-k_r}^r)$, for some $k_1, \ldots, k_r \in \{0, 1\}$. In this case the index of $\tilde{u}$ is 2.

Finally, suppose the edge $\tilde{u}v_{k_1}^1$ is a fold. The two edges of $lk(\tilde{u})$ incident with $v_{k_1}^1$ are above the same edge, say v^1v^2. Now every edge in $lk(u)$ has multiplicity 2 in $H_{\tilde{u}}$, and we have seen that v^1v^2 has multiplicity two because $lk(\tilde{u})$ has the form $(v_0^2, v_{k_1}^1, v_1^2, \ldots)$. It follows that the other edge v^1v^r in $lk(u)$ has multiplicity 2 and produces the string $v_0^r v_{1-k_1}^1 v_1^r$ in $lk(\tilde{u})$. Therefore, $\tilde{u}v_{1-k_1}^1$ is also a fold. Since $lk(\tilde{u})$ is connected, it is easy to see that these are the only folds incident with $\tilde{u}$. Thus, $lk(\tilde{u})$ has the form $(v_{k_1}^1, v_{k_2}^2, \ldots, v_{k_r}^r, v_{1-k_1}^1, v_{1-k_r}^r, v_{1-k_{r-1}}^{r-1}, \ldots, v_{1-k_2}^2)$. In this case, the index of $\tilde{u}$ is 0.

We use the notation $ind(\tilde{u})$ to denote the index of $\tilde{u}$ and we have the following summary, which is illustrated in Figure 2 in the particular case where the degree of $\tilde{u}$ is equal to 8.

Lemma 2.1 *Suppose G and $G_{(2)}$ are triangulations such that $G_{(2)}$ is a covering with folds of G. Let u be a vertex of G and let $\tilde{u}$ be a vertex of $G_{(2)}$ above u. If $ind(\tilde{u}) = 0$, then $\tilde{u}$ is incident with two folds, and they are of the form $\tilde{u}v_0$ and $\tilde{u}v_1$, for some vertex v of G adjacent to u. If $ind(\tilde{u}) = 1$, then u has even degree and every edge uv is covered by exactly two folds. If $ind(\tilde{u}) = 2$, then no edge incident with $\tilde{u}$ is a fold.*

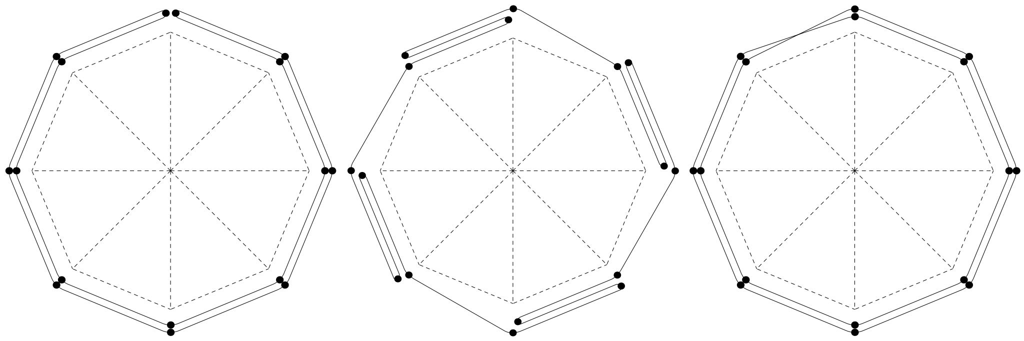

Figure 2: The three cases for a link of $G_{(2)}$ above a vertex of degree 8. From left to right the index is 0, 1, 2.

3 Parity of $G_{(2)}$

Let t be any triangle of G and let $c(t)$ denote the number of triangles of $G_{(2)}$ that cover t. We claim that the parity of $c(t)$ is independent of t. Let e be an edge of G incident with t and let s be the other triangle incident with e. Because G is connected, it suffices to prove that $c(t)$ and $c(s)$ are congruent modulo 2.

Let $\tilde{e}$ be any edge above e and let $\tilde{t}_1$ and $\tilde{t}_2$ be the triangles of $G_{(2)}$ incident with $\tilde{e}$. If $\tilde{e}$ is a regular edge, then one of $\tilde{t}_1$ and $\tilde{t}_2$ covers s and the other covers t. If $\tilde{e}$ is a fold, then $\tilde{t}_1$ and $\tilde{t}_2$ cover the same triangle of G. In either case, the number of times s and t are covered is the same modulo 2. This proves the claim.

We denote by $p(G_{(2)}) \in \{0, 1\}$ the common value of $c(t)$, taken modulo 2.

Continuing in this vein, let $\tilde{u}$ be a vertex of $G_{(2)}$ and let t be a triangle of G covered by a triangle incident with $\tilde{u}$. Let $n(\tilde{u}, t)$ denote the number of triangles of $G_{(2)}$ incident with $\tilde{u}$ and above t. The following three facts are an immediate consequence of the definitions and the discussion of the preceding section.

Lemma 3.1 *Let u be any vertex of G, let t be a triangle incident with u, let u_0 and u_1 be the two vertices above u, and let $\tilde{u} \in \{u_0, u_1\}$. Then:*

1. $p(G_{(2)}) \equiv n(u_0, t) + n(u_1, t) \pmod 2$;
2. $1 \leq n(\tilde{u}, t) \leq 3$; *and*

3. $ind(\tilde{u}) \equiv n(\tilde{u}, t) \pmod 2$.

Lemma 3.2 *If some vertex of G has odd degree, then $p(G_{(2)}) = 0$.*

Proof. Let u have odd degree. Suppose, by way of contradiction, that $p(G_{(2)}) = 1$. By Part 1 of Lemma 3.1, either $n(u_0, t)$ or $n(u_1, t)$ is odd. Part 3 implies that, for such an i, u_i has index 1. By Lemma 2.1, u has even degree, a contradiction. □

4 Triangles above a triangle

Let $t = [uvw]$ be a triangle of G and let $A(t)$ denote the set of triangles of $G_{(2)}$ above t. Then $A(t)$ is a subset of $\{[u_iv_jw_k] \mid i, j, k \in \{0, 1\}\}$, which has eight triangles, not all of which can be included in $G_{(2)}$, since then either $G_{(2)}$ consists of just these triangles or $G_{(2)}$ is not connected.

Lemma 4.1

1. $|A(t)| = n(u_0, t) + n(u_1, t)$.
2. *If G has a vertex of odd degree, then $|A(t)| = 4$.*

Proof. 1 is obvious. For 2, we note that Lemmas 3.1 and 3.2 imply $|A(t)|$ is even. Part 2 of Lemma 3.1 and Part 1 of Lemma 4.1 imply $|A(t)| = 2$, 4, or 6. We shall rule out 2 and 6.

Suppose u is a vertex of G incident with a triangle t such that $|A(t)| = 2$ and let v be any neighbor of u in G. We shall show that u has even degree and v is incident with a triangle s such that $|A(s)| = 2$. The fact that G is connected then implies that every vertex of G has even degree, the desired contradiction.

Evidently $n(u_0, t) = 1 = n(u_1, t)$, so both u_0 and u_1 have index 1, so, by Lemma 2.1, u has even degree. Let $t = t_0, t_1, t_2, \ldots$ be the cyclic sequence of triangles incident with u in G. For $i = 1, 2$, we have that $n(u_i, t_0) = 1$, which implies that $n(u_i, t_1) = 3$, from which we deduce that $n(u_i, t_2) = 1$. Continuing inductively, it is easy to see that $n(u_i, t_j)$ is 1 if j is even and 3 if j is odd.

For even j, then, we have that $|A(t_j)| = 2$. Since every neighbor v of u is incident with some such t_j, we have the desired contradiction.

Now we rule out $|A(t)| = 6$. By Part 1 of Lemma 4.1 and Part 2 of Lemma 3.1, $n(u_0, t) = n(u_1, t) = 3$. Thus, u_0 and u_1 are of index 1. Let s be a triangle sharing an edge with t and incident with u. Then $n(u_0, s) = n(u_1, s) = 1$, and we are back to the previous case, since $|A(s)| = 2$. $\square$

In [4] it is proved that if every vertex of G has even degree, then $G_{(2)}$ has a triangulation that covers G with folds.

5 General Necessary Conditions

If $G_{(2)}$ triangulates a surface so as to be a covering with folds of the triangulation G and $p(G_{(2)}) = 0$, then we have seen in Lemma 3.1 that the vertex set of G partitions into the sets $V_{0,0}$, $V_{0,2}$, $V_{2,2}$ and $V_{1,1}$, where $V_{i,j}$ is the set of vertices u of G such that if u is below u_1 and u_2, then $\{ind(u_1), ind(u_2)\} = \{i, j\}$.

Lemma 5.1 *Let $\tilde{u}$ cover u, let v_0 and v_1 cover v and suppose $ind(\tilde{u}) = 0$. If $\tilde{u}v_0$ is a fold, then $\tilde{u}v_1$ is a fold and, for $i = 1, 2$, $ind(v_i) \in \{0, 1\}$. Furthermore, if $u \in V_{0,2}$, then $v \in V_{1,1}$.*

Proof. Let $v_0 = \tilde{v}$. By assumption $\tilde{u}v_0$ is a fold, so v_0 is incident with a fold. Hence, it has index either 0 or 1. But $ind(\tilde{u}) = 0$, so Lemma 2.1 implies $\tilde{u}v_1$ is also a fold, so v_1 is incident with a fold. Hence, it has index either 0 or 1.

Furthermore, let u_0 be the vertex above u with index 0, so u_1 is the vertex above u with index 2. Let w and x be the neighbors of v in $lk(u)$. We shall show that both vw and vx are covered by folds. From this we see that v is incident with three edges covered by folds and so, by Lemma 2.1, must be in $V_{1,1}$.

We know that $lk(u_0)$ has the form $\ldots w_0v_0w_1 \ldots x_1v_1x_0 \ldots$ and $lk(u_1)$ has the form $\ldots w_iv_0x_j \ldots w_{1-i}v_1x_{1-j} \ldots$. Thus, we have, among others, the triangles $t_1 = [u_0v_0w_0]$, $t_2 = [u_0v_0w_1]$, $t_3 = [u_0v_1x_0]$, $t_4 = [u_0v_1x_1]$, $t_5 = [u_1v_0w_i]$, and $t_6 = [u_1v_1x_{1-j}]$.

The triangles t_1, t_2, t_5 show that v_0w_i is a fold. The triangles t_3, t_4, t_6 show that v_1x_{1-j} is a fold. Thus, each of uv, vw and vx is covered by a fold, as required. $\square$

Lemma 5.2 *If $u \in V_{1,1}$ and $e = uv$, then $v \notin V_{2,2}$.*

Proof. By Lemma 2.1, e is covered by two folds. $\square$

From Lemma 5.2, we see that the edges in the coboundary

$$\delta(V_{1,1}) = \{uv \in E(G) \mid u \in V_{1,1},\ v \notin V_{1,1}\}$$

are incident with vertices in $V_{0,0} \cup V_{0,2}$. For a vertex u in $V_{0,2}$, there is only one edge $e = uv$ incident with u and covered by a fold. Lemma 5.1 implies $v \in V_{1,1}$. Partition $V_{0,0}$ into two sets, namely $V_{0,0}^1$ consisting of vertices incident with edges in $\delta(V_{1,1})$ and $V_{0,0}^0 = V_{0,0} \setminus V_{0,0}^1$.

Lemma 5.3 *Let $u \in V_{1,1}$, $e = uv$ and $v \in V_{0,2}$. Let w be a neighbor of v in $lk(u)$. Then $w \notin V_{1,1}$.*

Proof. The only edge incident with v covered by a fold is e. If $w \in V_{1,1}$, then by Lemma 2.1 vw is covered by a fold. $\square$

Lemma 5.4 *Let $u, v \in V_{0,0}$ and suppose uv is covered by a fold. Then every edge above uv is a fold and uv is the only edge incident with u that is covered by a fold.*

Proof. Let w and x be the neighbours of v in $lk(u)$. Suppose u_i is the vertex above u such that u_iv_0 and u_iv_1 are folds. Then, up to relabelling w and x, Lemma 5.1 shows we must have the triangles $t_1 = [u_iv_0w_0]$, $t_2 = [u_iv_0w_1]$, $t_3 = [u_iv_1x_0]$ and $t_4 = [u_iv_1x_1]$.

Now the edge u_iv_0 is a fold and v_0 has index 0, so Lemma 2.1 implies $u_{1-i}v_0$ is also a fold. Since $ind(u_{1-i}) = 0$, it follows that $u_{1-i}v_1$ is also a fold, as required. $\square$

Every member u of $V_{0,0}$ is incident with an edge uv covered by a fold. Thus, by Lemma 5.1, the only possibilities are $v \in V_{0,0}$ and $v \in V_{1,1}$. We can distinguish these cases, since, for the former, Lemma 5.4 tells us uv is the only edge incident with u covered by a fold, while, for the latter, u is incident with two edges, each covered by two folds.

We now give our final necessary condition.

Lemma 5.5 *The degree of a vertex $u \in V_{2,2}$ is congruent modulo 2 to the number of folds covering edges in its link.*

Proof. Suppose v and w are consecutive in $lk(u)$. If vw is covered by a fold, then the triangles above $[uvw]$ are, without loss of generality, $[u_0v_0w_0]$, $[u_1v_0w_0]$, $[u_0v_1w_1]$ and $[u_1v_1w_1]$. In this case, the links of u_0 and u_1 look like

$$\begin{aligned} lk(u_0) &= \ldots v_0w_0 \ldots v_1w_1 \ldots \\ lk(u_1) &= \ldots v_0w_0 \ldots v_1w_1 \ldots \end{aligned}$$

If vw is not covered by a fold, then the triangles above $[uvw]$ are, without loss of generality, $[u_0v_0w_0]$, $[u_0v_1w_1]$, $[u_1v_1w_0]$ and $[u_1v_0w_1]$. In this case, the links of u_0 and u_1 look like

$$\begin{aligned} lk(u_0) &= \ldots v_0w_0 \ldots v_1w_1 \ldots \\ lk(u_1) &= \ldots v_0w_1 \ldots v_1w_0 \ldots \end{aligned}$$

The conclusion is that the occurrences of subscripts on v and w in consecutive terms differ in $lk(u_0)$ and $lk(u_1)$ if and only if vw is not covered by a fold. So the number of edges in $lk(u)$ covered by non-folds must be even, since we have to get from v_0 to v_1 in both $lk(u_0)$ and $lk(u_1)$. □

The preceding lemmas form the core of the following necessary conditions for $G_{(2)}$ to have a triangulation covering G. Note that here we do not assume $p(G_{(2)}) = 0$, since the case $p(G_{(2)}) = 1$ is covered as well (taking $V_{1,1} = V(G)$).

Theorem 5.6 General Necessary Conditions. *Let G be a triangulation. Suppose $G_{(2)}$ is a triangulation that is a covering with folds of G. Then $V(G)$ has a partition into $V_{1,1}$, $V^1_{0,0}$, $V^0_{0,0}$, $V_{0,2}$ and $V_{2,2}$ such that:*

1. *every vertex in $V_{1,1}$ has even degree;*
2. *the coboundary $\delta(V_{1,1})$ partitions into a set Δ of pairs and singletons such that each pair is incident with a common element of $V^1_{0,0}$, each singleton is incident with a member of $V_{0,2}$ and each member of $V^1_{0,0} \cup V_{0,2}$ is incident with the edges in exactly one member of Δ;*
3. *$V^0_{0,0}$ has a perfect matching M;*
4. *for each vertex $u \in V_{2,2}$, $deg_G(u) \equiv |lk(u) \cap M| \pmod 2$.*

Proof. If $p(G_{(2)}) = 1$, then every vertex has even degree and we may take $V_{1,1}$ to be all of $V(G)$. Therefore, we may assume $p(G_{(2)}) = 0$.

The partition is defined by setting $V_{i,j}$ to be the set of vertices u of G such that $\{ind(u_0), ind(u_1)\} = \{i, j\}$. We further refine the partition to let $V^1_{0,0}$ denote the set of vertices $v \in V_{0,0}$ for which there is a $u \in V_{1,1}$ with $uv \in E(G)$. Finally, set $V^0_{0,0} = V_{0,0} \setminus V^1_{0,0}$.

Conclusion 1 is Lemma 2.1.

For Conclusion 2, suppose $uv \in \delta(V_{1,1})$ with $u \in V_{1,1}$ and $v \notin V_{1,1}$. By Lemma 5.2, $v \notin V_{2,2}$.

If $v \in V_{0,2}$, then $\{uv\}$ is one of the singletons in Δ.

If $v \in V_{0,0}$, then, by definition, $v \in V^1_{0,0}$. As four edges above edges incident with v are folds, and only two edges above uv are folds, there is another neighbour w of v such that vw is covered by a fold. Lemma 5.4 implies $w \notin V_{0,0}$. By definition, $w \notin V_{2,2}$ and, by Lemma 5.1, $w \notin V_{0,2}$. Therefore, $w \in V_{1,1}$. It follows that uv and vw are the only edges incident with v covered by folds. We take $\{uv, vw\}$ to be one of the pairs in Δ.

For Conclusion 3, let $u \in V^0_{0,0}$ and let v be such that uv is covered by a fold. By definition of $V^0_{0,0}$, $v \notin V_{1,1}$. By Lemma 5.1, $v \notin V_{0,2}$. By definition, $v \notin V_{2,2}$. Hence $v \in V_{0,0}$.

By Lemma 5.4, all four edges above uv are folds. Hence $v \notin V^1_{0,0}$, as only four edges above those incident with v are folds. It follows that $v \in V^0_{0,0}$ and we can set M to be the set of all edges uv such that $u, v \in V^0_{0,0}$ and uv is covered by four folds.

Finally, Conclusion 4 is Lemma 5.5. □

6 Examples of triangulations G for which $G_{(2)}$ has no triangulation

In this section, we describe a simple type of triangulation G such that no embedding of $G_{(2)}$ is a triangulation.

The General Necessary Conditions show that the set $V_{1,1}$ plays a complicating role and if $V_{1,1} = \emptyset$, then also $V^1_{0,0}$ and $V_{0,2}$ are empty. We can force this to happen by making all vertices have odd degree.

In this case, the necessary conditions become much simpler: there is a matching M such that, for any vertex u not covered by M, $deg(u) \equiv |lk(u) \cap M| \pmod 2$. Since every vertex has odd degree, this implies that every vertex either is incident with an edge of M or has an odd number of matching edges in its link.

Let G_1 be any triangulation such that the vertex set of G_1 partitions into triangles $t_1, t_2, \ldots, t_k$. Thus, $|V(G_1)| = 3k$. Let G' be the triangulation obtained from G_1 by adding a new vertex v_t inside each triangle $t = [uvw]$ and joining v_t to each of u, v and w. Note that each v_t has degree 3 and if u is a vertex of G_1 with degree d in G_1, then u has degree $2d$ in G'. Finally, let G be obtained from G' by deleting the vertices v_{t_i}, $i = 1, 2, \ldots, k$, so that each vertex of G has odd degree.

We begin our analysis of G by showing that if T is a triangulation of any surface by $G_{(2)}$, then T is a covering of G with folds. To see this, note that the only 3-cycles of G incident with a v_t are the three triangles $s_1 = [uvv_t]$, $s_2 = [uwv_t]$ and $s_3 = [vwv_t]$, where $t = [uvw]$ is a triangle of G_1. It follows that the only triangles in $G_{(2)}$ incident with a vertex above v_t are above these three triangles.

Suppose there are σ_i triangles of $G_{(2)}$ above s_i. For each $e \in \{uv_t, vv_t, wv_t\}$ there are four edges of $G_{(2)}$ above e and, therefore, there are eight triangles incident with edges covering e. Each of these triangles is above one of the s_i and so we deduce that $\sigma_1 + \sigma_2 = 8$, $\sigma_1 + \sigma_3 = 8$ and $\sigma_2 + \sigma_3 = 8$.

The only solution to these equations has every $\sigma_i = 4$. Thus, any triangulation of $G_{(2)}$ has four triangles above each of the s_i, and this is true for every triangle t of G, other than the t_i.

If e is an edge of $G_{(2)}$ not above an edge of some t_i, then we have identified the two triangles containing e. As the t_i are pairwise disjoint, it follows that each of the remaining triangles of $G_{(2)}$ (i.e., those not incident with a vertex above a v_t) must be above some t_i. Thus, every triangle of T covers a triangle of G, and so T must be a covering with folds.

Now suppose $G_{(2)}$ has a triangulation that covers G with folds. From the General Necessary Conditions, $V(G)$ has a partition $V^0_{0,0} \cup V_{2,2}$ such that $V^0_{0,0}$ has a perfect matching M and, for each vertex $u \in V_{2,2}$, $deg_G(u) \equiv |lk(u) \cap M|$ (mod 2). Thus, for every vertex $u \in V_{2,2}$, M has an odd number of edges in $lk(u)$.

Let V' denote the set of vertices v_t of G. From the matching M, we obtain a set M' of edges by replacing all edges $uv \in M$ for which $u, v \in V(G_1) = V(G) \setminus V'$ by the edges uv_s and uv_t, where s and t are the triangles of the triangulation G_1 incident with uv, assuming neither s nor t is one of $t_1, t_2, \ldots, t_k$. In case s, say, is one of $t_1, t_2, \ldots, t_k$, then we simply do not add uv_s to M'.

Note that every edge of M' is incident with a vertex in V' and a vertex in $V(G_1) = V(G) \setminus V'$.

For every vertex $w \in V'$, either w is incident with an edge of M, and therefore, incident with an edge of M', or some edge uv of $lk(w)$ is in M and either uw or vw is in M'. Thus, every vertex of V' is incident with at least one edge of M'. Consequently, $|M'| \geq |V'|$.

On the other hand, if $u \in V(G_1)$, u is incident with no more edges of M' than it is incident with edges of M. Since M is a matching, no vertex of G_1 is incident with more than one edge of M'. Thus, $|V(G_1)| \geq |M'|$.

Since $V(G_1) = V(G) \setminus V'$, the two preceding paragraphs imply $|V(G_1)| \geq |V'|$.

Now $\chi(\Sigma) = |V(G_1)| - |E(G_1)| + |F(G_1)|$ and, since G_1 is a triangulation of Σ, $|E(G_1)| = (3/2)|F(G_1)|$. Also, $|V(G_1)| = 3k$ and $|F(G_1)| = k + |V'|$. Therefore, $\chi(\Sigma) = 3k - (1/2)(k + |V'|) = (5/2)k - (1/2)|V'|$.

We have that $|V'| \leq |V(G_1)| = 3k$. Thus, $\chi(\Sigma) = (5/2)k - (1/2)|V'| \geq (5/2)k - (1/2)(3k) = k$. Thus, as long as we choose $k > \chi(\Sigma)$, we have a contradiction. It follows that $G_{(2)}$ does not have a triangular embedding.

7 Examples of triangulations G for which $G_{(2)}$ has a triangulation that covers with folds

It is clear from the General Necessary Conditions Theorem 5.6 that the set $V_{1,1}$ complicates matters greatly. It is the purpose of this section to give a converse of Theorem 5.6 in the case $V_{1,1}$ can be chosen to be empty.

Theorem 7.1 *Let G be a triangulation and let $\mathcal{T}$ be the set of triangles of G. Suppose $V(G)$ partitions into two sets A and B such that:*

1. *the subgraph induced by A has a perfect matching M;*
2. *each vertex $u \in B$ satisfies $deg(u) \equiv |lk(u) \cap M| \pmod 2$; and*
3. *there is a function $\phi : \mathcal{T} \to \{0, 1\}$ such that, for each $e \in M$, the triangles t and t' containing e satisfy $\phi(t) \neq \phi(t')$ and, for each vertex $u \in B$,*

$$\sum_{u \in t \in \mathcal{T}} \phi(t) \equiv 1 \pmod 2.$$

Then there is a triangular embedding of $G_{(2)}$ above G.

Proof. We begin by describing the triangles of the triangular embedding of $G_{(2)}$. For each triangle $t = [uvw] \in \mathcal{T}$, if no edge of t is in M, then we select the triangles $[u_i v_j w_k]$ for which $i + j + k \equiv \phi(t) \pmod 2$, while if $uv \in M$, then we select the triangles $[u_i v_j w_k]$ for which $i + j \equiv \phi(t) \pmod 2$. It is easy to verify that each edge of $G_{(2)}$ is in exactly two of these triangles. Therefore, it suffices to show that the link of each vertex $\tilde{u}$ of $G_{(2)}$ is a cycle. Naturally, there are two cases.

Case 1: $u \in A$. Let $uv \in M$. Let $lk(u) = vwx \ldots y$.

Claim 1 *$lk(\tilde{u})$ is of the form $v_0 w_{i_w} x_{i_x} \ldots y_{i_y} v_1 y_{1-i_y} \ldots x_{1-i_x} w_{1-i_w}$.*

To see this, we begin by considering a substring abc of $wx \ldots y$. In G we have the triangles $t = [uab]$ and $t' = [ubc]$. There are four triangles of $G_{(2)}$ above each of them. There are two essentially different cases, depending on whether or not one of ab, bc is in M.

If neither ab nor bc is in M, then, taking all subscripts modulo 2, the four of these triangles incident with u_i are $[u_i a_{i+\phi(t)} b_0]$, $[u_i a_{1+i+\phi(t)} b_1]$, $[u_i c_{i+\phi(t')} b_0]$, and $[u_i c_{1+i+\phi(t')} b_1]$. This tells us that $lk(u_i)$ contains the substrings (possibly reversed) $a_{i+\phi(t)} b_0 c_{i+\phi(t')}$ and $a_{1+i+\phi(t)} b_1 c_{1+i+\phi(t')}$. In particular, the "$abc$" sequence is preserved, although possibly reversed.

A similar phenomenon occurs if, say, $ab \in M$. In this case, $lk(u_i)$ contains the strings $a_{\phi(t)} b_0 c_{i+\phi(t')}$ and $a_{1+\phi(t)} b_1 c_{1+i+\phi(t')}$.

Now consider the string vwx. Thus, G contains the triangles $t = [uvw]$ and $t' = [uwx]$. Again there are two cases, depending on whether or not $wx \in M$. If $wx \notin M$, then we have selected for $G_{(2)}$ the triangles $[u_i v_{i+\phi(t)} w_0]$, $[u_i v_{i+\phi(t)} w_1]$, $[u_i w_0 x_{i+\phi(t')}]$ and $[u_i w_1 x_{1+i+\phi(t')}]$. This implies that $lk(u_i)$ contains (up to reversal) the string

$$x_{1+i+\phi(t')} w_1 v_{i+\phi(t)} w_0 x_{i+\phi(t')}.$$

If, on the other hand, $wx \in M$, then we find $lk(u_0)$ contains the string $x_{1+\phi(t')} w_1 v_{\phi(t)} w_0 x_{\phi(t')}$ and $lk(u_1)$ contains a similar string.

The preceding paragraphs make it clear that $lk(u_i)$ is of the form

$$v_{i+\phi(t)} w_0 x_{i_x} \ldots y_{i_y} v_{i+\phi(t')} y_{1-i_y} \ldots x_{1-i_x} w_1,$$

where t and t' are the two triangles of G incident with uv. It is here that we require $\phi(t) \neq \phi(t')$, so that we get both v_0 and v_1 occurring in $lk(u_i)$.

Case 2: $u \in B$. Let $lk(u)$ be $vw \ldots x$.

Claim 2 *$lk(\tilde{u})$ is of the form*

$$v_{i_v} w_{i_w} \ldots x_{i_x} v_{1-i_v} w_{1-i_w} \ldots x_{1-i_x}.$$

To see this, let abc be any substring of $lk(u)$, so that $t = [uab]$ and $t' = [ubc]$ are the triangles of G containing ub.

If neither ab nor bc is in M, then in $G_{(2)}$ we have the triangles $[u_i a_j b_k]$, $i+j+k \equiv \phi(t) \pmod 2$ and $[u_i b_j c_k]$, $i+j+k \equiv \phi(t') \pmod 2$. Thus, we have in $lk(u_i)$, up to reversal, the strings

$$a_0 b_{i+\phi(t)} c_{\phi(t)+\phi(t')} \text{ and } a_1 b_{1+i+\phi(t)} c_{1+\phi(t)+\phi(t')}.$$

If we have $ab \in M$, then in $G_{(2)}$ we have the triangles $[u_i a_j b_k]$, $j+k \equiv \phi(t) \pmod 2$ and $[u_i b_j c_k]$, $i+j+k \equiv \phi(t') \pmod 2$. Thus, we have in $lk(u_i)$, up to reversal, the strings

$$a_0 b_{\phi(t)} c_{i+\phi(t)+\phi(t')} \text{ and } a_1 b_{1+\phi(t)} c_{1+i+\phi(t)+\phi(t')}.$$

From these remarks, it is clear that $lk(u_i)$ has a string of the form $v_{i_v} w_{i_w} \ldots x_{i_x}$ and a string of the form $v_{1-i_v} w_{1-i_w} \ldots x_{1-i_x}$. The question is, do these strings connect up to make two cycles or do they connect up to make one cycle?

The answer is found by considering the indices. We can compute the next index k of z in the sequence from the current index j of y as $k \equiv j+i+\phi(\hat{t})$ if the edge yz is not in M, and $k \equiv j+\phi(\hat{t})$ if yz is in M, where $\hat{t}$ is the triangle $[uyz]$. It follows that the successor of x_{i_x} has index congruent modulo 2 to $i_v + \sum_{u \in t \in \mathcal{T}} \phi(t) + i(deg(u) - |lk(u) \cap M|)$. Since the term $i(deg(u) - |lk(u) \cap M|)$ is 0 modulo 2 and the sum is 1 modulo 2, we see that the successor to x_{i_x} is v_{1+i_v}, and we have only one cycle. $\square$

Corollary 7.2 *If G is a triangulation with a perfect matching, then $G_{(2)}$ has a covering triangulation.*

8 Remarks on Theorem 7.1

If Conditions 1 and 2 of Theorem 7.1 are satisfied, a problem is to find a function ϕ satisfying Condition 3 or to prove that such a function does not exist. The problem amounts to solving the following system of linear equations over the field $GF(2)$, with unknowns $\phi(t)$, $t \in \mathcal{T}$:

$$\sum_{u \in t} \phi(t) = 1,\ u \in B, \tag{1}$$

$$\sum_{e \in t} \phi(t) = 1,\ e \in M. \tag{2}$$

This system has a solution if and only if there is no impossibility condition, which is a linear combination of the equations with a vanishing first member and a nonnull second member. This happens if and only if there is a subset $C \subseteq B$ and a subset $N \subseteq M$ satisfying the two following properties.

8.1 $|N| + |C|$ *is odd.*

8.2 *Every triangle* $t \in \mathcal{T}$ *is incident to an even number of vertices and edges in* $C \cup N$.

If M is empty, which implies by Condition 2 of Theorem 7.1 that G has even degrees, the necessary and sufficient conditions for ϕ to exist are stated in [2]. We now assume that M is nonempty and that ϕ does not exist, so that Properties 8.1 and 8.2 are satisfied.

Say that a triangle $t \in \mathcal{T}$ is *essential* if it is incident to a nonnull number of elements in $N \cup C$. It is easy to verify that every essential triangle is either incident to one edge in N and one vertex in C or to precisely two vertices in C. Moreover, if t is an essential triangle incident to the elements p and q in $C \cup N$, we verify that each of the three triangles t' sharing a common edge with t is also essential, and that the elements p' and q' of t' incident with $C \cup N$ can be deduced from p and q. By connectivity it follows that every triangle is essential and that, if p and q are known for some essential triangle t, C and N are determined. Now we notice that, if t is incident to an edge $e \in M$, then $\{p, q\}$ is made of e and the vertex of t not incident to e. It follows that $N = M$ and, since M is nonempty, C is unique.

An example of a triangulation satisfying the preceding conditions is K_4 in the plane, with M containing only one edge and $B = C$ made of the two vertices not incident to M. We have $|M| + |C| = 3$. This does not mean that $K_{4(2)}$ has no triangular imbedding. We can construct such an imbedding by taking a perfect matching in K_4 and applying Corollary 7.2. Similarly, in the case where G has even degrees and the linear system defined by 1 and 2 has no solution, it is proved in [4] that another construction can be used to find a triangular imbedding of $G_{(2)}$. Thus two problems arise.

Problem 8.3 *If the vertex-set of a triangulation G partitions into two subsets A and B such that Conditions 1 and 2 of Theorem 7.1 are satisfied, is there some construction of a triangular imbedding of $G_{(2)}$?*

Problem 8.4 *Is there a good characterization of the triangulations for which A and B can be found satisfying Conditions 1 and 2?*

9 Final Remarks

In order to make the General Necessary Conditions into sufficient conditions, it is necessary to have a function $\phi : \mathcal{T} \to \{0, 1\}$, generalizing the one required in Theorem 7.1. There are essentially three conditions on ϕ, two of which are described in Theorem 7.1 (thinking of A as $V^0_{0,0}$ and B as $V_{2,2}$). The third condition is for vertices $u \in V^1_{0,0}$. This vertex is incident with two edges $e = uv$ and $e' = uw$ in $\delta(V_{1,1})$.

If v and w are not consecutive in $lk(u)$, then ϕ must take on the same value on the two triangles incident with e and the same value on the two triangles incident with e', but these two values must be different. Thus, one of e and e' will have ϕ being 0 for both incident triangles and the other will have ϕ being 1 for both incident triangles.

If v and w are consecutive in $lk(u)$, then there are two triangles t_1 and t_2 of G, each containing u and exactly one of v and w. We require $\phi(t_1) \neq \phi(t_2)$.

A function ϕ that satisfies these three conditions is *compatible* with the partition. It turns out that the existence of a compatible function ϕ is also necessary for the existence of a triangulation of $G_{(2)}$ that covers G with folds. Thus, we actually have necessary and sufficient conditions. However, the proofs of both the necessity and sufficiency are quite technical and have many cases, as the reader will quickly verify. It would be interesting to find a proof of the sufficiency of the conditions that does not involve a huge case analysis.

The problem is that the vertices in $V_{1,1}$ are complicated by the fact that they can have three different types of neighbours. In [4], the case that they have only neighbours from $V_{1,1}$ is handled and that already is technically demanding. Our own work simply generalizes that argument to include all the possibilities.

We have not devoted any attention to looking for triangulations of $G_{(3)}$ or $G_{(4)}$.

Given that not every $G_{(2)}$ triangulates a surface, the hope to prove that $G_{(m)}$ always triangulates a surface by proving it only for prime m has disappeared. However, the fact that $G_{(m)}$ always triangulates when m is prime and at least 7 suggests the following question, first brought to our attention by Carsten Thomassen. We present it in the form of a conjecture.

Conjecture 9.1 *If $m \geq 5$ and G is a triangulation of a surface, then $G_{(m)}$ triangulates a surface that covers G with folds.*

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