Cycle spaces in topological spaces

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Abstract: We develop a general model of edge spaces in order to generalize, unify, and simplify previous work on cycle spaces of infinite graphs. We give simple topological criteria to show that the fundamental cycles of a (generalization of a) spanning tree generate the cycle space in a connected, compact, weakly Hausdorff edge space. Furthermore, in such a space, the orthogonal complement of the bond space is the cycle space. This work unifies the two different notions of cycle space as introduced by Diestel and Kühn (Combinatorica **24** (2004), 68–89 and Europ. J. Combin. **25** (2004), 835–862) and by Bonnington and Richter (J. Graph Theory **44** (2003), 132–147).

1. Introduction

In [1], the authors introduce the cycle space of a locally finite graph as the set of edge-sets of subgraphs in which every vertex has even degree. Motivated by this work, Richter asked when the cycle space of an infinite graph is generated by fundamental cycles. In response, Diestel and Kühn [6, 7, 8] pioneered the study a different cycle space of infinite graphs: the space of all combinations of "thin" families of (possibly infinite) circuits. A main goal of this work is to provide a study of cycle spaces of which these are both special cases.

For Diestel and Kühn, a *circuit* is the set of edges in $\tilde{G}$ that are contained in a homeomorph of the unit circle. Having added ends gives the possibility of "infinite circuits" in addition to the usual finite ones. Also, allowing restricted infinite symmetric differences gives quite a rich (and natural) cycle space.

We feel that the approach taken by Diestel and Kühn has several elements that make it seem quite restrictive. Although in [8] the proofs are applied in the case no two vertices are joined by infinitely many vertex-disjoint paths, the authors clearly prefer graphs in which no two vertices are joined by infinitely many edge-disjoint paths. This is in part due to their focus on circles. In the case vertices can be joined by infinitely many edge-disjoint paths, but not by infinitely many vertex-disjoint paths, there can be a circle contained in the set of vertices and ends. Furthermore, there can be a path in G that must be admitted as an element of the cycle space. These are counterintuitive. One of the goals of this work is to provide a context in which such events are less disconcerting.

In addition to these esthetic considerations, we are left with the feeling that there may be more general graphs for which one can say, for example, that the fundamental cycles generate the cycle space, or describe the spaces orthogonal to the cycle space or orthogonal to the bond space. While Diestel and Kühn discuss five different possible topologies, all obtained by adding ends to a graph as "points at infinity", and some of which are discarded for various reasons, the approach we take is to consider arbitrary pre-assigned topologies (with certain compatibility requirements), and to see what topological properties are required to make the proofs work. This provides scope for discussing such matters in greater generality.

In this work, we do not undertake to determine which graphs have suitable identifications or other modifications to make them fit our theory. However, we do show that the $\tilde{G}$ of Diestel and Kühn and the Alexandroff (or 1-point) compactification of a locally finite graph are both special cases of our theory. The 1-point compactification provides the connection to Bonnington and Richter, showing their theory is also a special case of our general theory.

As indicated above, we adopt a topologically biased approach to cycle space questions. In Section 2, we introduce the notion of an edge space and in Section 3 describe topological properties of the edge space which are used to prove the cycle space is the space orthogonal to the bond space. Also in this section is the fact that elements of the cycle space partition into cycles. In Section 4, we show that the fundamental cycles of any (generalization of) a spanning tree generate the cycle space. We then show in Section 5 that our results imply the Diestel-Kühn results related to cycle spaces, while we discuss in Section 7 the connection to the Bonnington-Richter cycle space. Finally, we conclude with two short technical sections that indicate how the results of this work can be applied to slightly more general spaces.

Our starting point, from which the theory presented here developed, was trying to reconcile graph-connection with topological connection. If $G = (V, E)$ is a graph, then there is a natural edge space (a general definition will follow shortly) $(V \cup E, E)$ associated with G, whereby the basic open sets are the singletons $\{e\}$, for $e \in E$, and the sets $N(v)$, consisting of a vertex v and all its incident edges. We call this topology the *classical topology*.[1] Note that the classical topology is *Alexandroff discrete*, that is, the arbitrary intersection of open sets is open (see [9]).

It is an easy exercise to show that G is connected in the graph-theoretic sense if and only if the classical topology on $V \cup E$ yields a space connected in the sense of point-set topology. It will turn out that, for the cycle space theory, it is in

[1]Note that this differs from the usual topology of a one-dimensional cell-complex associated with a graph. It is "classical" in the sense that it strictly preserves the graph-theoretical notion of connectedness.

some sense only the edges that really matter. This is perhaps not so surprising, in view of the fact that matroid theory successfully generalizes cycle spaces of finite graphs. Thus, we will generalize the topological context even further to emphasize this fact.

We find it interesting that we can deal directly with topologies on graphs in which the edges are taken to be open singletons. The closure of an edge is just the edge and its incident vertices. In this version, the topological spaces are not Hausdorff, but we will be dealing with natural extensions of this property suitable to the context. We believe that the structural insight gained in this model for cycle space problems outweighs the small price paid in having to rework a few topological theorems in this new setting.

We now describe the two topological concepts that we need for our main theorems. A topological space X is *weakly Hausdorff* if, for any two points x and y of X, there are open sets U_x and U_y, containing x and y, respectively, such that $U_x \cap U_y$ is finite. If G is a possibly infinite simple graph with the classical topology, then G is weakly Hausdorff but, as soon as some two vertices are adjacent, not Hausdorff.

A *hyperedge space* (X, E) is a topological space X and a subset $E \subseteq X$ consisting of points e such that $\{e\}$ is open but not closed. Henceforth we do not always distinguish between a point and the corresponding singleton; in particular we refer to "open" (or "closed") points, and write $X \setminus e$ for $X \setminus \{e\}$. The elements of E are *hyperedges* and any point of $X \setminus E$ is a *vertex*. Notice that a vertex need not be a closed singleton, nor need it be isolated from other vertices.

A hyperedge is an *edge* if it has at most two additional points in its closure. A hyperedge space (X, E) is an *edge space* if every hyperedge is an edge.

One of our main results is the following.

Theorem 1. *Let (X, E) be a connected, compact, weakly Hausdorff edge space. There is a minimal connected subset of X containing $X \setminus E$ and the fundamental cycles of such a subset generate the cycle space.*

At this point, the reader does not really know what the cycle space is or what a fundamental cycle is, but we hope the idea of what we are aiming for is clear. We point out here, and show in Section 5, that these theorems subsume essentially all the cycle space theory developed by Diestel and Kühn (although not that of later works by Diestel and his students, e.g., [2]).

This work is a part of the first author's Ph.D. dissertation written under the supervision of the second author.

2. Edge Spaces

In this section we provide some technical background about edge spaces that we will need in this work. The important point that we prove is Theorem 3, which proves the existence of certain minimal connecting sets. This is used to provide "spanning trees" and "fundamental cycles".

One of the advantages of edge spaces is that we do not need to work only with the classical topology. Indeed, Diestel and Kühn make it clear that there are good reasons for considering other possibilities. In particular, they suggest there are legitimate reasons for thinking an end should be a point treated on a par with the vertices and edges of the graph. A major goal of this work is to show how simple topological considerations subsume the graph-theoretically distasteful problems of circles in the vertices and ends, or of paths in the cycle space mentioned in the introduction, and provide insight into what is really going on.

Many basic facts from elementary topology about Hausdorff spaces have natural analogues for weakly Hausdorff spaces. For one that is of use to us, we need an additional notion. A topological space X is *weakly normal* if, for any two disjoint closed sets C and D in X, there are open sets U_C and U_D in X, containing C and D, respectively, such that $U_C \cap U_D$ is finite. The following fact is proved in the standard way.

Lemma 2. *Let X be a compact, weakly Hausdorff space. Then X is weakly normal.*

The crucial point for us in the context of cycle spaces will be to know there are "spanning trees" and "fundamental cycles". The following theorem, the main result of this section, gives these to us. This is a variation of a standard fact [3, Ch. 3, Ex. G 9].

Theorem 3. *Let X be a connected, compact, weakly Hausdorff space and let $A \subseteq X$. Then there is a minimal closed connected subset C of X such that $A \subseteq C$.*

Proof. In order to apply Zorn's Lemma, we let $\mathcal{C}$ be a chain of closed connected subsets of X containing A, ordered by inclusion. Let $\hat{C} = \bigcap \mathcal{C}$. Evidently $\hat{C}$ is closed and contains A, so it suffices to show $\hat{C}$ is connected.

Suppose to the contrary that $\hat{C}$ has a separation (K_1, K_2). Both K_i are closed in X. Since X is, by Lemma 2, weakly normal, there are open sets U_1 and U_2 containing K_1 and K_2, respectively, such that $|U_1 \cap U_2|$ is finite. Because the K_i are closed, we may assume that $U_1 \cap K_2$ and $U_2 \cap K_1$ are both empty. This implies that $U_1 \cap U_2$ is disjoint from $\hat{C}$. Since $U_1 \cap U_2$ is finite, there is a $C_1 \in \mathcal{C}$ such that $U_1 \cap U_2$ is disjoint from C_1.

Suppose $C_1 \nsubseteq U_1 \cup U_2$. For each $x \in C_1 \setminus (U_1 \cup U_2)$, $x \notin \hat{C}$, so there is a $C_x \in \mathcal{C}$ such that $x \notin C_x$. Since C_x is closed, there is an open set U_x containing x and disjoint from C_x. The open sets U_x, for $x \in C_1 \setminus (U_1 \cup U_2)$, cover the closed and compact set $C_1 \setminus (U_1 \cup U_2)$. Therefore, there is a finite subcover, say $\{U_x \mid x \in J\}$.

For each $x \in J$, U_x is disjoint from C_x. There are only finitely many such x, so that there is a $C_2 \in \mathcal{C}$ disjoint from all the sets U_x, for $x \in J$. This implies that $C_2 \subseteq C_1$ and, furthermore, C_2 is disjoint from $C_1 \setminus (U_1 \cup U_2)$. This implies that $C_2 \subseteq U_1 \cup U_2$.

We conclude that, in every case, there is a $C_2 \subseteq C_1$ such that $C_2 \subseteq U_1 \cup U_2$. It follows immediately that $(U_1 \cap C_2, U_2 \cap C_2)$ is a separation of C_2, contradicting the fact that C_2 is connected. ■

Given edge spaces (X, E), (Y, F), the latter is an *edge subspace* of the former if Y is a (topological) subspace of X, $F = E \cap Y$ and, for all $e \in F$, $\texttt{Cl}(e) \subseteq Y$. Note that, if $C \subseteq X$ is closed, $(C, E \cap C)$ is an edge subspace of (X, E). Hence we have the following corollary.

Corollary 4. *Let A be an edge subspace of a connected, compact, weakly Hausdorff edge space. Then there is a minimal closed, connected edge subspace C of X such that A is an edge subspace of C.*

Corollary 5. *Let (X, E) be a connected, compact, weakly Hausdorff edge space. Then there is a minimal subset E' of E such that $(X \backslash E) \cup E'$ is connected.*

The set $(X \backslash E) \cup E'$ is a generalization of a classical spanning tree; it includes all the non-edges of X and it keeps just enough edges to retain connection. We shall refer to the sets $(X \backslash E) \cup E'$ whose existence is guaranteed by Corollary 5 as *minimal connected spanning sets for* (X, E).

In fact, we can prove Corollary 5 for weakly normal hyperedge spaces that are not compact.

It is natural to wonder if, in the context of graphs plus ends, the minimal connected spanning sets may be taken to be classical spanning trees. Diestel and Kühn point out that this seems to be a difficult problem.

There are three interesting examples related to Theorem 3. Two are based on the Knaster-Kuratowski example of a subset K of the plane $\mathbb{R}^2$ which is connected, but for which there is a point p so that $K \setminus \{p\}$ is totally disconnected. In the first version, Example 4.2.9 in [13], the space is weakly Hausdorff (actually slightly more than that) and has no minimal connected spanning set. This shows (not surprisingly) that some additional hypothesis, such as compactness or weak normality, is required.

In the second example, the edge space is weakly normal and is already its own minimal connected spanning set. However, there are pairs of points for which there is no minimal closed connected set containing the two points of the pair. In the context of trying to find fundamental cycles in a superspace having additional edges, such pairs present difficulties.

The third example is an edge space (X, E) for which $X \backslash E$ consists of three closed points u, v, w. There are denumerably many edges in E, all having $\{u, v\}$ for boundary. The basic open sets containing w consist of w and all but finitely many edges. (They do not contain either u or v.) Such a space is connected and compact, but has no minimal connected spanning set. It is not weakly Hausdorff—any neighbourhood of u will contain at least one point from each of the infinitely many edges, so u, w can not be finitely separated by open sets. Note that we do not need to specify further the subspace topology on $X \setminus \{w\}$.

We shall need the following fact throughout this work.

Lemma 6. *Let H be a compact weakly Hausdorff space. Then, for any two components K and L of H, there is a separation (A, B) of H such that $K \subseteq A$ and $L \subseteq B$.*

Kuratowski [11] (Section 47, Theorems 1 and 2) proves this for compact Hausdorff spaces. His proof applies in our context, but with some small modifications. One relevant point for the modification to weakly Hausdorff spaces is that, if K is a component of H, then the intersection of all the open sets in H containing K is just K. Details are given in [13], Lemma 4.3.2 and Theorem 4.3.3.

3. Cycle Spaces

In this section, we introduce cycles in an edge space, the cycle and bond spaces of an edge space and prove that, for every compact weakly Hausdorff edge space: (1) every element of the cycle space is the edge-disjoint union of cycles; and (2) the space orthogonal to the bond space is the cycle space. The precise meanings of the ingredients of these theorems will be made clear through the rest of this section.

A *cycle* is a connected edge space (X, E) such that, for every $e \in E$, $X \backslash e$ is connected, but for every distinct $e, f \in E$, $X \setminus \{e, f\}$ is not connected.

This surprisingly simple definition is clearly a characterization of classical cycles in graphs. It gives a very combinatorial flavour to the meaning of cycle, and so at least provides some kind of answer to the problem posed by Diestel and Kühn to find a combinatorial meaning for cycles.

The following lemmas may reassure the reader that our cycles have properties comparable to cycles in graphs. Corollary 9 perhaps does so even more. As we do not need the corollary in this work, its proof is omitted.

Lemma 7. *Let (X, E) be a cycle and let $F \subseteq E$ be finite and non-empty. Then $X \setminus F$ has $|F|$ components and there is a cyclic order $(K_1, \cdots, K_{|F|})$ on these components so that, for each $i = 1, 2, \ldots, |F|$, there is an edge of F with an end in each of K_i and K_{i+1} (indices taken modulo $|F|$).*

The proof requires the following, expected, fact. As it is basic to the theory, we provide its proof.

Lemma 8. *Let (X, E) be an edge space, let $e \in E$ and let K be the component of X containing e. Then $K \backslash e$ has at most two components.*

Proof. Suppose (A, B) is a separation of $K \backslash e$ and that A is not connected. Then A has a separation (A', A''). There is one of A', A'', and B that does not contain an end of e. Let L be this one and let M be the union of the other two. Then (L, M) is a separation of $K \backslash e$ so that both ends of e are in M. It follows that $(L, M \cup e)$ is a separation of K, a contradiction. ∎

Proof of Lemma 7. We proceed by induction on $|F|$. The case $|F| = 1$ is trivial; we need separately the case $|F| = 2$.

In this case, let $e, f \in F$. The definition of cycle tells us that $X \backslash e$ is connected and $X \setminus \{e, f\}$ has precisely two components and it is easy to see that each of e and f has an end in each of the components.

If $|F| \geq 3$, delete an edge e from F to get F'. Then $X \setminus F'$ has $m = |F'|$ components and a cyclic order $(K_1, \ldots, K_m)$ of them. For $i = 1, 2, \ldots, m$, there is an $e_i \in F'$ having ends in both K_i and K_{i+1}. We may assume $e \in K_m$.

The definition tells us that $X \setminus \{e, e_m\}$ is disconnected; it has precisely two components L and M. Let L' be the component of $K_m \backslash e$ containing an end of e_{m-1}. Since $Y = (K_1 \cup \cdots \cup K_{m-1}) \cup L' \cup \{e_1, \cdots, e_{m-1}\}$ is a connected subset of $X \setminus \{e, e_m\}$, it must be contained in L, say. Thus, $M \subseteq K_m$.

It follows that $K_m \backslash e$ has components L' and M. We know that e_m has an end in M and an end in K_1. It follows that $(K_1, \cdots, K_{m-1}, L', M)$ is a cyclic sequence of the $|F|$ components of $X \setminus F$, such that consecutive components are joined by an edge of F. ∎

Corollary 9. *Let (X, E) be a connected edge space with $|E| \geq 4$. The following are equivalent:*

1. *(X, E) is a cycle; and*
2. *there is a cyclic order on E such that, if (w, x, y, z) is a cyclic subsequence, then x and z separate w and y.*

Lemma 10. *Let (X, E) be a connected compact weakly Hausdorff edge space. Then for every edge $e \in E$ either $X \setminus e$ is disconnected or else there exists an edge-cycle of X containing e, but not both.*

Proof. Given any edge e, the subset $X \setminus e$ can only be disconnected if e has two ends, which are in distinct components after deleting e. Since an edge-cycle containing e would arise from a cycle C containing both ends of e, and since $C \setminus e$ would be a connected subset of $X \setminus e$, the two assertions cannot hold simultaneously.

If e has only one end, then $\mathtt{Cl}(e)$ is the required cycle. Otherwise, let u, v be the ends of e, and assume that $X \setminus e$ is connected. Then, by Corollary 4 it contains a minimal closed connected edge subspace P containing $\{u, v\}$. We claim that, if $f \in E \cap P$, then u and v are in different components of $P \setminus f$. Otherwise, let K be the component of $P \setminus f$ containing u and v. Since f is open, $P \setminus f$ is closed in X, so K is closed in X, contradicting minimality. It follows that $P \cup e$ is the required cycle. ∎

We are now prepared to introduce the cycle space. Let (X, E) be an edge space. An *edge-cycle* in (X, E) is a subset E' of E for which there is an edge subspace (Y, F) of (X, E) that is a cycle. (Recall this implies $F = Y \cap E$, and for every $e \in F$, $\mathtt{Cl}(e) \subseteq Y$.)

A family $\mathcal{E}$ of subsets of a set E is *thin* if each element of E occurs in only finitely many elements of $\mathcal{E}$. The point is that if $\mathcal{E}$ is thin, then the symmetric difference D of the sets in $\mathcal{E}$ is well-defined: an element e of E is in D if and only if e is in an odd number of elements of $\mathcal{E}$.

We have the following sets of subsets of E generated by a family $\mathcal{E}$ of subsets of E (thin or not):

(1) the *weak span* $\mathcal{W}(\mathcal{E})$ of $\mathcal{E}$ is the set of all symmetric differences of finitely many elements of $\mathcal{E}$;
(2) the *algebraic span* $\mathcal{A}(\mathcal{E})$ of $\mathcal{E}$ is the set of all symmetric differences of thin subsets of $\mathcal{E}$; and
(3) the *strong span* $\mathcal{S}(\mathcal{E})$ is the smallest subset of the power set of E that contains $\mathcal{E}$ and is closed under symmetric differences of thin families.

Obviously we have $\mathcal{W}(\mathcal{E}) \subseteq \mathcal{A}(\mathcal{E}) \subseteq \mathcal{S}(\mathcal{E})$. The *cycle space* $\mathcal{Z} = \mathcal{Z}(X, E)$ of an edge space (X, E) is the strong span of the edge cycles of (X, E). We shall see in Corollary 15 that, when X is compact and weakly Hausdorff, the cycle space is also the algebraic span of the edge cycles of (X, E); this is not obvious and is an important part of the statement of Theorem 14. (We note that Diestel and Kühn define the cycle space as the algebraic span of the cycles. Corollary 15 shows the two definitions are equivalent. Their version of this corollary is that the cycle space is closed under sums of thin families of elements of the cycle space.)

It is interesting to consider Figure 5 from [8]. This graph has a set $\mathcal{P}$ of infinitely many edge-disjoint paths joining the vertices x (at the extreme left) and y (at the extreme right). No two vertices are joined by infinitely many internally-disjoint paths. We may take one member of $\mathcal{P}$ to be the path P of length 3 from x to y.

If $Q, R \in \mathcal{P}$, then $Q \cup R$ is the edge-disjoint union of finitely many finite cycles in G and, therefore, $E(Q \cup R)$ is in the cycle space. By letting $P = P_1, P_2, \dots$ be an infinite sequence of distinct paths in $\mathcal{P}$, we see that $E(P)$ is the symmetric difference of $E(P_1 \cup P_2)$, $E(P_2 \cup P_3)$, $E(P_3 \cup P_4)$, $\dots$. Thus $E(P)$ is in the cycle space.

However, this is not the only reason $E(P)$ is in the cycle space. In fact, it is a cycle of $\tilde{G}$! It is not obvious, but the set consisting of the ends and the vertices of infinite degree contains a set S that is homeomorphic to [0,1], with x and y corresponding to 0 and 1. Thus $P \cup S$ has the property that deleting any one edge will not disconnect this set, but deleting any two edges will. The ends are "hiding" the fact that the connection in the space is much richer than provided for by just considering the edges. This is where the "extra" cycles come from – even extra finite cycles.

It seems that in order to study cycles, we must also study edge cuts. Let A and B partition $X \backslash E$ into two closed sets. Then the set $\delta(A)$ of edges having one end in A and one end in B is an *edge cut*. It is not obvious that, when X is connected, edgecuts are cutsets; in fact, the third example in Section 2 (which is not weakly Hausdorff) provides a counterexample to this—namely the edge cut $\delta(w)$. We shall show that edge cuts are cutsets in Section 7 (Corollary 39), at a level of generality which extends that of compact weakly Hausdorff edge spaces.

A basic proposition, that is not trivial in this context, is the following.

Lemma 11. *Let (X, E) be a compact, weakly Hausdorff edge space. Let Z be an edge cycle of (X, E) and let B be an edge cut of (X, E). Then $|Z \cap B|$ is finite and even.*

We need the following crucial fact.

Theorem 12. *Every edge cut in a compact weakly Hausdorff edge space is finite.*

Proof. Let (X, E) be the given edge space and let A, B be a partition of $X \backslash E$ into two closed sets. Since X is compact, Lemma 2 implies X is weakly normal and, therefore, there are open sets U_A and U_B in X, containing A and B, respectively, such that $U_A \cap U_B$ is finite. But $\delta(A) \subseteq U_A \cap U_B$, so $\delta(A)$ is finite. ∎

Proof of Lemma 11. By Theorem 12, B is finite, so $Z \cap B$ is finite. Let (Y, Z) be a cycle. By Lemma 7, $Y \setminus (Z \cap B)$ has finitely many components, joined cyclically by the edges of $Z \cap B$. Let (A, A') be a partition of $X \backslash E$ such that $B = \delta(A)$.

The set $\mathcal{K}$ of components of $Y \setminus B$ is finite. Every edge of $Z \cap B$ has one end in A and one end in A'. Applying Lemma 7, we see that the components in $\mathcal{K}$ alternate being in A and being in A' and, therefore, $|\mathcal{K}|$ is even, as required. ∎

It is interesting to compare this with the classical case, for infinite graphs. In that case, a cycle z is necessarily finite, while the cuts b may be infinite. Of course $|z \cap b|$ is even. In our context, it is the cycles that may be infinite, while the cuts are finite and we also have $|z \cap b|$ even.

The *bond space* $\mathcal{B} = \mathcal{B}(X, E)$ of an edge space (X, E) is the set of all cuts. It is very easy to see that, given separations $(A_1, B_1), (A_2, B_2)$ of $X \backslash E$ (so A_i, B_i are complementary and both simultaneously open and closed in $X \backslash E$, for $i = 1, 2$), we have that $\delta(A_1 \triangle A_2) = \delta(A_1) \triangle \delta(A_2)$, where $S \triangle T$ is the symmetric difference of S and T. It follows that the bond space is a vector space over $\mathbb{Z}_2$.

Furthermore, Theorem 12 implies that $\mathcal{B}$ is the weak span of the inclusion-wise minimal cuts, which are often called *bonds*, whence the name "bond space" for $\mathcal{B}$.

For any set A of subsets of a set B, the set $A^\perp$ is defined to be the set of subsets C of B such that, for every $C' \in A$, $|C \cap C'|$ is even. Throughout this article, whenever we consider $A^\perp$ for a set A whose elements are sets of edges of an edge space, B will tacitly be assumed to be the set of all edges (and nothing more).

Corollary 13. *$\mathcal{Z} \subseteq \mathcal{B}^\perp$ and $\mathcal{B} \subseteq \mathcal{Z}^\perp$.*

Proof. The two claims are trivially equivalent. To prove the first, it suffices to show that every edge cycle is in $\mathcal{B}^\perp$ and that $\mathcal{B}^\perp$ is closed under sums of thin families. The former is the content of Lemma 11. For the latter, suppose $A \subseteq \mathcal{B}^\perp$ is a thin family and that $b \in \mathcal{B}$. The set A is thin and, by Theorem 12, b is finite, so the set $A' = \{a \in A \mid a \cap b \neq \emptyset\}$ is finite. Clearly $b \cap (\triangle A') = b \cap (\triangle A)$.

Since A' is finite, $|b\cap(\triangle A')| \equiv \sum_{a\in A'} |b\cap a| \pmod 2$. Since every $|b\cap a|$ is even, $|b\cap(\triangle A)|$ is even, as required. ∎

We are now ready to prove our first main result.

Theorem 14. *Let (X,E) be a compact, weakly Hausdorff edge space. Then:*

(1) *$\mathcal{B}^\perp = \mathcal{Z}$; and*
(2) *every element of the cycle space is the disjoint union of edge cycles.*

Proof. For both of these, it suffices to prove that every element of $\mathcal{B}^\perp$ is the disjoint union of edge-cycles. From this (1) follows, since it obviously implies $\mathcal{B}^\perp \subseteq \mathcal{Z}$ and Corollary 13 is the reverse inclusion. Conclusion (2) is now immediate from the two preceding sentences.

So let $F \in \mathcal{B}^\perp$ and let $\mathcal{P}$ denote the set of all sets of pairwise disjoint edge cycles contained in F. That is, an element of $\mathcal{P}$ is a set of pairwise disjoint edge cycles, each contained in F. The union of any chain of elements of $\mathcal{P}$ is trivially again an element of $\mathcal{P}$, so Zorn's Lemma implies $\mathcal{P}$ has a maximal element P.

Let $F' = \cup_{C\in P} C$. Corollary 13 implies that $F' \in \mathcal{B}^\perp$. Hence $F'\triangle F = F\setminus F' \in \mathcal{B}^\perp$.

Now suppose, by way of contradiction, that $F'\triangle F$ has an edge e, and let $X' = (X\backslash E)\cup(F'\triangle F)$. Let K be the component of X' containing e. If $K\backslash e$ is connected, then by Lemma 10 e is in an edge cycle C of X', which is necessarily contained in $F'\triangle F$. Since $F'\triangle F = F\setminus F'$, $P\cup\{C\}$ is a set of pairwise disjoint edge cycles, all contained in F, contradicting the maximality of P.

Thus, we may assume that $K\backslash e$ is not connected. We claim that $\{e\}$ is an edge-cut of X'. To see this, let $W = X'\backslash e$. Then by Lemma 8 K has turned into two components K_1 and K_2 of W. From Lemma 6 there is a separation (A,B) of W such that $K_1 \subseteq A$ and $K_2 \subseteq B$. Since $W = X'\backslash e$, $\{e\}$ is an edge-cut of X', as required.

Since W contains all of $X\backslash E$, (A,B) gives an edge cut of X, which is finite by Theorem 12. Only finitely many edge cycles in P can meet $\delta(A)$, and each one does so in an even number of edges. Thus, $F'\cap\delta(A)$ is even. On the other hand, F has only one more edge in $\delta(A)$, namely e, so $F\cap\delta(A)$ is odd, contradicting the assumption that F is orthogonal to every cut. ∎

Corollary 15. *Let (X,E) be a connected, compact, weakly Hausdorff edge space. Then the cycle space is the algebraic span of the cycles.*

4. Fundamental Cycles

In this section, we shall show that the fundamental cycles of a connected, compact, weakly Hausdorff edge space (X,E) algebraically generate the cycle space. Let T be a minimal connected spanning set for X, as guaranteed by Corollary 5, and let $e^*\in E\setminus E'$ have incident points u and v. Then $u,v\in T$ and $T\cup e^*$ is a closed subset of X, whence $Y := (T\cup e^*, E'\cup e^*)$ is another connected, compact, weakly Hausdorff edge space, where $E' = E\cap T$. Since T is connected, Lemma 10 implies that there exists an edge-cycle $F \subseteq E$ of Y (and therefore of (X, E)) containing e^*.

An easy argument similar to the one in the proof of Lemma 10 shows that, for any edge $e \in T \cap F$, the subset $T \backslash e$ is disconnected and u and v are in different components of $T \backslash e$. Therefore, the set $F \setminus e^*$ is precisely the set of edges of T that separate u and v (in T). Hence the edge-cycle F is uniquely determined by e^*.[2] We refer to F as the *fundamental cycle* of e^* with respect to T, and denote it by C_{e^*}.

Our next goal is to show that the set of fundamental cycles is thin. This is done by considering the fundamental bonds: for each edge $e \subseteq T$, $T \backslash e$ has precisely two components, which induce an edge cut B_e containing e. (In fact this edge cut is minimal, i.e., is a bond.) By Theorem 12, $|B_e|$ is finite.

The following is very simple and generalizes classical results.

Lemma 16. *Let (X, E) be a connected, compact, weakly Hausdorff edge space. Let T be a minimal connected spanning set for (X, E), e an edge in T and f an edge not in T. Then $e \in C_f$ if and only if $f \in B_e$.*

Theorem 17. *Let (X, E) be a connected, compact, weakly Hausdorff edge space. For any minimal connected spanning set T, the set of fundamental cycles is thin.*

Proof. If the edge e is not in T, then e is only in C_e. If $e \in T$ is in C_f, then $f \in B_e$ and, since $|B_e|$ is finite, there are only finitely many such C_f containing e, as required. ∎

And now we have one of our main theorems.

Theorem 18. *Let (X, E) be a connected, compact, weakly Hausdorff edge space. Let T be any minimal connected spanning set. If $z \in \mathcal{Z}(X, E)$, then*

$$z = \triangle_{e \in z \backslash T} C_e\,.$$

That is, the fundamental cycles algebraically generate the cycle space.

Proof. By Theorem 17, the set $z' = \triangle_{e \in z \backslash T} C_e$ is defined and is in the cycle space. Since the cycle space is closed under symmetric difference, $z \triangle z'$ is in the cycle space. By definition, if e is not in T, then $e \notin z \triangle z'$, so $z \triangle z' \subseteq T$. By Theorem 14 (2), if $z \triangle z' \neq \emptyset$, there is an edge cycle $C \subseteq z \triangle z'$. Thus, there is a connected edge-closed subset Y of X such that the edges in Y are precisely those of C and deleting any element of C does not disconnect Y. But $Y \subseteq T$ and deleting any edge of T disconnects T, a contradiction. ∎

We conclude this section with the following result, for which no analogue appears in the Diestel-Kühn works. A connected edge space (X, E) is *2-edge-connected* if, for every $e \in E$, $X \backslash e$ is connected.

[2] Although the same can not be said of the cycle that determines F, i.e., the subset C such that (C, F) is a cycle and an edge-subspace of Y.

Theorem 19. *Let (X, E) be a 2-edge-connected, compact, weakly Hausdorff edge space. Then $\mathcal{Z}^\perp = \mathcal{B}$.*

Proof. We have from Corollary 13 that $\mathcal{B} \subseteq \mathcal{Z}^\perp$. Let T be a minimal connected spanning set for (X, E) and let $\mathcal{F}$ be the set of fundamental cycles with respect to T.

Let $A \in \mathcal{Z}^\perp$. We claim that A is finite. By way of contradiction, suppose not. Let G be the simple bipartite graph with bipartition $(A, \mathcal{F})$ and an edge between $a \in A$ and $F \in \mathcal{F}$ if a is an edge of F.

Notice that every $F \in \mathcal{F}$ has finite even degree in G, since $A \in \mathcal{Z}^\perp$. Since $\mathcal{F}$ is a thin family, for every $e \in E$, and so, in particular, for every $e \in A$, e is in finitely many members of $\mathcal{F}$. Thus, every vertex in A has finite degree in G.

Since (X, E) is 2-edge-connected, by Lemma 10 every $e \in E$ is in some edge cycle and, therefore, in some element of $\mathcal{F}$. By assumption, A is infinite, so the preceding sentence and paragraph imply $\mathcal{F}$ is infinite. It follows easily that there is an infinite subset $\mathcal{F}'$ of $\mathcal{F}$ such that, if F and F' are distinct elements of $\mathcal{F}'$, $F \cap A$ and $F' \cap A$ are disjoint and not empty.

Since $\mathcal{F}$ is thin, so is $\mathcal{F}'$. Thus, the set $z = \triangle_{F \in \mathcal{F}'} F$ is in $\mathcal{Z}$. The definition implies $A \cap z$ is even; in particular it is finite. But $\mathcal{F}'$ was selected so that no two elements of $\mathcal{F}'$ have any elements of A in common. So z contains all the elements of A contained in some member of $\mathcal{F}'$. Since every element of $\mathcal{F}'$ has an element of A and there are infinitely many elements of $\mathcal{F}'$, z has infinitely many elements of A, a contradiction.

So A is finite. Let H be the graph whose vertices are the finitely many components of $X \setminus A$ and whose edges are the edges in A, having as ends the components containing their ends in X. It is easy to use Lemma 9 and Theorem 3 to transform graphical cycles (i.e., classical cycles in finite graphs) in H to edge cycles in (X, E). If the cycle in H has k edges, then the corresponding edge cycle D in (X, E) has $|D \cap A| = k$. Since $A \in \mathcal{Z}^\perp$, we conclude that k must be even. Thus, H has only even cycles, i.e., H is bipartite.

Let B be the union of components of $X \setminus A$ on one side of the bipartition of the vertices of H and let C be the union of the remaining components of $X \setminus A$. Then $(B \setminus E, C \setminus E)$ is a partition of $X \setminus E$ into two closed sets, and $A = \delta(B \setminus E)$, as required. ∎

An interesting example to ponder is the 1-way infinite path P, with its end point. This space is connected, compact and weakly Hausdorff. However, $\mathcal{Z} = \{\emptyset\}$, $\mathcal{B}$ is the set of finite sets of edges, while $\mathcal{Z}^\perp$ is the set of all subsets of edges. The following corollary shows that having infinitely many cut edges is the only time that $\mathcal{Z}^\perp$ is not $\mathcal{B}$.

Corollary 20. *Let (X, E) be a connected, compact, weakly Hausdorff edge space. Let $E' = \{e \in E \mid X \setminus e$ is disconnected$\}$. Then $\mathcal{Z}^\perp = \mathcal{B}(X, E \setminus E') \oplus 2^{E'}$.*

Here, the $\oplus$ means that every set in $\mathcal{Z}^\perp$ is the disjoint union of a set in $\mathcal{B}(X, E \setminus E')$ and a set in $2^{E'}$, which is the set of all subsets of E'. The proof is interesting in that it uses the freedom we have to declare which edges we consider. In this case, it is the cut-edges we ignore. The reader should realize that this is contraction of edges. A feature of our model, however, is that we do not have to change the space, just the set of "recognized" edges.

Proof. If $z \in \mathcal{Z}$, then $z \cap E' = \emptyset$, so every subset of E' is in $\mathcal{Z}^\perp$ (and may be freely added to any other set in $\mathcal{Z}^\perp$).

Furthermore, $\mathcal{Z}(X, E) = \mathcal{Z}(X, E \setminus E')$. Since $(X, E \setminus E')$ is 2-edge-connected, Theorem 19 implies $[\mathcal{Z}(X, E \setminus E')]^\perp = \mathcal{B}(X, E \setminus E')$. Putting these two paragraphs together, the result follows. ∎

5. APPLICATION I: Subsuming Diestel and Kühn

The purpose of this section is to indicate how this work subsumes the work on cycle spaces of Diestel and Kühn. This is done by showing that the topological spaces they consider are, from our point of view, compact and weakly Hausdorff. The compactness has been shown independently by Diestel [4]; in any case we shall not provide all the details.

Of course, for $\tilde{G}$, their results imply ours by the expedient of contracting to a point each of the open intervals representing the edges.

A graph G is *finitely connected* if, for every pair of vertices u and v of G, there is no set of infinitely many pairwise internally disjoint uv-paths in G. Analogously we may define *finitely edge-connected* by forbidding infinitely many edge-disjoint uv-paths.

Recall that an *end* is an equivalence class of rays, two rays being equivalent if the deletion of any finite set of vertices leaves the tails of the two rays in the same component. A vertex u *dominates* an end ω if, for some (and hence every) ray $R \in \omega$, there are infinitely many paths joining u to R, totally disjoint except for their common end u.

Given an infinite graph G, the space $\bar{G}$ is obtained by adding a new point for each end. We give $\bar{G}$ a topology by using our classical topology on the points of G, namely each edge is an open singleton and, for each vertex u, the set consisting of u and its incident edges is open. A basic neighbourhood of an end ω is obtained by deleting a finite set S from G and taking the component containing all the tails of the rays in ω, including also *all* the end points themselves (not just ω) for rays whose tails are in this component. We remark that, we may as well assume S to consist entirely of vertices, and in this case the basic neighbourhood includes any edges joining the component to S (but not any vertices in S). In the case that G is locally-finite, it is well-known that $\bar{G}$ is compact; in this case it is called the *Freudenthal compactification of* G.

If G is finitely connected, then no two vertices can dominate the same end. In this case, which obviously includes the locally-finite case, we produce the space $\hat{G}$ from $\bar{G}$ by identifying with a vertex u all the ends which are dominated by u. This space is effectively the space $\tilde{G}$ of [8], but we have replaced the arcs representing edges by singletons.

The two main results in this section are the following theorems, which show that our theory applies to the Diestel-Kühn spaces.

Theorem 21. *If G is finitely connected and 2-connected, then $\hat{G}$ is compact.*

Theorem 22. *If G is finitely connected, then $\hat{G}$ is weakly Hausdorff.*

We prove the second of these first.

Proof of Theorem 22. Let $p : \bar{G} \to \hat{G}$ be the identification map. There are four kinds of points in $\hat{G}$: edges, vertices not identified with an end, an end not identified with a vertex, and a vertex which has been identified with one or more ends. Any edge e is open, so if x is any other point of $\hat{G}$, then we may choose $\{e\}$ and $\hat{G}$ as the open sets containing e and x, respectively.

Let u and v be points of $\bar{G}$ that are not edges and such that $p(u) \neq p(v)$. We claim that there is a finite set F of vertices and edges so that u and v are in different components of $\bar{G} \setminus F$. This is essentially the definition if u and v are both ends, and it is finite-connection in the case u and v are nonadjacent vertices. (In these cases we do not need to delete any edges.) If u and v are adjacent, then they have only finitely many edges joining them. Clearly these edges and a finite set of vertices suffice to separate u and v.

If u is a vertex and v is an end, then, for any ray R in v, there are not infinitely many (u, R)-paths that are pairwise disjoint except for their common end u. Thus, there is a finite set F of vertices that separate u and the tail of R in G. That is, F separates u and v in $\bar{G}$.

Let K_u be the component of $\bar{G} \setminus F$ containing u and let U be the open set obtained by adding to K_u the edges that are in F, plus the edges from the vertices in F to $V(U)$ in U and similarly let V be the open set obtained from the component of $\bar{G} \setminus F$ containing v. Evidently, U and V are open in $\bar{G}$ and, any point identified with a point of U is already in U and similarly for V. Thus, $p(U)$ and $p(V)$ are open in $\hat{G}$. Since they have at most the finitely many edges in F in common in $\hat{G}$, we are done. ∎

Compactness is somewhat more difficult. The first few of the following lemmas are quite straightforward and probably well-known, so we omit their proofs.

Lemma 23. *Let G be a connected graph and let U be an infinite set of vertices in G. Then either there is a ray R and infinitely many totally disjoint (U, R)-paths or there is a vertex u of G such that there are infinitely many (u, U)-paths, disjoint except for their common end u.*

Lemma 24. *Let G be a finitely-connected, 2-connected graph and let U be an infinite set of vertices of G. Then there is a ray R and infinitely many totally disjoint (U, R)-paths.*

Lemma 25. *Let G be a connected graph and let A be any subset of $V(G)$. Then there is a minimal tree in G containing A.*

Lemma 26. *If G is a connected graph so that every vertex has countable degree, then G is countable.*

Lemma 27. *Let G be a finitely-connected, 2-connected graph. Then G is countable.*

Now we move in to the realm of $\hat{G}$.

Lemma 28. *Let G be a finitely-connected, 2-connected graph. Then every vertex of infinite degree dominates some end of G.*

Proof: Suppose v has infinite degree in G and let U be the set of neighbours of u in G. Lemma 24 implies there is a ray R and infinitely many totally disjoint (U, R)-paths. Evidently, v dominates R. ∎

Lemma 29. *Let G be a finitely-connected, 2-connected graph and let A be a finite set of vertices of G. Then $G - A$ has only finitely many components.*

Proof. Otherwise, there exists an infinite set U of vertices, no two of which belong to the same component of $G - A$. By Lemma 24 there is a ray R and infinitely many totally disjoint (U, R)-paths, which is clearly impossible. ∎

We now are ready for the compactness of $\hat{G}$.

Proof of Theorem 21. Because the quotient map $p : \bar{G} \to \hat{G}$ is continuous, it suffices to show $\bar{G}$ is compact. To this end, let $\mathcal{U}$ be any open cover of $\bar{G}$. If $V(G)$ is finite, then the result is trivial, so we assume $V(G)$ is infinite.

By Lemma 27, $V(G)$ is countable. Let $v_1, v_2, \dots$ be any enumeration of $V(G)$ and, for every integer $i \geq 1$, let $V_i = \{v_1, \dots, v_i\}$.

For each i, Lemma 29 shows there are finitely many components of $G - V_i$. Consider the set $\mathcal{C}_i$ of those components C of $G - V_i$ such that no element U of $\mathcal{U}$ is such that $C \subseteq U$. We set $\mathcal{C}_0 = \{G\}$ and let $\mathcal{C}$ denote the union $\bigcup_{i \geq 0} \mathcal{C}_i$.

For each $C \in \mathcal{C}$, if $C \neq G$, let $j = j(C)$ denote the largest integer such that C is properly contained in a component K of $G - V_j$. Evidently, $K \in \mathcal{C}$ (since any set in $\mathcal{U}$ containing K contains C). Trivially, if $j(C) > 0$, then $j(K) < j(C)$. We can define a tree T to have as vertices the elements of $\mathcal{C}$ and edges CK, where K is related to C as in the preceding sentences.

We claim there is an i such that $\mathcal{C}_i = \emptyset$. The alternative is that, for every $i \geq 0$, there is some $C_i \in \mathcal{C}_i$. Clearly T is infinite and Lemma 29 implies T is locally-finite. It follows from König's Infinity Lemma [5, Lemma 9.1.3], which is well-known and easy to prove, that T has a ray starting at G; this gives a nested sequence $K_1 \supset K_2 \supset \cdots$ of distinct components from $\mathcal{C}$ and vertices $w_i \in V(K_i) \setminus V(K_{i+1})$. Let $W = \{w_1, w_2, \dots\}$.

Lemma 24 implies there is a ray R and infinitely many totally disjoint (W, R)-paths. Let ω be the end containing R. Let $U \in \mathcal{U}$ contain ω. Then U is an open set in $\bar{G}$ containing ω, so there is a finite set S of vertices of G for which the component C of $G - S$ containing the tail of R is such that $C \subseteq U$.

Clearly, $W \cap V(C)$ is infinite. But there is an i such that $S \subseteq V_i$, while, for some j, $K_j \subseteq C$. But this implies $K_j \subseteq C \subseteq U$, a contradiction.

Therefore, there is some i such that $\mathcal{C}_i = \emptyset$. For such an i, $G - V_i$ has finitely many components, the closures of which are contained in finitely many sets from $\mathcal{U}$. These sets from $\mathcal{U}$, plus the finitely many more needed to cover the vertices of V_i, provide the required finite subcover of $\bar{G}$. ∎

The reduction of the Diestel-Kühn theory to ours is completed by showing that the two cycle spaces coincide. One simple way to see this is as follows. If T is a pretree of $\tilde{G}$ (Diestel and Kühn show one always exists), then there is a minimal connected spanning set $\hat{T}$ of $\hat{G}$ with the same edges as T. It is easy to see that the fundamental cycles are the same in the two cases. Since these fundamental cycles generate the two cycle spaces algebraically in the same way, the two cycle spaces coincide.

More interestingly, another way to show the spaces are the same is to recognize the connection between circuits in $\tilde{G}$ and edgecycles in $\hat{G}$, given in the next lemma. This lemma and the combinatorial flavor of the definition of cycle at least in part answers the question posed by Diestel and Kühn to give a combinatorial description of a circle [7].

Lemma 30. *A non-empty set of edges in $\tilde{G}$ is a circuit if and only if it is the set of edges of a* closed *cycle in $\hat{G}$.*

Proof: If C is a circuit in $\tilde{G}$, then it is the set of edges contained in a circle C', which obviously satisfies the definition of cycle given here, i.e., the deletion of any edge does not disconnect C', while the deletion of any two edges does disconnect C'. Thus C' is an edge cycle. Moreover, since C' is compact and $\tilde{G}$ is Hausdorff (this will be shown in Lemma 33), C' is closed in $\tilde{G}$, and therefore in $\hat{G}$.

Conversely, suppose C' is a closed cycle in $\hat{G}$ with edge set C. The only way that $\hat{G}$ and $\tilde{G}$ differ is that edges are singletons in the former and intervals in the latter. So replacing the singleton edges of C' with intervals produces a closed subset $\tilde{C}'$ of $\tilde{G}$. The deletion of any edge e from $\tilde{C}'$ does not disconnect. Diestel and Kühn prove the non-trivial fact that, since $\tilde{C}' \setminus \{e\}$ is closed and connected, it is arcwise connected and, therefore, contains an arc A joining the two ends of e. This arc, together with the interval e yields a circle. Since the deletion of any other edge f of C disconnects $\tilde{C}' \setminus \{e\}$, f is contained in the arc A, so that $A \cup \{e\}$ contains precisely the edges C. ∎

Note that our definition of an edge cycle does not require the corresponding cycle to be closed, but also our whole theory carries through if we impose this extra condition; in particular, the fundamental (edge) cycles we obtain (Lemma 10) happen to arise from closed cycles. At the level of generality of compact Hausdorff spaces, there certainly can exist edgecycles which are not circuits, but we do not know if this can occur in the specific case of $\tilde{G}$.

6. Aside About $\tilde{G}$

In trying to show that every cycle in $\hat{G}$ corresponds to a circle in $\tilde{G}$, we were naturally led to consider properties of $\tilde{G}$ related, in particular, to arcwise connection. We were able to prove the following.

Theorem 31. *If G is a 2-connected, finitely separated graph, then $\tilde{G}$ is a Peano space.*

We remind the reader that a Peano space is a compact, connected, locally connected, metric space. We are fortunate here to have already proved that $\tilde{G}$ is compact (Theorem 21 or [4]). It is obviously connected and, being a quotient of the locally connected space $|G|$, it is locally connected. In order to show it is metric, it suffices to show it is Hausdorff and second countable (since a compact space is metric if and only if it is Hausdorff and second countable – see for example [10], Ch. 4, Thm. 16). We remind the reader that a Peano space is arcwise connected.

Recall that, for a partition $\mathcal{P}$ of a topological space X:

(1) a set $Y \subseteq X$ is *saturated* if Y is the union of sets in $\mathcal{P}$; and
(2) $\mathcal{P}$ is *upper semi-continuous* if, for every part P and every open set U of X for which $P \subseteq U$, then there is a saturated open set V such that $P \subseteq V \subseteq U$.

Kelley provides a useful result, that says that if $\mathcal{P}$ is a partition of a second countable space X so that the members of $\mathcal{P}$ are compact, then the quotient space with pointset $\mathcal{P}$ is second countable [10], Ch. 5, Thm. 20.

For us, this means only that we must show that the sets in $|G|$ consisting of a vertex and its dominated ends are closed (since closed subsets of the compact set $|G|$ are compact), that the partition into these closed sets is upper semi-continuous, and that $\tilde{G}$ is Hausdorff. The first and third are both done in [8], the former being the easy Lemma 4.1 in [8]. The latter is Theorem 4.7 in [8]; we shall give a much shorter proof here in Lemma 33.

For the remainder of this section, $\mathcal{P}$ is the partition of $|G|$ in which each part is either a vertex and all ends it dominates, singletons from edges, and singletons consisting of undominated ends.

Lemma 32. *Let G be a finitely-separated graph and let S be a finite subset of $V(G)$. For each $s \in S$, let E_s be the part in $\mathcal{P}$ containing s. Let K be any component of $|G| \setminus S$. Then $K \setminus (\cup_{s\in S} E_s)$ is open and saturated in $|G|$.*

Proof. We have that K is open by definition. We know from above that each E_s is closed. Since S is finite, we see that $K \setminus (\cup_{s\in S} E_s)$ is open.

To see that $K \setminus (\cup_{s\in S} E_s)$ is saturated, let E be any part such that $E \cap K \neq \emptyset$. If E is degenerate, then it is trivial that $E \subseteq K \setminus (\cup_{s\in S} E_s)$. Otherwise, let v be the unique vertex in E. If $v \notin S$, then v and any end it dominates must be in the same component of $G \setminus S$. Hence $E \subseteq K \setminus (\cup_{s\in S} E_s)$. If $v \in S$, then $E = E_v$ and $E \cap \{K \setminus (\cup_{s\in S} E_s)\} = \emptyset$. ∎

The next fact we need is that $\tilde{G}$ is Hausdorff.

Lemma 33. *Let G be a finitely-separated graph. Then $\tilde{G}$ is Hausdorff.*

Proof. Let E and F be distinct parts in $|G|$. If E is just a point in the interior of an edge, then E has a basic neighbourhood whose closure is disjoint from F. The crucial point that remains is that if $x, y \in V(G) \cup \Omega$, then there is a finite set S of vertices such that x and y are in different (topological) components of $|G| \setminus S$ if and only if x and y are in different parts. If x and y are in different parts, then S exists (chosen for x and y vertex-representatives of their equivalence classes, if possible). It is then easy to see that S is disjoint from these two equivalence classes, so Lemma 32 provides the desired open sets. ∎

Lemma 34. *Let G be a finitely-separated graph. Then $\mathcal{P}$ is an upper semicontinuous decomposition.*

Proof. Let U be open in $|G|$ and let E be a part contained in U. If E is a point in the interior of an edge, that there is a saturated open set V containing E and contained in U is trivial. If E is just a vertex, then some basic neighbourhood of v is contained in U and all such neighbourhoods are saturated.

So we can assume E contains an end ω. Since U is open, there is some finite set S of vertices such that the component K of $|G| \setminus S$ containing ω is contained in U. If ω is not dominated by any vertex in S, then Lemma 32 gives the desired saturated open subset of U. If ω is dominated by a vertex $v \in S$, then U contains a basic neighbourhood N of v. In this case, $N \cup (K \setminus (\cup_{s \in S \setminus \{v\}} E_s))$ is open, saturated, and contained in U. ∎

It would be nice to prove that $\tilde{G}$ is hereditarily locally connected; this would imply that every closed connected subset of $\tilde{G}$ is arcwise connected, which is so far a difficult result [8], Thm. 5.3.

7. APPLICATION II: Bonnington-Richter Cycle Space

As another application of our theory, let G be a locally-finite graph. Notice that the closed and compact sets (in the classical topology defined in the introduction) are precisely the finite subgraphs of G. Thus, the Alexandroff (or 1-point) compactification gives, for the single new point ∞, the basic open neighbourhoods being the complements of such finite sets (including ∞ in the complement, of course).

It is easy to see that the cycles in this case are precisely the finite cycles and the 2-way infinite paths (with ∞). Thus, the cycle space is exactly the set of edge-sets of those subgraphs in which every vertex has even degree, giving a very natural extension to infinite graphs of the finite case. This is the cycle space introduced by Bonnington and Richter [1], particularly for planar graphs. This is somewhat larger than the cycle space of the Freudenthal compactification of G.

The bond space of the Alexandroff compactification is the set of cuts $\delta(A)$ for finite subsets A of $V(G)$. This is correspondingly smaller than the bond space of the Freudenthal compactification.

In this section, we only make the point that our theory, developed in Sections 2-4, applies equally well to the Alexandroff compactification. Thus, there is a minimal connected spanning set; for any such set, its fundamental cycles algebraically generate the cycle space; the cycle space is orthogonal to the bond space; and, if the compactification is 2-edge-connected, the bond space is orthogonal to the cycle space.

We hope in a later work to discuss the relations between the cycle spaces of different compactifications of the same locally-finite graph.

8. More general edges

In the literature on continuum theory, when X is a locally connected continuum (compact, connected Hausdorff space), a closed subset V of X is referred to as a T-set if the connected components (which are open because X is locally connected) of its complement each have precisely two points in their boundary (see [12]). Thus a T-set naturally determines an edge space structure, except the edges are not singletons. Conversely, given E so that (X, E) is an edge space, note that $X \setminus E$ is closed, and so may fail to be a T-set only in that the edges are allowed to have only one boundary point (not necessarily two). Note that the latter construction does not require X to be compact or locally connected.

In the Diestel-Kühn topology, edges are again not singletons, but rather are connected open subsets homeomorphic to the real line. In this section, we provide a general quotient operation that turns a compact, weakly Hausdorff space with connected open sets as "edges" into a compact, weakly Hausdorff edge space. The main point is that the cycle spaces of the two spaces are isomorphic and the cuts are in 1-1 correspondence. Thus, the theory developed in Sections 3 and 4 extend to include such spaces. We note that we could not simply refer to these developments when discussing Diestel and Kühn, since their circuits were defined in terms of circles; we had to show their circuits were the "same" as our edge cycles.

The main point of this section is to introduce the quotient operation that will simplify the structure of a space with open connected sets for edges and allow the application of the theory developed in Sections 3 and 4 to these more general spaces.

A *generalized edge space* is a pair $(X, \mathcal{E})$ consisting of a topological space X together with a set $\mathcal{E}$ of pairwise disjoint, connected, open, but not closed, subsets of X, each having at most two additional points in its closure. We denote the subspace $X \setminus (\cup_{e \in \mathcal{E}} e)$ by $X - \mathcal{E}$. The concepts of edges, edge-cycles, cycles, cycle spaces, fundamental cycles, cuts, bonds and bond spaces extend in the obvious way to their "generalized" version.

Let $(X, \mathcal{E})$ be a generalized edge space. The *edge-vertex decomposition* $\mathcal{D}$ *of* X is the partition of X into the connected open sets in $\mathcal{E}$ and the connected closed sets that are the components of $X - \mathcal{E}$. The *simplification* of $(X, \mathcal{E})$ is the pair (Y, F), where Y is the topological quotient $X/\mathcal{D}$ whose points are the sets in $\mathcal{D}$, and $F \subseteq Y$ being precisely the parts from $\mathcal{E}$.

Theorem 35. *Let $(X, \mathcal{E})$ be a compact, weakly Hausdorff generalized edge space. Then the decomposition of $X - \mathcal{E}$ into components is upper semicontinuous. If (Y, F) is the simplification of $(X, \mathcal{E})$ with $p : X \to Y$ being the quotient map, then:*

(1) *(Y, F) is a compact, weakly Hausdorff edge space; and*
(2) *if $A \subseteq Y$, then A is connected in Y if and only if $p^{-1}(A)$ is connected in X.*

Recall that a decomposition $\mathcal{D}$ of a topological space X is *upper semicontinuous* if, for every $D \in \mathcal{D}$ and every open set U such that $D \subseteq U$, there is an open set U' in X that is the union of sets from $\mathcal{D}$ such that $D \subseteq U' \subseteq U$. While it is not true that the edge-vertex decomposition of an edge space $(X, \mathcal{E})$ is upper semicontinuous, it is true that the decomposition of $X - \mathcal{E}$ is upper semicontinuous, when X is compact and weakly Hausdorff. This is Theorem 8.F.14 in [3] for compact metric spaces. The proof is readily adapted to compact weakly Hausdorff spaces (Theorem 4.3.16 in [13]).

In order to prove the rest of Theorem 35, we need some preliminaries.

Lemma 36. *Let $(X, \mathcal{E})$ be a compact generalized edge space and (A_1, A_2) a separation of $X - \mathcal{E}$. For $i = 1, 2$, let U_i be an open set containing A_i and disjoint from A_{3-i}. Let $\mathcal{M}, \mathcal{N}$ denote the sets of generalized edges $\{e \in \mathcal{E} \mid e \setminus (U_1 \cup U_2) \neq \emptyset\}$ and $\{e \in \mathcal{E} \mid \mathtt{Cl}(e) \cap U_1 \neq \emptyset$ and $\mathtt{Cl}(e) \cap U_2 \neq \emptyset\}$ respectively. Then $\mathcal{M}$ is finite, and if $U_1 \cap U_2$ is finite, then so is $\mathcal{N}$.*

Proof: Suppose first that the set $\mathcal{M}$ is infinite, and for every $e \in \mathcal{M}$ choose $x_e \in e \setminus (U_1 \cup U_2)$. Since X is compact, the set $\{x_e\}_{e \in \mathcal{M}}$ has an accumulation point x. Since the (generalized) edges are open and pairwise disjoint, x is not in any edge, that is, $x \in X - \mathcal{E}$. Thus, $x \in A_1$ or $x \in A_2$. For the sake of definiteness, we assume $x \in A_1$. Then U_2 is an open set containing x, but no x_e is in U, contradicting the fact that x is an accumulation point for $\{x_e\}_{e \in \mathcal{E}}$.

Hence $\mathcal{M}$ is finite. Suppose now that $U_1 \cap U_2$ is finite. It is sufficient to show that $\mathcal{L} := \mathcal{N} \setminus \mathcal{M}$ is finite. For any edge e and $i = 1, 2$, since U_i is open, $\mathtt{Cl}(e) \cap U_i \neq \emptyset$ implies that $e \cap U_i \neq \emptyset$. So for $e \in \mathcal{L}$ we have that $e \subseteq (U_1 \cup U_2)$, $e \cap U_1 \neq \emptyset \neq e \cap U_2$. Since e is connected, $(U_1 \cap e, U_2 \cap e)$ is not a separation of e, that is, there exists $y_e \in (e \cap U_1 \cap U_2)$. Since $U_1 \cap U_2$ is finite and the generalized edges are disjoint, $\mathcal{L}$ is finite. ∎

In the next result, for a separation (A, B) of $X - \mathcal{E}$, $\delta(A)$ denotes the generalized cut consisting of all the edges such that $(\mathtt{Cl}(e) \cap A) \neq \emptyset$ and $(\mathtt{Cl}(e) \cap B) \neq \emptyset$.

Corollary 37. *Let $(X, \mathcal{E})$ be a compact weakly Hausdorff generalized edge space, and let (A_1, A_2) be a separation of $X - \mathcal{E}$. Then $\delta(A)$ is finite.*

Proof: Since X is weakly normal, there are open sets U_1 and U_2 in X containing A_1 and A_2, respectively, such that $U_1 \cap U_2$ is finite. Furthermore, A_1 and A_2 are closed in X, so we may assume $U_1 \cap A_2 = \emptyset$ and $U_2 \cap A_1 = \emptyset$. Then the set $\mathcal{M}$, as defined in Lemma 36 is finite (by the same lemma). But $\delta(A) \subseteq \mathcal{M}$. Hence $\delta(A)$ is finite. ∎

We will need the following relation between the topology on $X - \mathcal{E}$ and X. For a set A that is open and closed in $X - \mathcal{E}$, we set $A^\square$ to be the set $A \cup \{e \in \mathcal{E} \mid \texttt{Cl}(e) \cap A \neq \emptyset\}$.

Lemma 38. *Let $(X, \mathcal{E})$ be a compact, weakly Hausdorff generalized edge space and let $A \subseteq X - \mathcal{E}$ be both open and closed in $X - \mathcal{E}$. Then $A^\square$ is open in X.*

In fact, we can describe the technical property that is required for Lemma 38. Let (X, E) be an edge space and let $E' \subseteq E$. A set $W \subseteq X$ is a *transversal* of E' if every edge in E' has an end in W. The edge space (X, E) is *quasi-regular* if, for any $E' \subset E$ and any transversal W of E', $\texttt{Cl}(\bigcup_{e \in E'} e) \setminus (\bigcup_{e \in E'} \texttt{Cl}(e)) \subseteq \texttt{Cl}(W)$. Theorem 3.4.3 in [13] shows that Lemma 38 holds if (X, E) is a quasi-regular edge space. It is also true that any weakly normal edge space is quasi-regular. We do not prove either of these facts here.

Proof of Lemma 38: Let A' be the complement of A in $X - \mathcal{E}$. As in the proof of Corollary 37, there exist open sets U, U' containing A, A' and disjoint from A', A, respectively. We partition $\mathcal{E}$ into the three (possibly empty) sets $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$, where $\mathcal{F}_1$ is the set of edges $e \in \mathcal{E}$ such that $e \not\subseteq U \cup U'$, and $\mathcal{F}_2$ the set of edges e such that $e \subseteq U \cup U'$ and $\texttt{Cl}(e) \cap U \neq \emptyset \neq \texttt{Cl}(e) \cap U'$. By Lemma 36 $\mathcal{F}_1$ and $\mathcal{F}_2$ are finite.

Note that any edge of $\mathcal{F}_3$ is entirely contained in precisely one of U, U' (and is disjoint from the other), according to whether its ends are in A or A' respectively. Thus, every such edge is in $M^\square$ if and only if it is contained in U. As for the edges in $\mathcal{F}_1 \cup \mathcal{F}_2$, we observe that if such an edge e is not (entirely) in $A^\square$, then $\texttt{Cl}(e)$ is disjoint from $A^\square$. Hence

$$A^\square = \left(U \cup \bigcup_{e \in \mathcal{F}_1 \cup \mathcal{F}_2 \;:\; e \subseteq A^\square} e \right) \setminus \left(\bigcup_{e \in \mathcal{F}_1 \cup \mathcal{F}_2 \;:\; e \not\subseteq A^\square} \texttt{Cl}(e) \right).$$

Since $\mathcal{F}_1 \cup \mathcal{F}_2$ is finite, $A^\square$ is open. ∎

Corollary 39. *Let $(X, \mathcal{E})$ be a compact weakly Hausdorff generalized edge space. If $\mathcal{F}$ is an edge cut, then $X - \mathcal{F}$ is disconnected.*

Proof: If $F = \delta(A) = \delta(B)$ for some separation (A, B) of $X - \mathcal{E}$, then $A^\square - \mathcal{F}, B^\square - \mathcal{F})$ is a separation of $X - \mathcal{F}$. ∎

The following gives the final ingredient for the proof of Theorem 35.

Lemma 40. *Given* $(X, \mathcal{E}), (Y, F)$ *and* $p : X \to Y$ *as in Theorem 35, for every* $f \in F, u \in Y \setminus F$ *we have that* $u \in \texttt{Cl}(f)$ *if and only if there exists some point* $v \in X$ *such that* $p(v) = u$ *and* $v \in \texttt{Cl}(p^{-1}(f))$.

Proof: One direction holds simply by virtue of the continuity of p. For the other direction, suppose that $u \in \texttt{Cl}(f)$. Let $e \in \mathcal{E}$ be such that $p(e) = f$ and let K be the component of $X - \mathcal{E}$ be such that $p(K) = u$. It suffices to show that $\texttt{Cl}(e) \cap K \neq \emptyset$.

Suppose not. Let W be a component of $X - \mathcal{E}$ such that $\texttt{Cl}(e) \cap W \neq \emptyset$. By Lemma 6, there is a separation (A_W, A'_W) of $X - \mathcal{E}$ such that $K \subseteq A_W$ and $W \subseteq A'_W$. There are at most two such W, so there is an open and closed (in $X - \mathcal{E}$) set A containing K and disjoint from every component containing an end of e.

Lemma 38 implies $A^\square$ is open in X. Since $A^\square$ is the union of parts of $\mathcal{D}$, $p(A^\square)$ is open in Y. But this is an open set in Y containing $u = p(K)$ and disjoint from f, contradicting the assumption that $u \in \texttt{Cl}(f)$. ∎

Proof of Theorem 35: We have already discussed the fact that the decomposition of $X - \mathcal{E}$ into components is upper semicontinuous.

Let $\mathcal{D}$ be the edge-vertex decomposition of X. Since p is continuous, $Y = X/\mathcal{D}$ is compact.

The points of F are open in Y and the points in $Y \setminus F$ are closed in Y. To show that (Y, F) is an edge space, we must show that each $f \in F$ has one or two additional points in its closure; this follows immediately from Lemma 40 (note that no point in F can be in the closure of any other point).

Next, we show Y is weakly Hausdorff. Let u_1, u_2 be distinct points in Y. If u_i is open in Y, then trivially $\{u_i\}$ and Y are open sets, containing u_i and u_{3-i}, respectively, such that $\{u_i\} \cap Y$ is finite. So we can assume both u_1 and u_2 are closed. For $i = 1, 2$, let K_i be the component of $X - \mathcal{E}$ such that $p(K_i) = u_i$.

Since K_1 and K_2 are disjoint components of $X - \mathcal{E}$, Lemma 6 implies there is a separation (A_1, A_2) of $X - \mathcal{E}$ such that, for $i = 1, 2$, $K_i \subseteq A_i$. Again, $p(A_i^\square)$ is open in Y and $p(A_1^\square) \cap p(A_2^\square)$ consists of precisely those edges f such that $p^{-1}(f)$ is a generalized edge in $\delta(A_1)$. By Lemma 37 $\delta(A_1)$ is finite. Therefore, Y is weakly Hausdorff.

Finally, if $p^{-1}(A)$ is connected, then the continuity of p implies A is connected. Conversely, suppose A is connected and (U, V) is a separation of $p^{-1}(A)$. Let U', V' be open in X such that $U = U' \cap p^{-1}(A)$ and $V = V' \cap p^{-1}(A)$.

Since the decomposition of $X - \mathcal{E}$ into components is upper semicontinuous, for each component K of $X - \mathcal{E}$, if $K \subseteq U'$, then there is an open set $U_K \subseteq U'$ of X containing K and every component of $X - \mathcal{E}$ that U_K meets. Let U'' be the union of the U_K, over all components K contained in U'. Similarly, obtain V'' from the components of $X - \mathcal{E}$ contained in V'.

If e is an edge of A, then $p^{-1}(e)$ is contained in one of U and V, since e is connected. Let U^* be the set obtained from U'' by adding all the edges of X which have non-empty intersection with U'', and similarly obtain V^* from V''.

Then U^* and V^* are open (being the union of open sets) and $U^* \cap p^{-1}(A)$ and $V^* \cap p^{-1}(A)$ are disjoint. Moreover, for any edge $e \in A$, unless $A = \{e\}$, since A is connected A must contain an end of e, whence $p^{-1}(A)$ contains an end of $p^{-1}(e)$, which is contained in one of U, V, and therefore one of U'' or V''. Hence $p^{-1}(e) \subseteq U^* \cup V^*$. So U^*, V^* together cover $p^{-1}(A)$, $p(U^*)$ and $p(V^*)$ are open in Y and $(p(U^*) \cap A, p(V^*) \cap A)$ is a separation of A, a contradiction. ∎

Notice that, if $(X, \mathcal{E})$ is a compact, weakly Hausdorff generalized edge space, with simplification (Y, F), then Theorem 35 implies that, for every cycle (C, F') in (Y, F), $(p^{-1}(C), p^{-1}(F'))$ is a cycle in $(X, \mathcal{E})$. However, it is not generally true that if $(C, \mathcal{E}')$ is a cycle in $(X, \mathcal{E})$, then $(p(C), p(\mathcal{E}'))$ is a cycle in $(Y, \mathcal{F})$. Consider, for example, an edge space $(X, \mathcal{E})$ consisting of two edges and one large component (say a disc) of $X - \mathcal{E}$. We can get a cycle using both edges and arcs in the disc joining appropriately the ends of the edges, but the quotient of such a cycle is not a cycle in the simplification.

The point of this section is the following.

Theorem 41. *Let $(X, \mathcal{E})$ be a compact, weakly Hausdorff generalized edge space and let (Y, F) be its point simplification, with quotient map $p : X \to Y$. Let $\hat{p} : 2^{\mathcal{E}} \to 2^F$ be defined by $\hat{p}(E') = \{p(e) \mid e \in E'\}$. Then $\hat{p} : \mathcal{Z}(X, \mathcal{E}) \to \mathcal{Z}(Y, F)$ and $\hat{p} : \mathcal{B}(X, \mathcal{E}) \to \mathcal{B}(Y, F)$ are isomorphisms.*

Proof: Part (2) of Theorem 35 implies that $\hat{p}$ maps the edgecuts of $(X, \mathcal{E})$ bijectively onto those of (Y, F), and, as observed above, if $C \subseteq F$ is an edge-cycle in (Y, F), then $\hat{p}^{-1}(C)$ is an edge-cycle in $(X, \mathcal{E})$. We complete the proof by showing that, if $\mathcal{D} \subseteq \mathcal{E}$ is an edge-cycle in $(X, \mathcal{E})$, then $\hat{p}(\mathcal{D})$ is in the cycle space of (Y, F). By part (1) of Theorem 35, (Y, F) satisfies the assumptions of Theorem 14. It is therefore sufficient to show that, for every edgecut $A = \delta(M)$ of (Y, F), $\hat{p}(\mathcal{D}) \cap A$ is finite and even.

By Lemma 40 $\hat{p}^{-1}(A)$ is the edgecut $\delta(p^{-1}(M))$. By Corollary 37 $\mathcal{D} \cap \hat{p}^{-1}(A)$ is finite. We now observe that Lemma 7 (as well as Lemma 8, upon which it depends) holds for *generalized* edges—the proofs carry over simply by inserting "generalized" in front of "edges" throughout. It now follows, just as in the proof of Lemma 11, that $\mathcal{D} \cap \hat{p}^{-1}(A)$ is even. We conclude that $\hat{p}(\mathcal{D}) \cap A$ is finite and even. ∎

9. Feebly Hausdorff Spaces

In the context of edge spaces, any open set containing a vertex must also contain all the edges incident with that vertex. Thus, if we are trying to separate two vertices u and v by open sets, the intersection must contain all the edges incident with both u and v. Let X be a topological space and let $v \in X$. We set $v^\diamond$ to be the intersection of all the open sets containing v (so a topological space is Alexandroff discrete if and only if, for every point x, the set $x^\diamond$ is open). The space X is *feebly Hausdorff* if, for any two points $u, v \in X$, there exist open sets U_u and U_v containing u and v, respectively, such that $U_u \cap U_v \subseteq u^\diamond \cap v^\diamond$.

That is, there are open sets which intersect just in those points that they must contain, as the intersection of any pair of open sets contains these points. (Thus, we could replace $\subseteq$ by $=$.)

A simple example of a feebly Hausdorff space which is not weakly Hausdorff is given by the graph with two vertices joined by infinitely many edges, equipped with the classical topology. Note that this space is compact, but replacing the edges with arcs, to obtain a Hausdorff space, gives a non-compact topology.

It turns out, and the theory is developed fully in [13], that we can extend many of our arguments to the case of connected, compact feebly Hausdorff edge spaces (X, E). The proofs are typically quite a bit more involved and the statements are not so clean. In particular, in this context we need to define the bond space to be the weak span of the *finite* bonds (equivalently, the space of all finite edgecuts). Note that, for weakly Hausdorff spaces, this definition is equivalent to the one we used previously. Here we only wish to indicate the flavour of the theory.

Firstly, there still exist minimal connected spanning subsets and the appropriate analogue of Lemma 3 holds, so that we may obtain, as before, fundamental cycles C_e, with respect to any minimal connected spanning set. Secondly, it is still true that the fundamental cycles strongly generate the cycle space.

In order to deal with orthogonality, the main tool used is to express the cycle space of a compact feebly Hausdorff edge space (X, E) in terms of that of a certain quotient (Y, F). In this case, we construct an intermediate edge space (X, F), where $F \subseteq E$ consists precisely of the edges whose parallel class is finite, that is, edges which do not share two ends with infinitely many other edges. The edge space (Y, F) is the simplification (as in Section 8) of (X, F) (since edges are points, we do not distinguish between the edges of (X, F) and their formally distinct images in the quotient). We show that (Y, F) is weakly Hausdorff, and that the bond spaces in all three edge spaces are the same. Moreover, the edge sets of the minimal spanning sets, and the corresponding fundamental cycles, are precisely the same in (X, F) and (Y, F). Hence, as a corollary of the fact that the fundamental cycles strongly generate the cycle space, we have that the corresponding cycle spaces are isomorphic (as in Section 8, although the proof is different). In order to bridge the gap between the cycle spaces of (X, E) and (X, F), we take a minimal spanning set of T of (X, F), which we then consider as an edge subspace $T' := (T, T \cap E)$ of (X, E). Finally we use a minimal spanning set of T' to extend the fundamental cycles of (X, F) to fundamental cycles of (X, E).

Theorem 42. *Let (X, E) be a connected compact feebly Hausdorff edge space. With Y and F as in the preceding paragraph, the cycle space of (X, E) is $\mathcal{Z}(Y, F) \oplus 2^{E \setminus F}$. For any minimal connected spanning set of (X, E), the set $\mathcal{F}$ of fundamental cycles is such that $\mathcal{A}^2(\mathcal{F})$ is the cycle space of (X, E).*

We remark that, as can be seen in the above discussion, the arguments depend heavily on the fact that we are free to declare what the edges are, that is, we need not include all the open singletons in the edge set of an edge space.

Theorem 43. *Let (X, E) be a connected compact feebly Hausdorff edge space, with cycle space $\mathcal{Z}(X, E)$ and bond space $\mathcal{B}(X, E)$. Then $\mathcal{B}(X, E)^{\perp} = \mathcal{Z}(X, E)$ and, letting $E' = \{e \in E \mid X \setminus e\}$ is connected$\}$, $\mathcal{Z}(X, E)^{\perp} = \mathcal{B}(X, E') \oplus 2^{E \setminus E'}$. In particular, if (X, E) is 2-edge-connected, then $\mathcal{Z}(X, E)^{\perp} = \mathcal{B}(X, E)$.*

Again, the flexibility to ignore the edges in $E \setminus E'$ makes for a cleaner statement and simplifies the arguments.

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