Cayley Maps

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Abstract

We present a theory of Cayley maps, i.e., embeddings of Cayley graphs into oriented surfaces having the same cyclic rotation of generators around each vertex. These maps have often been used to encode symmetric embeddings of graphs. We also present an algebraic theory of Cayley maps and we apply the theory to determine exactly which regular or edge-transitive tilings of the sphere or plane are Cayley maps or Cayley graphs. Our main goal, however, is to provide the general theory so as to make it easier for others to study Cayley maps.

1 Introduction

One of the central topics in the algebraic theory of graphs and maps is transitivity. In graphs, there are two extreme cases. The weakest kind of transitivity is vertex transitivity (or edge-transitivity). In a way, the simplest vertex-transitive graphs are Cayley graphs, i.e., graphs whose automorphism group contains a subgroup acting regularly on their vertex set. The importance of Cayley graphs for the study of transitivity was probably first realized in the pioneering paper [37]. It is by far not true that every vertex-transitive graph is a Cayley graph (for a survey of constructions of vertex-transitive non-Cayley graphs we refer to [33, 34]); however, Cayley graphs play an important role here in that every vertex-transitive graph is a quotient of some Cayley graph.

At the other extreme, the strongest kind of transitivity is commonly measured in two ways: the number of elements of the automorphism group (where complete graphs are undisputed winners) and in terms of s-transitivity. Recall that a graph is *s-transitive* if for any two s-arcs there is a graph automorphism that maps one s-arc onto the other. It has been known for a long time [43] that a 3-valent graph can be at most 5-transitive, and infinitely many examples of such graphs have been constructed by coverings in [10] (see also [6]). In general, by a remarkable result of [46] based on sophisticated group theoretic machinery, a graph of valence ≥ 4 can be at most 7-transitive.

For maps on oriented surfaces, the investigation of transitivity is similar at the lower end of the "transitivity spectrum" but quite different at the upper end. Again, the weakest kind of transitivity in maps is represented by vertex-transitive maps. The simplest examples of such maps are Cayley maps (the subject of this paper), which are oriented maps whose map automorphism group contains a subgroup that acts regularly on the vertex set. But as the map automorphism group of any oriented map acts semiregularly on the dart set (i.e., the set of edges with directions) of the map, the highest degree of transitivity is achieved by the *regular maps* such that for any two darts there exists exactly one orientation-preserving map automorphism that takes one dart onto the other. In the language of s-transitivity, oriented maps can be at most 1-transitive, and the 1-transitive oriented maps are regular maps.

Regular maps can be (sometimes surprisingly) encountered in a number of areas within mathematics. In geometry they appear as regular tessellations of the hyperbolic plane. The groups of orientation-preserving hyperbolic isometries that leave such tessellations invariant are the hyperbolic triangle groups that were well known to leading mathematicians of the 19th century. Hyperbolic triangle groups also act on the complex upper half-plane as discontinuous groups of conformal isometries; therefore quotients of such triangle groups turn out to be (conformal) map automorphism groups of the corresponding quotient maps on the resulting Riemann surfaces. Maps on Riemann surfaces are further related to complex curves defined over fields of algebraic numbers and actions of the Galois groups of these fields on the maps; more information about such connections can be found in the excellent survey papers [23, 25].

The term "Cayley maps" appeared first in the 1972 paper of Biggs [5] as a name for a class of maps that are now known as "balanced Cayley maps". However, maps that in the current terminology are known as Cayley maps have been used in constructions of regular maps prior to that date (in fact, since the end of the 19th century [30]). Perhaps the most important example is the solution of the famous Heawood map coloring problem (see [36] and references therein): the cases $n \equiv 0, 4$ and 7 (mod 12) of the solution rely (in current terminology) on constructing a triangular Cayley map of the complete graph of order n (see also [13]). Another example involving complete graphs where Cayley maps play an important role is the classification of regular maps of complete graphs; by a result in [21], all such maps are balanced Cayley maps. A substantial discussion of Cayley maps (under the name "strong symmetric embeddings of Cayley graphs") can be found in [42, 13] in the context of determining the genus of a group. We also note that a number of constructions of genus embeddings of some important classes of graphs (see e.g. [13]) are actually Cayley maps.

In the beginning of the 1990's a two-part series [41, 40] was devoted to a systematic study of Cayley maps, with emphasis on balanced and antibalanced Cayley maps. For example, it was shown in [41] that regularity of balanced Cayley maps is equivalent to the existence of certain automorphisms of the Cayley group (thus extending earlier results of [7]). A similar characterization of regular antibalanced Cayley maps can be found in [40]; in both papers the corresponding map automorphism groups are determined. In general, map automorphism groups of arbitrary Cayley maps have been described in [17, 18]. Balanced Cayley maps were also used in [1] to provide a self-dual embedding of the $(2n)$-dimensional cube.

All these previous works on or using Cayley maps appear to have been done in isolation from one another. In this work we present a substantial and coherent algebraic theory of Cayley maps. In Sections 2 and 3 we lay the elementary algebraic foundations for our study, with particular emphasis on homomorphisms of maps (or, equivalently, branched coverings). Section 2 provides the characterization of Cayley maps by the existence of a group of automorphisms acting regularly on the vertex set of the map, while Section 3 explains the algebraic context of maps forming the backdrop for this entire work with emphasis on the "rotation" and "direction-reversing" permutations on the set of darts.

In Section 4, d-valent Cayley maps are characterized by the existence of certain kinds of homomorphisms from their "dart group" and automorphism group to the symmetric group on d symbols. Section 5 gives an introduction to the special classes of balanced and antibalanced Cayley maps and relates our results to these maps.

Section 6 contains a detailed study of the automorphism group of a Cayley map, thereby generalizing the results in [20, 17, 41, 40]. Section 7 deals with homomorphisms of Cayley maps onto arbitrary maps and, furthermore, shows that every finite map is the homomorphic image of a finite regular Cayley map.

Section 8 provides a discussion of existence of covering maps with specified valences and covalences relative to the valences and covalences of the covered map. This section is a survey of material relevant to, but beyond the scope of, this work. We include in Section 8 a method for constructing Cayley maps with given valence and covalences.

The same construction is then used in Section 9 to prove the sufficiency of some necessary conditions for the existence of a Cayley map with given valence and covalences. In Section 10 these conditions are applied to characterize the regular or edge-transitive planar tilings that are Cayley maps, and, furthermore, which have Cayley graphs as their underlying graph. We conclude with a list of open problems for future research.

This work is an outgrowth of an article written by the first two authors. Conversations with the others made it clear that there was a much broader scope possible and this work is the consequence.

2 Cayley graphs and Cayley maps

Informally, a Cayley map is an embedding of a Cayley graph of a group H in which H has a natural vertex-transitive action on the resulting map. The goal of this section is to show that there is a strong analogy between fundamental characterizations of Cayley graphs and Cayley maps.

Graphs in this work are connected and undirected but there are no other restrictions; we allow parallel (i.e., multiple) edges, (multiple) loops, and also (multiple) semiedges, that is, edges with just one vertex (the other end of a semiedge is "dangling", and is not considered as a vertex of the graph). Our graphs may be finite as well as infinite, but in the latter case they are of bounded valence, that is, for some c all valences are bounded above by c. Each edge that is not a semiedge can be assigned one of two possible directions (from one endvertex to the other); if e is an edge with a direction, then $e\lambda$ denotes the same edge but with the opposite direction. Semiedges are not assigned any direction; however, it is convenient to extend the mapping λ by setting $e\lambda = e$ if e is a semiedge. For brevity, we refer to edges with directions together with semiedges as *darts*; the dart $e\lambda$ is the *reverse of* e. Throughout this work D denotes the set of darts of a graph Γ, that is, D contains all the directed edges, two for each edge of Γ and one for each semiedge. The involutory permutation λ of D leaves the semiedges invariant and reverses the non-semiedge darts.

An important link between graph theory and group theory is provided by Cayley graphs. Formally, let H be a group and let X be a finite sequence $x_1, x_2, \ldots, x_d$ of elements of H that generate H. We emphasize here the word *sequence*, which means that it may happen that $x_i = x_j$ for some $i \neq j$. We also allow that some of the x_i's are equal to 1_H, the unit element of the group H. Assume further that an involution τ on the set $[d] = \{1, 2, \ldots, d\}$ is given, with the property that $x_{i\tau} = x_i^{-1}$ for each $i \in [d]$. The *Cayley graph* $C(H, X, \tau)$ has vertex set H, and for each $g \in H$ and each $i \in [d]$ there is a dart (g, i) emanating from the vertex g. The dart (g, i) is a semiedge if $x_i = 1_H$ and $i = i\tau$; it is a directed loop at g if $x_i = 1_H$ and $i \neq i\tau$; and it is a directed edge incident with two distinct vertices if $x_i \neq 1_H$. In the last two cases, the terminal vertex of the dart (g, i) is the vertex gx_i and the corresponding reverse dart is $(gx_i, i\tau)$. For the dart-reversing involution λ we therefore have $(g, i)\lambda = (gx_i, i\tau)$; note that this formula is valid for semiedges as well. Except for the semiedge case $x_i = 1_H$ and $i = i\tau$, the pair of darts (g, i) and $(g, i)\lambda$ gives rise to an undirected edge; hence the Cayley graph $C(H, X, \tau)$ is an undirected graph (possibly with semi-edges) whose darts are labelled by the indices of the generating sequence X. The group H is the *Cayley group* and the involution τ is the *distribution of inverses*.

Note that the sequential arrangement of the elements in X is inessential in the following sense. Let θ be any permutation of the set $[d]$ and let X' denote the sequence $x'_1, x'_2, \ldots, x'_d$ where $x'_i = x_{i\theta}$ for each $i \in [d]$. Furthermore, let τ' be the involution on $[d]$ defined by $\tau' = \theta\tau\theta^{-1}$. Then the Cayley graphs $C(H, X, \tau)$ and $C(H, X', \tau')$ are identical. (To check this, note that $x'_{i\tau'} = x_{i\tau'\theta} = x_{i\theta\tau\theta^{-1}\theta} = x_{i\theta\tau} = (x_{i\theta})^{-1} = (x'_i)^{-1}$.)

Cayley graphs are sometimes introduced in two other (and equivalent) ways, for which we need to recall a few more concepts. A (right) group action G on a set D is *semi-regular* if, for any $e, e' \in D$, there is at most one $\mu \in G$ such that $e\mu = e'$. Some authors use the term *free* as a synonym of semi-regular. The action is *transitive* if, for any $e, e' \in D$, there is at least one $\mu \in G$ such that $e\mu = e'$. The action is *regular* if it is both semi-regular and transitive. (We will return to group actions in Section 3. See [9] or [32] for more on group actions.)

We pause here to comment that we have made every effort to be consistent in the use of notation in this work. Thus, H (along with H_1, etc.) is only used for the Cayley group of a Cayley graph or Cayley map. The symbol D (along with D', D_1, etc.) is reserved for the set of darts of a graph or map. There are many other examples of this effort, which we hope will help the reader.

Observe that for each $h \in H$ the *left translation* A_h that sends a dart (g, i) onto the dart (hg, i) is an automorphism of the Cayley graph $\Gamma = C(H, X, \tau)$. This shows, for example, that every Cayley graph is vertex-transitive (but not vice versa, the smallest counterexample being the notorious Petersen graph). The group of left translations $H_L = \{A_h : h \in H\}$ (isomorphic to H) is a subgroup of the automorphism group $Aut(\Gamma)$ of the Cayley graph acting regularly on the vertex set. (Of course, strictly speaking, the automorphisms act on the darts, rather than the vertices. However, they naturally induce an action on the vertices; we shall always use this induced action when speaking of the action of automorphisms on the vertices. It will be helpful to the reader to realize that a vertex determines and is determined by the set of darts emanating from it.)

Let $\tilde{\Gamma}$ and Γ be graphs with dart sets $\tilde{D}$ and D and dart-reversing involutions $\tilde{\lambda}$ and λ, respectively. A function $f : \tilde{D} \to D$ is a *graph homomorphism* if (1) $(f(\tilde{e}))\lambda = f(\tilde{e}\tilde{\lambda})$ for all $\tilde{e} \in \tilde{D}$ and (2) for all $\tilde{e}_1, \tilde{e}_2 \in \tilde{D}$ emanating from the same vertex of $\tilde{\Gamma}$, $f(\tilde{e}_1)$ and $f(\tilde{e}_2)$ emanate from the same vertex of Γ. Note that a graph homomorphism can collapse parallel edges into a single edge and an edge into a semi-edge or a loop. We sometimes write $f : \tilde{\Gamma} \to \Gamma$ rather than the more formal $f : \tilde{D} \to D$.

A graph homomorphism $f : \tilde{\Gamma} \to \Gamma$ is a *graph covering* if it is surjective and, for each vertex $v \in V(\Gamma)$ and each vertex $\tilde{v}$ in the preimage $f^{-1}(v)$, the set of darts emanating from $\tilde{v}$ is mapped by f bijectively onto the set of darts emanating from v. (We note that if Γ is connected, then the assumption about preimages implies that f is surjective.) The notion of graph covering corresponds to the topological notion of covering except when f takes an edge of $\tilde{\Gamma}$ to a semi-edge of Γ; then the edge is folded at its midpoint. The covering f is *regular* if there exists a subgroup $\tilde{K}$ of $Aut(\tilde{\Gamma})$ acting semi-regularly on $f^{-1}(e)$ for every dart e such that, for any two darts $\tilde{e}_1, \tilde{e}_2$ of $\tilde{\Gamma}$, $f(\tilde{e}_1) = f(\tilde{e}_2)$ if and only if $\tilde{e}_2 = \tilde{A}(\tilde{e}_1)$ for some automorphism $\tilde{A} \in \tilde{K}$.

We now consider an important special case. Let $C(H, X, \tau)$ be a Cayley graph, let $\langle 1_H \rangle$ be the trivial group and let $Y = y_1, y_2, \ldots, y_d$, where $y_i = 1_H$ for all $i \in [d]$. The Cayley graph $C(\langle 1_H \rangle, Y, \tau)$ has just one vertex with loops and/or semiedges attached to it. It will help the reader later to recognize that, in $C(\langle 1_H \rangle, Y, \tau)$, if $i\tau = i$, then the dart $(1_H, i)$ is the dart of a semi-edge, while if $i\tau \neq i$, then $(1_H, i)$ and $(1_H, i\tau)$ are the two darts associated with a loop.

Note that the graphs $C(H, X, \tau)$ and $C(\langle 1_H \rangle, Y, \tau)$ have the same distribution of inverses τ. The homomorphism f that sends the dart (g, i) to the dart $(1_H, i)$ is clearly a regular graph covering $C(H, X, \tau) \to C(\langle 1_H \rangle, Y, \tau)$ associated with the group $H_L = \{A_h : h \in H\} \simeq H$ of left translations.

We have seen that each Cayley graph contains a group of automorphisms that is regular on the vertex set; at the same time, each Cayley graph gives rise to a regular covering of a one-vertex graph. These two properties are, in fact, characteristic for Cayley graphs, as is formally stated below in Theorem 2.1. We note that the equivalence of (1) and (2) below is a classical result by Sabidussi [37]; for the equivalence of (1) and (3) see [13]. In both cases, the prior work considered classes of graphs more restricted than we have here, but the proofs readily extend to graphs with multiple edges, loops, and semiedges.

Theorem 2.1 *Let Γ be a connected graph. Then the following three statements are equivalent:*
(1) The graph Γ is a Cayley graph.
(2) The group $Aut(\Gamma)$ contains a subgroup that acts regularly on the vertex set of Γ.
(3) The graph Γ regularly covers a one-vertex graph. $\otimes$

For the purpose of this article, a *surface* is a connected (not necessarily compact) oriented 2-manifold without boundary. A *map* M is a 2-cell embedding of a graph Γ as a closed subset of a surface $\mathcal{S}$, with all valences and covalences being finite. The graph Γ is the *underlying graph* of M. (The term "2-cell" means that each component of the complement of the graph in the surface is homeomorphic to the interior of the unit disc and "closed" implies that the graph has no accumulation points as a topological subspace of the surface.) We note that the underlying graph of a map is necessarily connected; a disconnected graph does not have a 2-cell embedding. Equivalently, M can be viewed as a cellular decomposition of the surface $\mathcal{S}$. The 0-cells, 1-cells and 2-cells are the *vertices*, *edges* and *faces* of the map, and the 1-skeleton (that is, the union of 0-cells and 1-cells) is the underlying graph of the map. A map is *d-valent* if all its vertices have valence d in the underlying graph. Let D be the set of darts of the underlying graph Γ of M (which we will also refer to as the *darts of M*).

A map M is described algebraically in terms of two permutations of the dart set D [13]. For definiteness, we assume throughout that the surface $\mathcal{S}$

is oriented clockwise. The first permutation is λ, the dart-reversing involution associated with the underlying graph Γ (so λ is independent of the embedding). The other permutation is the *rotation* ρ defined by: for each dart $e \in D$ emanating from a vertex, $e\rho$ is the clockwise next dart of the embedded graph emanating from the same vertex. The pair ρ, λ of permutations of the dart set D carries full information about the embedding in the following sense. Vertices, edges, and faces (in terms of face boundaries) are in a 1-1 correspondence with orbits (i.e., cycles) of the permutations ρ, λ, and $\lambda\rho$, respectively; incidence between these elements coincides with nonempty intersection of the corresponding orbits. In what follows we use the extended notation $M = M(\rho, \lambda)$ for a map whenever necessary for clarity. In the case M is infinite (that is, when its underlying graph is infinite), we will assume that M has *bounded covalence*, i.e., all orbits of $\lambda\rho$ have bounded size.

Example. *The following two examples are used throughout this work to help illustrate the theory. The first is a one-vertex map $\mathcal{M}_1$ with two loops embedded in the torus. We can label the four darts e^U, e^R, e^D and e^L. (The superscripts are for "Up", "Right", "Down" and "Left".) In this case, $\rho = (e^U\, e^R\, e^D\, e^L)$ and $\lambda = (e^U\, e^D)(e^R\, e^L)$.*

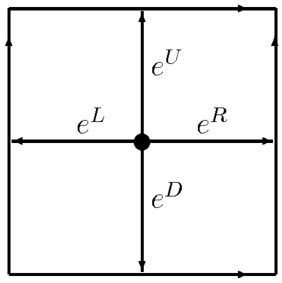

Fig.1

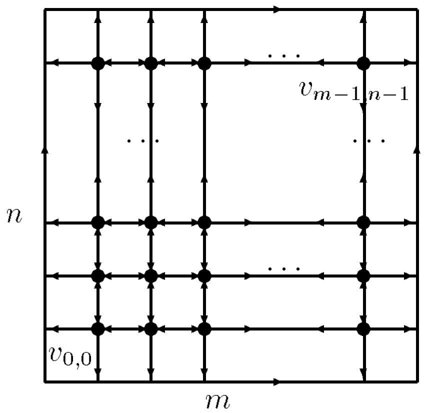

Fig.2

The second is the map $\mathcal{M}_{m,n}$, whose underlying graph is the Cartesian product $C_m \times C_n$ of two cycles, also embedded in the torus. The vertices of $\mathcal{M}_{m,n}$ are labelled $v_{i,j}$, $(i,j) \in \mathcal{Z}_m \times \mathcal{Z}_n$, with $v_{0,0}$ in the lower left-hand corner and $v_{m-1,n-1}$ in the upper right-hand corner. (The cyclic group of order d is $\mathcal{Z}_d$, with elements $1, 2, \ldots, d \equiv 0$.) The darts of $\mathcal{M}_{m,n}$ incident with $v_{i,j}$ are labelled $e^U_{i,j}, e^R_{i,j}, e^D_{i,j}, e^L_{i,j}$, for $i \in [m]$ and $j \in [n]$. The rotation ρ is the product of all the 4-cycles $(e^U_{i,j}\ e^R_{i,j}\ e^D_{i,j}\ e^L_{i,j})$ and λ is the product of all the transpositions $(e^U_{i,j}\ e^D_{i,j+1})$ and $(e^R_{i,j}\ e^L_{i+1,j})$.

It is easy to verify that $(\lambda\rho)^4 = 1$ (corresponding to the faces of the map). For later consideration it will be helpful to know the cycle structure of $\lambda\rho^2$ and its conjugate $\rho\lambda\rho$. We have $e^U_{i,j}\lambda\rho^2 = e^U_{i,j+1}$, $e^R_{i,j}\lambda\rho^2 = e^R_{i+1,j}$, $e^D_{i,j}\lambda\rho^2 = e^D_{i,j-1}$, and $e^L_{i,j}\lambda\rho^2 = e^L_{i-1,j}$. That is, $\lambda\rho^2$ moves U-darts up one step, R-darts right one step, D-darts down one step and L-darts left one step. Thus, $\lambda\rho^2$ has n cycles of length m consisting of R-darts, n cycles of length m of L-darts, m cycles of length n of U-darts, and m cycles of length n of D-darts. Similarly, $\rho\lambda\rho$ moves U-darts right one step, R-darts down one step, D-darts left one step, and L-darts up one step. Thus, $\rho\lambda\rho$ has the same cycle structure as $\lambda\rho^2$ (it is a conjugate) with U replacing R, R replacing D, D replacing L, and L replacing U. Note that this implies $\lambda\rho^2$ and $\rho\lambda\rho$ commute with each *other. (This last assertion can also be verified using the fact that $(\lambda\rho)^4 = 1$.)*

It follows that $\lambda\rho^2$ and $\rho\lambda\rho$ both have order the least common multiple of m and n.

Among the many ways that a Cayley graph may be embedded in a surface we focus on those embeddings that "look the same" at each vertex. A precise formulation of this type of local homogeneity leads to the concept of a Cayley map, which is the subject of this exposition. Let $C(H, X, \tau)$ be a Cayley graph with Cayley group H, a generating sequence $X = x_1, x_2, \ldots, x_d$, and a distribution of inverses τ. The *Cayley map* $CM(H, X, \tau)$ is the map $M(\rho, \lambda)$ whose underlying graph is the Cayley graph $C(H, X, \tau)$ with the obvious dart-reversing involution $(g, i)\lambda = (gx_i, i\tau)$ and with rotation ρ given at each vertex $g \in H$ by $(g, i)\rho = (g, i+1)$, where the second coordinate is understood modulo d. Thus, in a Cayley map, the clockwise cyclic order of "generators emanating from a vertex" is the same at every vertex. As with Cayley graphs, H is the *Cayley group* of the map.

Example. *The map $\mathcal{M}_{m,n}$ is the Cayley map $CM(H, X, \tau)$, where $H = \mathcal{Z}_m \times \mathcal{Z}_n$, $X = (x_1, x_2, x_3, x_4) = ((0, 1), (1, 0), (0, -1), (-1, 0))$, and $\tau =$* $(1\ 3)(2\ 4)$.

Clearly, unlike the situation for Cayley graphs, different sequential arrangement of the generators appearing in the generating sequence may now lead to different embeddings and hence to different Cayley maps. We shall use the notation $X = (x_1, x_2, \ldots, x_d)$ for the (cyclic) generating sequence of a Cayley map. Setting $\eta = (1\ 2\ \ldots\ d)$, it is easy to see that the permutation $\eta\tau\eta^{-1}$ is the same as τ except that all the numbers are increased by $1 \pmod d$, i.e., for $i \in [d]$, $i(\eta\tau\eta^{-1}) = (i+1)\tau - 1$. Thus, $CM(H, (x'_1, x'_2, \ldots, x'_d), \eta\tau\eta^{-1})$ with $x'_i = x_{i+1}$ is the same Cayley map as $CM(H, X, \tau)$, with $X = (x_1, x_2, \ldots, x_d)$. More generally, the Cayley map $CM(H, (x_{1+j}, x_{2+j}, \ldots, x_{d+j}), \eta^j\tau\eta^{-j})$ is the same as $CM(H, X, \tau)$.

For an arbitrary $\theta \in Sym[d]$, the Cayley graph $C(H, X', \theta\tau\theta^{-1})$ obtained by setting $x'_i = x_{i\theta}$ is isomorphic to the Cayley graph $C(H, X, \tau)$, but the Cayley maps $CM(H, X, \tau)$ and $CM(H, X', \theta\tau\theta^{-1})$ are typically not isomorphic as maps. The above comment shows, however, that for θ a power of η the Cayley maps are isomorphic.

Given two maps $\tilde{M} = \tilde{M}(\tilde{\rho}, \tilde{\lambda})$ and $M = M(\rho, \lambda)$ with dart sets $\tilde{D}$ and D, respectivley, a *map homomorphism* is a graph homomorphism $f : \tilde{D} \to D$ of the underlying graphs such that $f(\tilde{e}\tilde{\rho}) = (f(\tilde{e}))\rho$ for each dart $\tilde{e} \in \tilde{D}$. We will often write $f : \tilde{M} \to M$ rather than $f : \tilde{D} \to D$. From a topological viewpoint, a map homomorphism $f : \tilde{M} \to M$ defines a branched covering from the surface underlying $\tilde{M}$ to the surface underlying M with branch points possibly at the vertices, "centers" of faces and the "dangling" ends of semiedges of M. Conversely, such a branched covering determines a map homomorphism. For more about branched coverings, see Section 8. The influence of the branched covering terminology is reflected in our saying that, if $f : \tilde{M} \to M$ is a map homomorphism, then the map $\tilde{M}$ is a *cover of* and *covers* M, while M is *covered by* $\tilde{M}$.

Note that, since the underlying graph of M is connected, a map homomorphism is automatically onto; if it is also one-to-one, then it is an *isomorphism* of the maps $\tilde{M}$ and M. Furthermore, if $\tilde{M} = M$, then an isomorphism is an *automorphism* of the map M. The *automorphism group* $Aut(M)$ of $M = M(\rho, \lambda)$ is thus formed by all permutations A of the dart set D that commute with both ρ and λ, that is, $A(e\rho) = (A(e))\rho$ and $A(e\lambda) = (A(e))\lambda$, for each $e \in D$. It is easy to see (cf [13]) that, since M is connected, the action of an automorphism on any given dart in D determines uniquely its action on all of D and we emphasize, for later use, that every dart has a trivial stabilizer in $Aut(M)$. We also note for later reference that the equation $A(e\rho) = (A(e))\rho$ implies that automorphisms preserve the orientation of the surface.

Example. *The function from the darts of $\mathcal{M}_{m,n}$ to $\mathcal{M}_1$ defined by $e^J_{i,j} \mapsto e^J$,*

$J \in \{U, R, D, L\}$, $i \in [m]$, $j \in [n]$ is a map homomorphism.

The functions A and B on the dart set of $\mathcal{M}_{m,n}$ defined by $e^J_{i,j} \mapsto e^J_{i+1,j}$ and $e^J_{i,j} \mapsto e^J_{i,j+1}$, respectively, are automorphisms of $\mathcal{M}_{m,n}$; the automorphism A is a translation by one step right, while B translates one step up.

Also, the function $C(e^J_{i,j}) = e^{J\alpha}_{-i,-j}$, where $\alpha = (U\ D)(R\ L)$ is also an automorphism; it corresponds to a 180-degree rotation about the vertex $v_{0,0}$. If $m = n$, there is a further automorphism E corresponding to a 90 degree rotation about $v_{0,0}$. The formula for E is not so simple.

The concept of a regular homomorphism of maps (or, equivalently, of a regular branched covering of the corresponding surfaces, discussed in Section 8) can be introduced in a way analogous to what we saw for graphs; note that the semi-regularity is now automatic and need not be imposed. That is, a map homomorphism $f : \tilde{M} \to M$ is *regular* if there exists a subgroup $\tilde{K}$ of $Aut(\tilde{M})$ such that, for any two darts $\tilde{e}_1, \tilde{e}_2$ of $\tilde{M}$, $f(\tilde{e}_1) = f(\tilde{e}_2)$ if and only if $\tilde{e}_2 = \tilde{A}(\tilde{e}_1)$ for some map automorphism $\tilde{A} \in \tilde{K}$. Again this means that $\tilde{K}$ is regular on each fiber $f^{-1}(e)$, $e \in D$. In such a case the target map M is isomorphic to the *quotient map* $\tilde{M}/\tilde{K}$ whose dart set is the set of orbits $\tilde{K}\tilde{e}$, $\tilde{e} \in \tilde{D}$, and whose rotation $\rho_{\tilde{K}}$ and dart-reversing involution $\lambda_{\tilde{K}}$ are defined by $(\tilde{K}\tilde{e})\rho_{\tilde{K}} = \tilde{K}(\tilde{e}\tilde{\rho})$ and $(\tilde{K}\tilde{e})\lambda_{\tilde{K}} = \tilde{K}(\tilde{e}\tilde{\lambda})$; the isomorphism $\tilde{M}/\tilde{K} \to M$ is simply given by $\tilde{K}\tilde{e} \mapsto f(\tilde{e})$.

Conversely, for any subgroup K of $Aut(M)$. the natural projection $f : M \to M/K$, $f(e) = Ke$ is a regular homomorphism of maps. Given the regular homomorphism f, K is the subgroup of $Aut(M)$ *associated with* f.

We have seen in Theorem 2.1 that Cayley graphs can be conveniently characterized in three equivalent ways. The same holds true for Cayley maps, as we shall soon see. Observe first that, for each $h \in H$, the left translation A_h given by $A_h(g, i) = (hg, i)$ is a map automorphism of the Cayley map $M = CM(H, X, \tau)$, and so the automorphism group $Aut(M)$ of the Cayley map contains a subgroup (namely, the group $H_L \simeq H$ of left translations) that is regular on the vertex set.

Example. *In $\mathcal{M}_{m,n}$, the automorphism A described above is left-addition by $(1, 0)$ and B is left-addition by $(0, 1)$. Therefore, we may choose $H_L = \langle A, B\rangle$.*

Second, we may embed the one-vertex Cayley graph $C(\langle 1_H\rangle, Y, \tau)$ as a Cayley map with rotation ρ' and dart-reversing involution λ' defined as expected, that is, $(1_H, i)\rho' = (1_H, i + 1)$ and $(1_H, i)\lambda' = (1_H, i\tau)$. Observe that the dart set of $CM(\langle 1_H\rangle, Y, \tau)$ can be identified with the set $[d]$ and the permutations ρ' and λ' then correspond to the cyclic permutation $\eta = (1\ 2\ \dots\ d)$ and the involution τ, respectively. The Cayley graph homomorphism from $C(H, X, \tau)$ to $C(\langle 1_H\rangle, Y, \tau)$ given by $(g, i) \mapsto i$ is a regular homomorphism of the map $CM(H, X, \tau)$ onto the one-vertex Cayley map $CM(\langle 1_H\rangle, Y, \tau)$ of the same valence. As indicated, these two properties are characteristic for Cayley maps.

Theorem 2.2 *Let M be a d-valent map. Then the following three statements are equivalent:*
(1) M is a Cayley map.
(2) $Aut(M)$ contains a subgroup that acts regularly on the vertex set.
(3) There is a regular map homomorphism from M to a one-vertex d-valent map.

Proof. The implication (1)⇒(3) is clear from the above discussion.

(3)⇒(2): Let $f : M(\rho, \lambda) \to M'(\rho', \lambda')$ be a regular homomorphism of d-valent maps, where M' has just one vertex. By definition of regularity, there is a subgroup K of $Aut(M)$ acting regularly on each fiber $f^{-1}(e')$, e' a dart of M'. Since $f(e)$ and $f(e\rho)$ are incident with the same vertex of M' and both maps are d-valent, the darts in $f^{-1}(e')$ emanate from distinct vertices of M and, since M' has just one vertex, every vertex of M is the tail of precisely one dart in $f^{-1}(e')$. Therefore, K acts regularly on the vertices of M.

(2)⇒(1): Let H be the subgroup of $Aut(M)$ that acts regularly on the vertex set of M. Since automorphisms of M are automorphisms of its underlying graph Γ, Theorem 2.1 implies that Γ is a Cayley graph for H with generating sequence X. In this case, the darts may be represented in the form (A, i), with $A \in H$.

The generators from X label the darts incident with, in particular, the vertex 1_M, the identity automorphism of M. Let e_1 be any dart and let $x_1 \in X$ be the label on this dart. For $1 < i \leq d$, let $e_i = e_1\rho^{i-1}$ be labelled with x_i. Thus, $X = (x_1, x_2, \ldots, x_d)$ is the cyclic sequence of generator-labels on darts incident with 1_M.

Let $A \in H$ be the map automorphism that, as a graph automorphism, is left multiplication by A. Thus, for any $A' \in H$, $A(A', i) = (AA', i)$. In particular, $(A, i) = A(1_M, i)$. Because A is a map automorphism, $(A, i)\rho = (A(1_M, i))\rho = A((1_M, i)\rho)$. By the preceding paragraph, $(1_M, i)\rho = (1_M, i + 1)$. Hence, $(A, i)\rho = (A, i + 1)$, so M has Cayley group H. $\otimes$

3 Maps and homomorphisms

In order to be able to build on Statement (3) of Theorem 2.2, we now introduce more concepts and facts related to maps and (regular) map homomorphisms. Further results on Cayley maps to be presented in Section 4 turn out to be consequences of the general theory of maps and map homomorphisms. The theory itself (at various levels and in various terms) can be found, for example, in [24, 13, 44]. Some of the facts are conveniently explained in the language of group actions, for which we refer to [9, 32]. The group of interest to us is the *dart group* of a map, which is the permutation group $G = \langle \rho, \lambda \rangle$ generated by the rotation ρ and the involution λ. Others [22, 23] call G the monodromy group or the cartographic group.

Example. *In Section 2 we showed that in $\mathcal{M}_{m,n}$ we have $\rho^4 = 1$, $\lambda^2 = 1$, $(\lambda\rho)^4 = 1$ and $(\lambda\rho^2)^k = 1$, where $k = \mathrm{lcm}\{m, n\}$. It follows that the dart* *group of* $\mathcal{M}_{m,n}$ *is a quotient of*

$$G^{(k)} = \langle x, y : x^4 = y^2 = (yx)^4 = (yx^2)^k = 1 \rangle.$$

By definition, the underlying graph of any map $M = M(\rho, \lambda)$ is connected. Thus, if e, e' are darts of M, it is easy to find a sequence of darts $e = e_0, e_1, \ldots, e_m = e'$ such that e_i is obtained from e_{i-1} either by applying some power of ρ, because they have a common end, or by applying λ, to move to an adjacent vertex nearer to the tail of e'. Therefore the dart group $G = \langle \rho, \lambda \rangle$ is transitive on the dart set D. This group acts faithfully on D, i.e., if $e\mu = e$ for all $e \in D$, then $\mu = 1_G$.

The pair (D, G) — and hence the map M itself — is just a particular instance of a *right G-space*, that is, a set endowed with a transitive right group action (which, in our case, is given by permutations in G of the set D). The general theory of group actions [9, 32] is available to us, but we shall make use only of more elementary aspects, which we include next for the sake of completeness.

We henceforth let $G_e = \{\nu \in G : e\nu = e\}$ denote the stabilizer in G of a fixed dart $e \in D$. Then $e\mu = e\mu'$ if and only if $\mu'\mu^{-1} \in G_e$, that is, $G_e\mu = G_e\mu'$. Thus, the mapping $e\mu \mapsto G_e\mu$ defines a bijection between the elements of D and the right cosets of G_e in G, and this correspondence respects the right action by G. This correspondence shows $M = M(\rho, \lambda)$ (i.e., the right G-space (D, G)) is isomorphic to the right coset space $(G/G_e, G)$.

An *automorphism* of the right coset space $(G/G_e, G)$ is a permutation A of G/G_e such that $A(G_e\mu) = (A(G_e))\mu$ for every $\mu \in G$. The *automorphism group* of $(G/G_e, G)$ is the group of all automorphisms of $(G/G_e, G)$.

If $A \in Aut(M)$, then, for any $e \in D$, one has $A(e\rho) = (A(e))\rho$ and $A(e\lambda) = (A(e))\lambda$, or, equivalently,

$$A(e\mu) = (A(e))\mu, \qquad \text{for each } \mu \in G.$$

Thus, automorphisms of M correspond to automorphisms of $(G/G_e, G)$. We shall take advantage of this identification and show $Aut(M) \simeq N(G_e)/G_e$, where $N(G_e)$ is the normalizer of G_e in G.

Let $\nu \in N(G_e)$. Then $\nu G_e = G_e \nu$, which means that left multiplication by ν of G permutes the right cosets of G_e by $\nu(G_e\mu) = G_e(\nu\mu)$. Such a left multiplication respects the right action of G on the right cosets of G_e and hence, by the above formula, induces an automorphism of the coset space $(G/G_e, G)$. Formally, using the abbreviated notation $\bar{\nu} = G_e\nu$ for $\nu \in N(G_e)$, we denote this automorphism by $A_{\bar{\nu}}$; we thus have $A_{\bar{\nu}}(G_e\mu) = G_e(\nu\mu)$ for each $\mu \in G$. Since $\nu G_e = G_e$ if and only if $\nu \in G_e$, the assignment $\bar{\nu} \mapsto A_{\bar{\nu}}$ defines a mapping from $N(G_e)/G_e$ into $Aut(G/G_e, G)$, the automorphism group of $(G/G_e, G)$. This function is a group homomorphism, since $A_{\bar{\nu}}(A_{\bar{\kappa}}(G_e\mu)) = A_{\bar{\nu}}(G_e\kappa\mu) = G_e\nu\kappa\mu = A_{\overline{\nu\kappa}}(G_e\mu)$, for all $\nu, \kappa \in N(G_e)$.

To show that the function is onto, let $A \in Aut(G/G_e, G)$ and let $\nu \in G$ be such that $A(G_e) = G_e\nu$ (since A takes right cosets of G_e to right cosets of G_e, such a ν exists). We prove that $\nu \in N(G_e)$ and $A = A_{\bar{\nu}}$. Indeed, if $\mu \in G_e$, we have $G_e\nu = A(G_e) = A(G_e\mu) = (A(G_e))\mu = G_e\nu\mu$. In particular, $\nu\mu\nu^{-1} \in G_e$, so $\nu G_e \nu^{-1} \subseteq G_e$. The proof of the reverse inclusion follows from the observation that if $A(G_e) = G_e\nu$, then we have $G_e = A^{-1}(G_e\nu) = (A^{-1}(G_e))\nu$, that is, $A^{-1}(G_e) = G_e\nu^{-1}$. The same argument as above shows $\nu^{-1}G_e\nu \subseteq G_e$, so we conclude $\nu \in N(G_e)$. For $\mu \in G$ we have $A(G_e\mu) = (A(G_e))\mu = (G_e\nu)\mu = A_{\bar{\nu}}(G_e\mu)$, so $A = A_{\bar{\nu}}$, as claimed. In view of all the correspondences explained earlier we have proved the first two items in the following [9, 32]:

Proposition 3.1 *Let $M = M(\rho, \lambda)$ be a map with dart group G and let e be any dart. Then:*

(1) $Aut(M) \simeq Aut(G/G_e, G) \simeq N(G_e)/G_e$;

(2) an isomorphism from $N(G_e)/G_e$ onto $Aut(M)$ is given by $\bar{\nu} \mapsto A_{\bar{\nu}}$ where $A_{\bar{\nu}}(e\mu) = e\nu\mu$ for any $\mu \in G$; and

(3) for any two darts e, e', there is an $A \in Aut(M)$ such that $A(e) = e'$

if and only if $G_e = G_{e'}$.

Proof. (3): If $A(e) = e'$, then $\mu \in G_e$ implies $e'\mu = (A(e))\mu = A(e\mu) = A(e) = e'$. Thus, $G_e \subseteq G_{e'}$. Using A^{-1}, we obtain the reverse inclusion.

Conversely, suppose $G_e = G_{e'}$. Since G acts transitively on the dart set, there is a $\mu \in G$ such that $e\mu = e'$. For any $\nu \in G_e$, $e'\mu^{-1}\nu\mu = e\nu\mu = e\mu = e'$, so $\mu^{-1}\nu\mu \in G_{e'} = G_e$. Thus, $\mu \in N(G_e)$, so $A(e\nu) = e\mu\nu$ defines an automorphism that maps e to e'. $\otimes$

Example. *With the aid of Proposition 3.1, we can compute the dart group G of $\mathcal{M}_{m,n}$, from which we will compute its automorphism group.*

Let $k = \mathrm{lcm}\{m, n\}$. We have already shown that the dart groups of the maps $\mathcal{M}_{m,n}$ and $\mathcal{M}_{k,k}$ are both quotients of the group

$$G^{(k)} = \langle x, y : x^4 = y^2 = (yx)^4 = (yx^2)^k = 1 \rangle.$$

On the other hand, consider a vertical dart $e = e^J_{i,j}$, $J \in \{U, D\}$. We noted when we introduced the map $\mathcal{M}_{m,n}$ that $e = e(\lambda\rho^2)^n = e(\rho\lambda\rho)^m$. Recall that $\lambda\rho^2$ and $\rho\lambda\rho$ commute and both have order k. Let $m' = k/m$ and $n' = k/n$. Then G_e contains a subgroup isomorphic to the abelian group $\mathcal{Z}_{m'} \times \mathcal{Z}_{n'}$.

We begin by showing that $|G^{(k)}| \leq 4k^2$. Let $u = yx^2$ and $v = xyx$. There are three crucial properties we need. We note that the first two of these can be "seen" by considering how particular group elements act on an arbitrary dart of $\mathcal{M}_{m,n}$, but we need to prove them for $G^{(k)}$.

Property 1. *$uv = vu$.*

Note that $vu = (xyx)(yx^2) = (xyxyx)x$. Since $1 = (xy)^4$, $(xyxyx) = (yxy)^{-1} = yx^{-1}y$. Therefore, $vu = (yx^{-1}y)x = (yx^3y)x = (yx^2)(xyx) = uv$.

Property 2. *(a) $x^{-1}ux = v^{-1}$; (b) $x^{-1}vx = u$; (c) $yuy = u^{-1}$; and (d) $yvy = v^{-1}$.*

For (a), $x^{-1}ux = x^{-1}yx^3 = x^{-1}yx^{-1} = v^{-1}$. For (b), $x^{-1}vx = yx^2 = u$. For (c), $yuy = x^2y = u^{-1}$. For (d), $yvy = yxyxy = x^{-1}yx^{-1} = v^{-1}$.

Property 3. *Let $K = \langle u, v\rangle$ and $K_{m,n} = \langle u^m, v^n\rangle$. Then K is normal in $G^{(k)}$ and if $m \neq n$, then $N(K_{m,n})$ is generated by u, v and x^2.*

That K is a normal subgroup of $G^{(k)}$ follows immediately from Property 2. For $K_{m,n}$, note that Property 2 implies $x^{-1}K_{m,n}x = K_{n,m}$ (indices are switched) and $yK_{m,n}y = K_{m,n}$ (indices are not switched). Thus, if w is a word in x and y and $m \neq n$, then $w \in N(K_{m,n})$ if and only if $w^{-1}K_{m,n}w = K_{m,n}$ if and only if the sum of the exponents of x in w is even. A simple induction on the number of y's in w shows all such w are generated by $u = x^2y$, $v = xyx$ and x^2.

Note that u and v are conjugates and the order of u is obviously a divisor of k, so they both have order a divisor of k. By Property 1, K is abelian. Therefore, $|K| \leq k^2$. Since $yx^2 = u \in K$, $Kyx^2 = K$, or $Ky = Kx^2$. Therefore the quotient $G^{(k)}/K$ is generated by Kx, so $G^{(k)}/K$ has order at most 4. Thus, $|G^{(k)}| = |G^{(k)}/K||K| \leq 4k^2$, as required.

Clearly $|G| \leq |G^{(k)}|$ and $|G_e| \geq m'n'$. Since (D, G) is isomorphic to $(G/G_e, G)$, we know that $4mn = |D| = |G|/|G_e| \leq 4k^2/(m'n') = 4mn$. Therefore, equality holds throughout, namely $|G| = |G^{(k)}| = 4k^2$ and $|G_e| = |\mathcal{Z}_{m'} \times \mathcal{Z}_{n'}|$. Consequently, $G = G^{(k)}$ and $G_e = \mathcal{Z}_{m'} \times \mathcal{Z}_{n'}$.

It also follows that $|K| = k^2$, $|G^{(k)}/K| = 4$, each of u and v has order k, $G_e \simeq K_{m,n}$ and $|K_{m,n}| = k^2/(mn)$. Moreover, $G^{(k)}$ is the split extension of the normal abelian subgroup $K \simeq \mathcal{Z}_k \times \mathcal{Z}_k$ by the cyclic subgroup $\langle x\rangle \simeq \mathcal{Z}_4$, where conjugation by x takes u to v and v to u^{-1}. Furthermore, if $m \neq n$, then Property 3 shows $N(K_{m,n})$ is generated by u, v, and x^2. It follows that $Aut(\mathcal{M}_{m,n}) \simeq N(G_e)/G_e \simeq N(K_{m,n})/K_{m,n}$ is the split extension of the abelian group $\mathcal{Z}_m \times \mathcal{Z}_n$ (generated by the images of u and v) by $\mathcal{Z}_2$ (generated by x^2), where conjugation by $\mathcal{Z}_2$ inverts all the elements of $\mathcal{Z}_m \times \mathcal{Z}_n$. Thus, the automorphisms A, B, and C of $\mathcal{M}_{m,n}$ given in the previous section generate $Aut(\mathcal{M}_{m,n})$, as expected.

Cayley maps for which there is an edge e with $G_e = \{1_G\}$ will be of particular interest to us later. For such group actions we have the following facts.

Proposition 3.2 *Let G be a group that acts transitively on a set D. Consider the following five statements.*

(1) For some $e \in D$, $G_e = \{1_G\}$.

(2) For every $e \in D$, $G_e = \{1_G\}$.

(3) The correspondence $\nu \leftrightarrow A_{\bar{\nu}}$ is an isomorphism between G and $Aut(D, G)$.

(4) $Aut(D, G)$ acts transitively on D.

(5) For every $e, e' \in D$, $G_e = G_{e'}$.

Then (1), (2) and (3) are equivalent, (4) and (5) are equivalent, (1) implies (4), and if G is faithful, then (4) implies (1).

Proof. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (3). Suppose that, for the dart e, $G_e = \{1_G\}$. Then the action of G is faithful and $N(G_e) = G$, so $Aut(D, G) \simeq N(G_e)/G_e \simeq G$. In particular, the function f defined by $f(\nu) = A_{\bar{\nu}}$ is an isomorphism between G and $Aut(D, G)$.

(3) $\Rightarrow$ (2). Let $e \in D$. If f is an isomorphism, then, in particular, it is an injection. Since $A_{\bar{\mu}} = A_{\bar{\nu}}$ whenever $G_e\mu = G_e\nu$, f is an injection only if $G_e = \{1_G\}$. Thus, if f is an isomorphism, then $G_e = \{1_G\}$.

(1) $\Rightarrow$ (4). Suppose G_e is trivial. For $e' \in D$, by transitivity there is a $\mu \in G$ such that $e' = e\mu$. Then $A_{\bar{\mu}}(e) = A_{\bar{\mu}}(e1_G) = e\mu 1_G = e\mu = e'$, so $Aut(D, G)$ acts transitively on D.

(4) $\Leftrightarrow$ (5). This is immediate from Proposition 3.1 (3).

(5) $\Rightarrow$ (1). Since the action by G is faithful, $\cap_{e \in D} G_e = \{1_G\}$, and the result follows. $\otimes$

Let $\tilde{M}, M$ be two maps with dart groups $\tilde{G} = \langle \tilde{\rho}, \tilde{\lambda} \rangle$ and $G = \langle \rho, \lambda \rangle$, respectively. Let $f : \tilde{M} \to M$ be a map homomorphism. The first observation is that f gives rise to a group homomorphism $\phi : \tilde{G} \to G$: given $\tilde{\mu} \in \tilde{G}$, we simply write $\tilde{\mu}$ as a word in $\tilde{\rho}$ and $\tilde{\lambda}$ and define $\phi(\tilde{\mu})$ to be the same word but in ρ and λ.

To see that that ϕ is well-defined, suppose that $\tilde{\omega}$ is a word in $\tilde{\rho}$ and $\tilde{\lambda}$ such that $\tilde{\omega} = 1_{\tilde{G}}$. We show that for the corresponding word ω in ρ and λ we have $\omega = 1_G$. For any dart $\tilde{e} \in \tilde{D}$, we have $f(\tilde{e}) = f(\tilde{e}\tilde{\omega}) = f(\tilde{e})\omega$ and hence ω fixes every dart in D. But the action of the dart group G on D is faithful and therefore $\omega = 1_G$.

Therefore by Dyck's Theorem [11], ϕ is also a group homomorphism. (We comment that we shall use Dyck's theorem several times and will not make specific reference to it again.) Note also that $f(\tilde{e}\tilde{\mu}) = f(\tilde{e})\phi(\tilde{\mu})$, for each dart $\tilde{e} \in \tilde{D}$ and each $\tilde{\mu} \in \tilde{G}$.

If $e = f(\tilde{e})$, then the relation $f(\tilde{e}\tilde{\mu}) = e\phi(\tilde{\mu})$, for all $\tilde{\mu} \in \tilde{G}$, shows that $\tilde{\mu} \in \tilde{G}_{\tilde{e}}$ implies $\phi(\tilde{\mu}) \in G_e$, that is, $\phi(\tilde{G}_{\tilde{e}}) \leq G_e$. Conversely, if $\phi : \tilde{G} \to G$ is a group homomorphism such that $\phi(\tilde{\rho}) = \rho$, $\phi(\tilde{\lambda}) = \lambda$ and $\phi(\tilde{G}_{\tilde{e}}) \leq G_e$, then, for any $\tilde{\mu}, \tilde{\mu}' \in G$, from $\tilde{e}\tilde{\mu} = \tilde{e}\tilde{\mu}'$ we have $\tilde{\mu}'\tilde{\mu}^{-1} \in \tilde{G}_{\tilde{e}}$, and as $\phi(\tilde{\mu}'\tilde{\mu}^{-1}) \in G_e$ we see that $e\phi(\tilde{\mu}) = e\phi(\tilde{\mu}')$. This shows that the assignment $\tilde{e}\tilde{\mu} \mapsto e\phi(\tilde{\mu})$, $\tilde{\mu} \in \tilde{G}$, gives a well-defined map homomorphism $\tilde{M} \to M$ that sends $\tilde{e}$ onto e.

Note that since ρ and λ generate G, ϕ is a surjection.

The preceding paragraphs are summarized in the first part of the following result. We use the usual notation $K \trianglelefteq G$ to indicate K is a normal subgroup of G.

Proposition 3.3 *Let $\tilde{M}$ and M be maps with dart sets $\tilde{D}$ and D and dart groups $\tilde{G} = \langle \tilde{\rho}, \tilde{\lambda} \rangle$ and $G = \langle \rho, \lambda \rangle$, respectively. Let $\tilde{e} \in \tilde{D}$ and $e \in D$. There exists a map homomorphism $f : \tilde{M} \to M$ with $f(\tilde{e}) = e$ if and only if there is a group epimorphism $\phi : \tilde{G} \to G$ taking $\tilde{\rho}$ onto ρ and $\tilde{\lambda}$ onto λ such that*

$\phi(\tilde{G}_{\tilde{e}}) \leq G_e$, *in which case the following diagram commutes:*

$$\begin{array}{ccc} \tilde{D} & \xrightarrow{f} & D \\ \mu \downarrow & & \downarrow \phi(\mu) \\ \tilde{D} & \xrightarrow{f} & D \end{array}$$

Moreover, f is a regular homomorphism if and only if $\tilde{G}_{\tilde{e}} \trianglelefteq \phi^{-1}(G_e)$, in which case $M \simeq \tilde{M}/\tilde{K}$ where $\tilde{K} \simeq \phi^{-1}(G_e)/\tilde{G}_{\tilde{e}}$.

Proof. We need only prove the "Moreover" assertion. Suppose the map homomorphism $f : \tilde{M} \to M$ is regular, with associated subgroup $\tilde{K}$ of $Aut(\tilde{M})$; in particular, for $\hat{e} \in \tilde{D}$, $f(\tilde{e}) = f(\hat{e})$ if and only if $\hat{e} = \tilde{A}(\tilde{e})$ for some automorphism $\tilde{A} \in \tilde{K}$.

Invoking Proposition 3.1 (2) and the first part of this proposition, there exists a uniquely determined subgroup $\tilde{G}_{\tilde{K}}$ of $N(\tilde{G}_{\tilde{e}})$ such that $\tilde{G}_{\tilde{K}}/\tilde{G}_{\tilde{e}}$ is isomorphic to $\tilde{K}$ via the assignment $\bar{\nu} \mapsto \tilde{A}_{\bar{\nu}}$, where $\tilde{A}_{\bar{\nu}}(\tilde{e}\mu) = \tilde{e}\nu\mu$ for all $\mu \in \tilde{G}$.

Now $f(\tilde{e}) = f(\hat{e})$ if and only if $\hat{e} = \tilde{A}(\tilde{e})$ for some $\tilde{A} \in \tilde{K}$ if and only if $\hat{e} = \tilde{e}\nu$ for some $\nu \in \tilde{G}_{\tilde{K}}$. Note that if also $\hat{e} = \tilde{e}\mu$, for some $\mu \in \tilde{G}$, then $\mu\nu^{-1} \in \tilde{G}_{\tilde{e}} \subset \tilde{G}_{\tilde{K}}$ and so $\mu \in \tilde{G}_{\tilde{K}}$ as well. We therefore have $f(\tilde{e}) = f(\tilde{e}\nu)$ if and only if $\nu \in \tilde{G}_{\tilde{K}}$.

Furthermore, for $\nu \in \tilde{G}$, $f(\tilde{e}\nu) = f(\tilde{e})\phi(\nu)$, and hence $f(\tilde{e}) = f(\tilde{e}\nu)$ if and only if $\phi(\nu) \in G_{f(\tilde{e})}$. It follows that $\nu \in \tilde{G}_{\tilde{K}}$ if and only if $\phi(\nu) \in G_{f(\tilde{e})}$, that is, $\tilde{G}_{\tilde{K}} = \phi^{-1}(G_{f(\tilde{e})})$, and hence $\tilde{G}_{\tilde{e}} \trianglelefteq \phi^{-1}(G_{f(\tilde{e})})$.

Conversely, let ϕ have the property that $\tilde{G}_{\tilde{e}} \trianglelefteq \phi^{-1}(G_{f(\tilde{e})})$. Using the relation $f(\tilde{e}\mu) = f(\tilde{e})\phi(\mu)$, we see that $f(\tilde{e}\mu) = f(\tilde{e}\mu')$ if and only if $\phi(\mu'\mu^{-1}) \in G_{f(\tilde{e})}$, that is, $\mu'\mu^{-1} \in \phi^{-1}(G_{f(\tilde{e})})$. Consequently, for $\nu = \mu'\mu^{-1}$, $\bar{\nu} = \tilde{G}_{\tilde{e}}\nu \in \phi^{-1}(G_{f(\tilde{e})})/\tilde{G}_{\tilde{e}} \leq N(\tilde{G}_{\tilde{e}})/\tilde{G}_{\tilde{e}}$. Let $\tilde{A}_{\bar{\nu}}$ be the corresponding automorphism. From above, we have $f(\tilde{e}\mu) = f(\tilde{e}\mu')$ if and only if $\tilde{A}_{\bar{\nu}}(\tilde{e}\mu) = \tilde{e}\mu'$, which shows that f is a regular homomorphism associated with the subgroup

$\tilde{K} = \{\tilde{A}_{\bar{\nu}} : \bar{\nu} \in \phi^{-1}(G_{f(\tilde{e})})/\tilde{G}_{\tilde{e}}\}$ of $Aut(\tilde{M})$. $\otimes$

Example. *The map homomorphism $e^J_{i,j} \mapsto e^J$ from $\mathcal{M}_{m,n}$ to $\mathcal{M}_1$ is associated with the group homomorphism from the dart group $G = \langle \rho, \lambda : \rho^4 = \lambda^2 = (\rho\lambda)^4 = (\lambda\rho^2)^k = 1 \rangle$ to $\mathcal{Z}_4$ obtained by adding the relation $\lambda\rho^2 = 1$.*

It should be noted in Proposition 3.3 that ϕ need not take G_e *onto* $G'_{e'}$; in fact, it is possible for $\phi : G \to G'$ to be an isomorphism and *not* induce a map automorphism.

Example. *Let M be the icosahedron described as follows. We identify the darts of M with the elements of $A_5 = \langle \eta, \tau \rangle$, the alternating group on the set $\{1, 2, 3, 4, 5\}$, where $\eta = (1\,2\,3\,4\,5)$ and $\tau = (1\,2)(3\,4)(5)$. Define ρ and λ on D so that, for $\mu \in D$, $(\mu)\rho = \mu\eta$ and $(\mu)\lambda = \mu\tau$. Then ρ is the rotation and λ is the dart-reversing involution. Observe that the orbits of ρ are the twelve 5-cycles (corresponding to the 12 vertices of M), the orbits of λ are 30 2-cycles (corresponding to the 30 edges of M), and the orbits of $\lambda\rho$ are 20 3-cycles (corresponding to the 20 triangular faces).*

Let M' be the one-vertex, five-valent map with dart set $D' = \{1, 2, 3, 4, 5\}$, rotation $\rho' = (1\,2\,3\,4\,5)$ and dart-reversing involution $\lambda' = (1\,2)(3\,4)(5)$. Since G and $G' = \langle \rho', \lambda' \rangle$ are both isomorphic to A_5, the group homomorphism $\phi : G \to G'$ defined by $\phi(\rho) = \rho'$ and $\phi(\lambda) = \lambda'$ is a group isomorphism. Given $e \in D$ and $e' \in D'$, G_e is trivial while $G'_{e'}$ consists of the elements of G' that fix e', and hence $G'_{e'} \simeq A_4$. Thus, ϕ takes G_e into a proper subgroup of $G'_{e'}$. Since G_e is trivially normal in $\phi^{-1}(G'_{e'})$, by Proposition 3.3, ϕ induces a regular map homomorphism from M onto M'. Moreover, as ϕ preserves valences, by Theorem 2.2 we see that M, the map of an icosahedron, is a Cayley map (with Cayley group $\phi^{-1}(G'_{e'})/G_e \simeq A_4$).

The map of the icosahedron, like the map of any Platonic solid, has the property that its automorphism group acts transitively on its dart set. We shall focus on maps with this property for the remainder of this section. As mentioned earlier the automorphism group of any map acts semi-regularly on its dart set. A map M is a *regular* if its map automorphism group $Aut(M)$ acts transitively (and hence regularly) on D. Thus, if M is a regular map with dart group G, then, for any $e \in D$, there is a bijection $Aut(M) \to D$ given by $A \mapsto A(e)$ for all $A \in Aut(M)$. We point out that Proposition 3.2 provides several statements about a map M that are all equivalent to M being regular.

Example. *We claim that the map $\mathcal{M}_{m,n}$ is regular if and only if $m = n$. In particular, suppose there is an automorphism F of $\mathcal{M}_{m,n}$ that takes $e^U_{0,0}$ to $e^R_{0,0}$. Then $e^R_{0,0} = F(e^U_{0,0}) = F(e^U_{0,0}(\lambda\rho^2)^n) = (F(e^U_{0,0}))(\lambda\rho^2)^n = e^R_{0,0}(\lambda\rho^2)^n = e^R_{n,0}$ and so $m|n$. On the other hand, $e^R_{0,0} = F(e^U_{0,0}) = F(e^U_{0,0}(\rho\lambda\rho)^m) = e^R_{0,-m}$, so $n|m$. Hence $m = n$.*

From what we have seen earlier, the map $\mathcal{M}_{n,n}$ is regular.

For a regular map M, we may identify the dart set of M with its dart group G so that the specific dart e corresponds to the unit element of G; the right action of G on itself is then the usual action of the dart group $G = \langle \rho, \lambda \rangle$ on the dart set $D = G$, whereas the left action of G on $D = G$ (by left multiplication) can be identified with the action of the automorphism group $Aut(M)$. In other words, we may identify the regular map M with the right G-space (D, G) where $D = G$, that is, with the G-space (G, G). Clearly, in a regular map all vertices have the same valence d and all faces have the same covalence ℓ; the dart group G of M has a presentation of the form $\langle \rho, \lambda : \rho^d = \lambda^2 = (\lambda\rho)^\ell = \ldots = 1_G \rangle$.

Conversely, let G be any group with a presentation of the form $\langle x, y : x^d = y^2 = (yx)^\ell = \ldots = 1_G \rangle$. The map $Map\langle x, y \rangle$ has as darts the elements of G, and ρ and λ are defined by $z\rho = zx$ and $z\lambda = zy$. The corresponding right G-space is (G, G), and so $Map\langle x, y \rangle$ is regular. In this sense, the study of regular maps is equivalent to the study of groups generated by two elements, one of which is an involution. This connection is used throughout the remainder of this work.

Now let M be an arbitrary map (with arbitrary finite valences and covalences) with dart group $G = \langle \rho, \lambda \rangle$. Following [22], the *generic regular cover* of M is the map $Map\langle \rho, \lambda \rangle$. (This use of the word "cover" will be justified by Proposition 3.4.)

Example. *The icosahedron is the generic regular cover of the one-vertex map having* $\rho = (1\,2\,3\,4\,5)$ *and* $\lambda = (1\,2)(3\,4)(5)$. *The generic regular cover of* $\mathcal{M}_{m,n}$ *is* $\mathcal{M}_{k,k}$, *where* $k = \mathrm{lcm}\{m, n\}$.

We note that if $M = M(\rho, \lambda)$ is not regular, then the map homomorphism from the generic regular cover $Map\langle \rho, \lambda \rangle$ to M induces a group isomorphism between the (identical) dart groups, but M and $Map\langle \rho, \lambda \rangle$ are not isomorphic (cf. Proposition 3.3).

Proposition 3.4 *Let M be a map with dart group $G = \langle \rho, \lambda \rangle$.*

(1) The identity group isomorphism $\iota_G : G \to G$ defines a regular map homomorphism $f_{\iota_G} : Map\langle \rho, \lambda \rangle \to M$. The homomorphism f_{ι_G} is an isomorphism if and only if M is regular.

Suppose further that $\tilde{M}$ is a regular map and $\tilde{f} : \tilde{M} \to M$ is a map homomorphism. Then:

(2) $\tilde{f}$ is a regular homomorphism;

(3) $\tilde{f}$ factors through $Map\langle \rho, \lambda \rangle$, i.e., there is a map homomorphism $f : \tilde{M} \to Map\langle \rho, \lambda \rangle$ such that $\tilde{f} = f_{\iota_G} \circ f$. (In other words, $Map\langle \rho, \lambda \rangle$ is the smallest regular map that covers M.)

(4) Let K be the subgroup of $Aut(\tilde{M})$ associated with $\tilde{f}$. Then M is regular if and only if $K \trianglelefteq Aut(\tilde{M})$. Moreover, if $K \trianglelefteq Aut(\tilde{M})$, then $Aut(M) \simeq Aut(\tilde{M})/K$ and if A is an automorphism of M, then there is an automorphism $\tilde{A}$ of $\tilde{M}$ such that $\tilde{f}\tilde{A} = A\tilde{f}$.

Proof. (1): Since $M' = Map\langle \rho, \lambda \rangle$ is regular, $G_e = \{1_G\}$ for every dart

e of M'. Thus, $G_e \trianglelefteq \iota^{-1}(G_{f(e)})$. By Proposition 3.3, f_{ι_G} is a regular map homomorphism.

If f_{ι_G} is an isomorphism, then the regularity of $Map\langle\rho,\lambda\rangle$ obviously implies that M is regular. Conversely, if M is regular, Proposition 3.3 implies $M \simeq Map\langle\rho,\lambda\rangle/K$, with $K = \iota^{-1}(G_{f(e)})/G_e$. By regularity of M, $G_{f(e)}$ is $\{1_G\}$, showing that $K = \{1_G\}$, whence f_{ι_G} is an isomorphism.

(2) Let $\phi : \tilde{G} \to G$ be the group homomorphism from the dart group $\tilde{G}$ of $\tilde{M}$ to G. Since $\tilde{M}$ is regular, G_e is trivial for every e in the dart set $\tilde{D}$ of $\tilde{M}$. Thus, $\phi(G_e) \trianglelefteq G_{f(e)}$, for every $e \in \tilde{D}$. By Proposition 3.3, $\tilde{f}$ is regular.

(3) By (2), $\tilde{f}$ is regular. Fix $\tilde{e}_1$ and let $\tilde{f}(\tilde{e}_1) = e_1$. By Proposition 3.3, there is a group homomorphism $\phi : \tilde{G} \to G$ such that $\phi(\tilde{G}_{\tilde{e}_1}) \leq G_{e_1}$. Since $\tilde{G}_{\tilde{e}_1} = \{1_{\tilde{G}}\}$, the same proposition implies there is a map homomorphism $f : \tilde{M} \to Map\langle\rho,\lambda\rangle$. Let $\hat{e}_1$ be a dart of $Map\langle\rho,\lambda\rangle$ such that $f_{\iota_G}(\hat{e}_1) = e_1$. Since $\tilde{M}$ is regular, we may choose f so that $f(\tilde{e}_1) = \hat{e}_1$, from which it follows that $\tilde{f} = f_{\iota_G} \circ f$.

(4) Since $\tilde{f}$ is regular, by Proposition 3.3 the associated subgroup K of $Aut(M)$ is isomorphic to $\phi^{-1}(G_{f(e)})/\tilde{G}_e$. Since $\tilde{M}$ is regular, $\tilde{G}_e = \{1_{\tilde{G}}\}$. If M is regular, then $G_{f(e)} = \{1_G\}$, so $K \simeq \mathrm{Ker}(\phi) \trianglelefteq \tilde{G} \simeq (Aut(\tilde{M}))$.

Conversely, suppose $K \trianglelefteq (Aut(\tilde{M}))$. The darts of $M \simeq \tilde{M}/K$ are the orbits of the darts of $\tilde{M}$ under the left action by K. If $\tilde{e}, \tilde{e}' \in \tilde{D}$, then there exists an $\tilde{A} \in Aut(\tilde{M})$ such that $\tilde{A}(\tilde{e}) = \tilde{e}'$. Since $K \trianglelefteq Aut(\tilde{M})$, $\tilde{A}(K\tilde{e}) = K(\tilde{A}(e)) = K\tilde{e}'$. Thus, M is regular.

Moreover, this last argument shows how automorphisms of M lift to automorphisms of $\tilde{M}$ and, more precisely, $Aut(M) \simeq Aut(\tilde{M})/K$. $\otimes$

Automorphism groups of (not necessarily regular) Cayley maps are studied more fully in Section 6.

4 Characterizations of Cayley maps

In this section, we present various characterizations of Cayley maps. In general, the same map M may be isomorphic to a Cayley map $CM(H, X, \tau)$ for different groups H of the same order; even for the same group H, different generating sequences X (and different distributions of inverses τ) may still yield isomorphic maps. Therefore, in what follows, the statement "*M is a Cayley map*" means that there exists a group H, a generating sequence X and a distribution of inverses τ such that M is isomorphic to the Cayley map $CM(H, X, \tau)$.

We start with a general characterization of d-valent Cayley maps in terms of certain homomorphisms into $Sym[d]$, the symmetric group on the set $[d]$. For a subgroup L of $Sym[d]$, we denote by L_j the stabilizer in L of the symbol $j \in [d]$, that is, the subgroup of all $\theta \in L$ such that $j\theta = j$. The identity of $Sym[d]$ is denoted $1_{[d]}$ and we continue to use η for the d-cycle $(1\,2\,\ldots\,d)$.

Theorem 4.1 *Let M be a d-valent map with dart group $G = \langle \rho, \lambda \rangle$, and let e be a dart of M. Then M is a Cayley map if and only if there is a group homomorphism $\phi : G \to Sym[d]$ taking ρ to the cycle η such that $G_e \trianglelefteq \phi^{-1}(\phi(G)_j)$ for some $j \in [d]$. In such a case, M is a Cayley map with Cayley group $\phi^{-1}(\phi(G)_j)/G_e$ and distribution of inverses $\tau = \phi(\lambda)$.*

Proof. By Theorem 2.2, M is a Cayley map if and only if there is a regular homomorphism f from M onto a d-valent one-vertex map $M' = M'(\rho', \lambda')$. However, as pointed out in the paragraph preceding Theorem 2.2, the dart set of M' can be identified with the set $[d]$ and we may set $\rho' = \eta$; the dart group $G' = \langle \rho', \lambda' \rangle$ of M' then becomes a subgroup of $Sym[d]$. Let $j \in [d]$ be the dart for which $f(e) = j$. According to Proposition 3.3, a regular homomorphism $f : M \to M'$ with $f(e) = j$ exists if and only if there is a group homomorphism $\phi : G \to Sym[d]$ such that $\phi(\rho) = \rho'$, $\phi(\lambda) = \lambda'$, and $G_e \trianglelefteq \phi^{-1}(\phi(G)_j)$; moreover, f is then associated with the subgroup $H \simeq \phi^{-1}(\phi(G)_j)/G_e$. Recalling the proof of Theorem 2.2, the map

M has Cayley group H and distribution of inverses given by the (involutory) permutation $\lambda' = \phi(\lambda) \in Sym[d]$. $\otimes$

Example. *We know from the example following Proposition 3.1 that the dart group G of $\mathcal{M}_{m,n}$ is*

$$G^{(k)} = \langle \rho, \lambda : \rho^4 = \lambda^2 = (\rho\lambda)^4 = (\lambda\rho^2)^k = 1 \rangle,$$

where $k = \text{lcm}\{m, n\}$, that $K = \langle \rho^2\lambda, \rho\lambda\rho \rangle \trianglelefteq G^{(k)}$ and that $K \simeq \mathcal{Z}_k \times \mathcal{Z}_k$. Since G/K is isomorphic to the cyclic group $\langle \eta \rangle$, we can view the canonical isomorphism as mapping ρ to η and λ to $\eta^2 = (1\ 3)(2\ 4)$. We have also shown for a U-dart e that $G_e = \langle \lambda\rho^2, \lambda\rho\lambda \rangle$. Therefore, as expected, $\mathcal{M}_{m,n}$ is a Cayley map $CM(H, X, \tau)$ with $H \simeq K/G_e \simeq \mathcal{Z}_m \times \mathcal{Z}_n$ and distribution of inverses $\tau = (1\ 3)(2\ 4)$.

If we wish to specify the distribution of inverses in advance, we have the following obvious corollary.

Corollary 4.2 *Let M be a d-valent map with dart group $G = \langle \rho, \lambda \rangle$ and let e be a dart of M. Then M is a Cayley map with a given distribution of inverses τ if and only if there is a group homomorphism $\phi : G \to Sym[d]$ for which $\phi(\rho) = \eta$ and $\phi(\lambda) = \tau$ and such that $G_e \trianglelefteq \phi^{-1}(\phi(G)_j)$ for some $j \in [d]$.* $\otimes$

The above results can be turned into a characterization of Cayley maps stated purely in terms of dart groups, without reference to any homomorphisms, as follows.

Theorem 4.3 *Let M be a map with dart group $G = \langle \rho, \lambda \rangle$ and let e be a dart of M. Then, M is a Cayley map if and only if there exists a subgroup K of G such that $K \cdot \langle \rho \rangle = G$, $K \cap \langle \rho \rangle = \{1_G\}$, and $G_e \trianglelefteq K$. In such a case the Cayley group is K/G_e.*

Proof. Let M be a d-valent Cayley map. By Theorem 4.1 there is group homomorphism $\phi : G \to Sym[d]$ for which $\phi(\rho) = \eta$ and such that, for some $j \in [d]$, $G_e \trianglelefteq \phi^{-1}(\phi(G)_j)$. Set $K = \phi^{-1}(\phi(G)_j)$; then $G_e \trianglelefteq K$.

Suppose $\rho^i \in K$, for some $i \in [d]$. Then we have $\eta^i = \phi(\rho^i) \in \phi(G)_j$, that is, $j = j(\phi(\rho^i)) = j\eta^i$. This is possible only if $i = d$, whence $K \cap \langle\rho\rangle = \{1_G\}$.

Let μ be an arbitrary element of G and let $i \in [d]$ be such that $j\phi(\mu) \equiv j+i \pmod d$. Then, $j\phi(\mu\rho^{-i}) = (j+i)\phi(\rho^{-i}) = j$, and so $\phi(\mu\rho^{-i}) \in \phi(G)_j$, that is, $\mu\rho^{-i} \in K$, and $\mu \in K\rho^i$. This shows that $\bigcup_{i\in[d]} K\rho^i = G$, i.e., $K \cdot \langle\rho\rangle = G$.

Conversely, let G contain a subgroup K such that $K \cdot \langle\rho\rangle = G$, $K \cap \langle\rho\rangle = \{1_G\}$, and $G_e \trianglelefteq K$; let d be the order of ρ in G. Define $\phi : G \to Sym[d]$ by letting $\phi(\mu)$ be the permutation ϕ_μ that is defined by $K\rho^i\mu = K(\rho^{i\phi_\mu})$. It is easy to see that ϕ is a group homomorphism. Taking $\mu = \rho$, we get $i\phi_\mu = i+1$, so $\phi(\rho) = \eta$.

Moreover, since $\rho^d = 1$, for each $\mu \in G$ we have $d\phi_\mu = d$ if and only if $\mu \in K$. Thus, K coincides with $\phi^{-1}(\phi(G)_d)$. As $G_e \trianglelefteq K = \phi^{-1}(\phi(G)_d)$, the homomorphism ϕ has all the properties stated in Theorem 4.1, which proves that M is a Cayley map. $\otimes$

Let M be a d-valent Cayley map with dart group. It is not completely obvious how to recover the generating sequence X (i.e., a representation of M as a Cayley map $CM(H, X, \tau)$) from the homomorphism $\phi : G \to Sym[d]$. Before we explain how to recover X, note that Theorem 4.1 implies $H = K/G_e$, where $K = \phi^{-1}(\phi(G)_j)$. Therefore, the left translations in the representation $M = CM(H, X, \tau)$ to be constructed will correspond to the map automorphisms $A_{\bar\nu} : e\mu \mapsto e\nu\mu$, for $\nu \in K$ (and for any $\mu \in G$). In particular, if we make u the vertex that is labelled 1_H in the Cayley map, then the generators in X will correspond to the left translations that take u onto its neighbors.

More explicitly, let $e' = e\rho^i$ be a dart from u to some other vertex v. Our goal is to compute the generator that labels e'. Let $\nu \in K$ be such that $\bar{\nu} = \nu G_e$ yields the left translation $A_{\bar{\nu}}$ that takes u to v. Since $e'\lambda$ emanates from v, $A_{\bar{\nu}}^{-1}(e'\lambda)$ emanates from u, so there is an $i' \in [d]$ such that $A_{\bar{\nu}}^{-1}(e'\lambda) = e\rho^{i'}$.

By the definition of $A_{\bar{\nu}}$, $e\rho^i\lambda = e'\lambda = A_{\bar{\nu}}(e\rho^{i'}) = e\nu\rho^{i'}$. Thus, $e\nu = e\rho^i\lambda\rho^{-i'}$. Since G_e is normal in K, $\nu G_e = (\rho^i\lambda\rho^{-i'})G_e$. Thus, in the Cayley representation of M, the dart $e\rho^i$ is assigned the label $\bar{\nu} = (\rho^i\lambda\rho^{-i'})G_e$.

But we can be more precise and determine i'. The group homomorphism $\phi : G \to Sym[d]$ induces a regular map homomorphism $f : M \to M_1$, where M_1 is the one-vertex map with dart set $\{1, 2, \ldots, d\}$, rotation $\rho_1 = \eta$, and dart-reversing involution $\lambda_1 = \phi(\lambda) = \tau$. Explicitly, f is given by first choosing $j \in [d]$ such that $G_e \trianglelefteq \phi^{-1}(\phi(G)_j)$, defining $f(e)$ to be j, and, for $\mu \in G$, $f(e\mu) = f(e)\phi(\mu)$. (To see that f is well-defined, suppose $e\mu = e\mu'$. Then $\mu'\mu^{-1} \in G_e$, so $j\phi(\mu'\mu^{-1}) = j$, i.e., $f(e)\phi(\mu'\mu^{-1}) = f(e)$. Thus, $f(e\mu) = f(e\mu')$, as required.)

If A is any left translation, then $A(e\rho^{i'}) = e\rho^i\lambda$ if and only if $f(e\rho^{i'}) = f(e\rho^i\lambda)$. Since $f(e) = j$ and $f(e\mu) = f(e)\phi(\mu)$, we have (modulo d) that $j + i' = j\rho_1^{i'} = f(e)\phi(\rho^{i'}) = f(e\rho^{i'}) = f(e\rho^i\lambda) = f(e)\phi(\rho^i\lambda) = j\rho_1^i\tau = (j+i)\tau$. Thus, $i' = (j+i)\tau - j$, and the generator on the dart $e\rho^i$ is $\rho^i\lambda\rho^{(j-(j+i)\tau)}G_e \in H$.

If $G_e \trianglelefteq \phi^{-1}(\phi(G)_j)$ and $e' = e\rho^{d-j}$, then $G_{e'} = \rho^j G_e \rho^{-j}$ and $\phi^{-1}(\phi(G)_d) = \rho^j\phi^{-1}(\phi(G)_j)\rho^{-j}$, so $G_{e'} \trianglelefteq \phi^{-1}(\phi(G)_d)$. Therefore, we can choose e so that $j = d$, in which case the generator on the dart $e\rho^i$ has the form $\rho^i\lambda\rho^{-i\tau}G_e$. Now the cyclic sequence of generators around the vertex u is simply $\rho\lambda\rho^{-1\tau}G_e$, $\rho^2\lambda\rho^{-2\tau}G_e$, …, $\rho^d\lambda\rho^{-d\tau}G_e$.

We can get even more information from this. It is an easy induction to verify that $\{\rho^i\lambda\rho^{-i\tau} : i = 1, 2, \ldots, d\}$ is a generating set for $\phi^{-1}(\phi(G)_d)$. Furthermore, assuming G_e is a subgroup of $\phi^{-1}(\phi(G)_d)$, it is a normal subgroup if and only if, for each $\mu \in G_e$ and each $i \in [d]$, $e\rho^i\lambda\rho^{-i\tau}\mu = e\rho^i\lambda\rho^{-i\tau}$. Letting $e_i = e\rho^i\lambda\rho^{-i\tau}$, this is equivalent to $e_i\mu = e_i$. Thus, G_e is a normal subgroup of $\phi^{-1}(\phi(G)_d)$ if and only if, for each i, $G_e = G_{e_i}$. This in turn is equivalent, by Proposition 3.1 (3), to the existence, for each i, of automorphisms taking e to each e_i.

In summary, we have:

Proposition 4.4 *For any map $M = M(\rho, \lambda)$ with dart group G and group homomorphism $\phi : G \to Sym[d]$ such that $\phi(\rho) = \eta$ and $\phi(\lambda) = \tau$:*

(1) The elements $\rho^i \lambda \rho^{-i\tau}$ generate $\phi^{-1}(\phi(G)_d)$.

(2) Let e be a dart such that $G_e \leq \phi^{-1}(\phi(G)_d)$. Then G_e is a normal subgroup of $\phi^{-1}(\phi(G)_d)$ if and only if, for each i, there is an automorphism of M taking e to $e\rho^i \lambda \rho^{-i\tau}$.

(3) If $G_e \trianglelefteq \phi^{-1}(\phi(G)_d)$, then $M = CM(H, X, \tau)$, with $H = \phi^{-1}(\phi(G)_d)/G_e$ and $X = (x_1, x_2, \ldots, x_d)$, where, for $i = 1, 2, \ldots, d$, $x_i = \rho^i \lambda \rho^{-i\tau} G_e$.

Example. *We can use the above information to determine all the ways in which the maps $\mathcal{M}_{m,n}$ can be Cayley maps. By Corollary 4.2, it suffices to determine all the homomorphisms $\phi : G \to Sym[4]$ such that $\eta = \phi(\rho)$. Since λ is an involution, $\tau = \phi(\lambda)$ (which, together with η, completely determines the homomorphism) must also be an involution. Up to conjugation by η, there are only the five possibilities $\tau = 1_{[4]}$, $(1\,2)$, $(1\,3)$, $(1\,2)(3\,4)$ and $(1\,3)(2\,4)$. In each case, we must determine whether ϕ actually is a homomorphism and if $G_e \trianglelefteq \phi^{-1}(\phi(G)_4)$.*

Cayley maps in the torus have a long history and are discussed in detail by Gross and Tucker in [13]. In particular, Theorem 6.3.3 in [13] classifies by partial presentations all groups having a toroidal Cayley map, and the corollary of this result relates these groups to Euclidean space (or crystallographic) groups. In each of the cases below, we give a partial presentation for the underlying Cayley group H, together with a reference to the corresponding presentation from [13] (e.g., GTb_1 means presentation (b_1) from Theorem 6.3.3) and the corresponding Euclidean space group (using the standard notation $p1$, $p2$, etc., as in [13] or [8]).

For $J \in \{U, R, D, L\}$, let $\mathcal{J}$ denote the set of all darts of the form $e^J_{i,j}$, $i \in [m]$, $j \in [n]$. If $e, e' \in \mathcal{U} \cup \mathcal{D}$ or $e, e' \in \mathcal{R} \cup \mathcal{L}$, then there is an automorphism of $\mathcal{M}_{m,n}$ taking e to e'.

The function $e \mapsto e\lambda$ stabilizes both $\mathcal{U} \cup \mathcal{D}$ and $\mathcal{R} \cup \mathcal{L}$, while $e \mapsto e\rho$ interchanges $\mathcal{U} \cup \mathcal{D}$ and $\mathcal{R} \cup \mathcal{L}$. Thus $\mu \in G$ stabilizes both $\mathcal{U} \cup \mathcal{D}$ and $\mathcal{R} \cup \mathcal{L}$ if and only if the sum of the exponents of ρ in μ (when expressed as a word in ρ and λ) is even. Recall that $k = \mathrm{lcm}\{m, n\}$.

CASE 1: $\tau = 1_{[4]}$. We see that $(\eta\tau)^4 = 1_{[4]}$. However,

$$(\tau\eta^2)^k = \eta^{2k},$$

which is $1_{[4]}$ if and only if k is even. Since $\eta \notin \phi(G)_4$, $\phi^{-1}(\phi(G)_4)$ is generated by λ, $\rho\lambda\rho^3$, $\rho^2\lambda\rho^2$ and $\rho^3\lambda\rho$. Note that this implies $\mu \in \phi^{-1}(\phi(G)_4)$ if and only if the sum of the ρ-exponents of μ is divisible by 4.

For G_e to be a subgroup of $\phi^{-1}(\phi(G)_4)$, we must have both $(\lambda\rho^2)^n$ and $(\rho\lambda\rho)^m$ to be in $\phi^{-1}(\phi(G)_4)$. This happens exactly when m and n are both even. Since all the elements of $\phi^{-1}(\phi(G)_4)$ have even sum of ρ-exponents, G_e is a normal subgroup of $\phi^{-1}(\phi(G)_4)$.

It is possible to show that the Cayley group $H = \phi^{-1}(\phi(G)_4)$ is isomorphic to the iterated split extension $((\mathcal{Z}_{m/2} \times \mathcal{Z}_{n/2}) *_1 \mathcal{Z}_2) *_2 \mathcal{Z}_2$, where $(((a,b),c),d) * (((a',b'),c'),d') = (((a + (-1)^c((-1)^d a' + dc'), b + (-1)^c((-1)^d b' - dc')), c + c'), d + d')$, and $X = (x_1, x_2, x_3, x_4)$, where $x_1 = (((0,0),0),1)$, $x_2 = (((0,0),1),0)$, $x_3 = (((1,0),0),1)$, and $x_4 = (((0,1),1),0)$.

Thus, a partial presentation of H is:

$$H = \langle x_1, x_2, x_3, x_4 : x_1^2 = x_2^2 = x_3^2 = x_4^2 = x_1x_2x_3x_4 = 1, \ \ldots \rangle.$$

[This is presentation GTb_1 with $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_4 = xyz$; it is space group type $p2$.]

CASE 2: $\tau = (1\,2)$. Note that $\eta\tau = (2\,3\,4)$, so $(\eta\tau)^4 \neq 1_{[4]}$. Thus, ϕ is not a homomorphism.

CASE 3: $\tau = (1\,3)$. *In this case,* $\eta\tau = (1\,2)(3\,4)$ *and* $\tau\eta^2 = (2\,4)$*, so* ϕ *is a homomorphism if and only if* k *is even. We know that* $\phi^{-1}(\phi(G)_4)$ *is generated by* λ*,* $\rho\lambda\rho$*,* $\rho^2\lambda\rho^2$ *and* $\rho^3\lambda\rho^3$*. Hence* $(\rho\lambda\rho)^t$ *is in* $\phi^{-1}(\phi(G)_4)$ *for every* t*; in order for* G_e *to be a subgroup of* $\phi^{-1}(\phi(G)_4)$*, we need* $(\lambda\rho^2)^n$ *(or* $(\lambda\rho^2)^m$*) to be in* $\phi^{-1}(\phi(G)_4)$*. We note that* $(\lambda\rho^2)^{2s} = (\lambda\rho^2\lambda\rho^2)^s$ *is in* $\phi^{-1}(\phi(G)_4)$*, while* $(\lambda\rho^2)^{2s+1} = (\lambda\rho^2)^{2s}\lambda\rho^2$ *is not in* $\phi^{-1}(\phi(G)_4)$*, since* ρ^2 *is not in* $\phi^{-1}(\phi(G)_4)$*. Since* k *is even, at least one of* m *and* n *is even, so at least one of* $G_{e^U_{0,0}}$ *and* $G_{e^R_{0,0}}$ *is a subgroup of* $\phi^{-1}(\phi(G)_4)$*.*

Moreover, every element of $\phi^{-1}(\phi(G)_4)$ *has even sum of* ρ*-exponents, so any* G_e *that is a subgroup of* $\phi^{-1}(\phi(G)_4)$ *is a normal subgroup.*

In this case, we may assume without loss of generality that n *is even. The group* H *is the semidirect product* $(\mathcal{Z}_m \times \mathcal{Z}_{n/2}) * \mathcal{Z}_2$*, where* $((a,b),c) * ((a',b'),c') = ((a + (-1)^c a', b + (-1)^c b'), c + c')$*,* $X = (x_1, x_2, x_3, x_4)$*, with* $x_1 = ((-1,0),0)$*,* $x_2 = ((0,0),1)$*,* $x_3 = ((1,0),0)$*, and* $x_4 = ((0,1),1)$*.*

Thus, a partial presentation for H *is*

$$\langle x_1, x_2, x_4 : x_2^2 = x_4^2 = (x_2x_1)^2 = (x_4x_1)^2 = 1, \ \ldots\rangle.$$

[This is presentation GTb_2 *with* $x_1 = x_3^{-1} = x$*,* $x_2 = y$*, and* $x_4 = z$*; it is space group type p2.]*

CASE 4: $\tau = (1\,2)(3\,4)$. *It is easy to check that* ϕ *is a homomorphism as long as* k *is even. Here* $\phi^{-1}(\phi(G)_4)$ *is generated by* $\lambda\rho$*,* $\rho\lambda\rho^2$*,* $\rho^2\lambda\rho^3$ *and* $\rho^3\lambda$*. Since these do not have even sum of* ρ*-exponents,* $\mathcal{M}_{m,n}$ *can be a Cayley map only if it is regular, i.e., only if* $m = n$*. Since* k *is even,* $m = n$ *must be even.*

In the regular case, all the G_e *are trivial, so they are automatically normal subgroups of* $\phi^{-1}(\phi(G)_4)$*.*

It turns out the Cayley group H *is the semidirect product* $(\mathcal{Z}_{m/2} \times \mathcal{Z}_{m/2}) * \mathcal{Z}_4$*, where* $((a,b),c) * ((a',b'),c') = ((a + (-1)^{c/2}a', b + (-1)^{c/2}b'), c + c')$ *if* c *is even, and* $((a,b),c) * ((a',b'),c') = ((a + (-1)^{(c-1)/2}b', b + (-1)^{(c+1)/2}a'), c + c')$ *if* c *is odd. The generating sequence* $X = (x_1, x_2, x_3, x_4)$ *has* $x_1 = ((0,0),1)$*,*

$x_2 = ((0,0),3)$, $x_3 = ((1,0),1)$, *and* $x_4 = ((0,1),3)$. *[This is partial presentation GTd with* $x_1 = x_2^{-1} = x$ *and* $x_3 = x_4^{-1} = u$*, where the generator* y *in GTd is given by* $y = xu$*; it is space group type p4.]*

CASE 5: $\tau = (1\,3)(2\,4)$. *This case has been studied earlier, following Theorem 4.1. The Cayley group* $H \simeq \mathcal{Z}_m \times \mathcal{Z}_n$ *is abelian and is generated by* $x_1 = x_3^{-1} = x$ *and* $x_2 = x_4^{-1} = y$ *[which is presentation GTa and space group p1].*

Theorem 4.3 has a "mate" stated in terms of subgroups of the map automorphism group; however, one has to assume vertex-transitivity.

Theorem 4.5 *Let* M *be a vertex-transitive map and let* $Q \leq Aut(M)$ *be the stabilizer of some vertex of* M*. Then* M *is a Cayley map if and only if there is a subgroup* K *of* $Aut(M)$ *such that* $K \cdot Q = Aut(M)$ *and* $K \cap Q = \{1_M\}$*.*

Proof. If $M = CM(H, X, \tau)$, then we may take $K = H_L$, the group of left translations.

Conversely, suppose Q stabilizes the vertex v, and w is any vertex of M. There is an automorphism of the form kq, for some $k \in K$ and $q \in Q$, such that $w = kq(v) = k(v)$, so K is transitive on vertices. Now suppose that $k \in K$ and $k(w) = w$. There is a $\bar{k} \in K$ such that $\bar{k}(v) = w$, so $\bar{k}^{-1}k\bar{k}(v) = v$. Hence $\bar{k}^{-1}k\bar{k} \in K \cap Q$, so $\bar{k}^{-1}k\bar{k} = 1_M$, whence $k = 1_M$. $\otimes$

Theorem 4.5 yields the "automorphism mate" of Theorem 4.1, as follows.

Theorem 4.6 *Let* M *be a vertex-transitive map, let* $Q \leq Aut(M)$ *be the stabilizer of some vertex of* M *and let* $q = |Q|$*. Then* M *is a Cayley map if and only if there is a homomorphism* $\varphi : Aut(M) \to Sym[q]$ *and* $Q = \langle B \rangle$ *for some automorphism* $B \in Q$ *such that* $\varphi(B) = \eta$*.*

Proof. By Theorem 4.5, M is a Cayley map if and only if $Aut(M)$ contains a subgroup K such that $K \cdot Q = Aut(M)$ and $K \cap Q$ is trivial, where Q

is the stabilizer in $Aut(M)$ of some vertex v of the map. Since map automorphisms preserve orientation, the only map automorphisms in Q are rotations about v, and hence the group Q is cyclic; let B be a generator of Q (of order q).

Using the same method as in the proof of Theorem 4.3 it is now easy to show that the existence of the subgroup K and $Q = \langle B \rangle$ is equivalent to the existence of a group homomorphism $\varphi : Aut(M) \to Sym[q]$ such that $\varphi(B) = \eta$. $\otimes$

Example. *Consider the map* $\mathcal{M}_{m,n}$*, for* $m \neq n$*. Its automorphism group is generated by the automorphisms* A $(e^J_{i,j} \mapsto e^J_{i+1,j})$*,* B $(e^J_{i,j} \mapsto e^J_{i,j+1})$*, and* C $(e^J_{i,j} \mapsto e^{J\alpha}_{-i,-j})$ *(recall* $\alpha = (U\ D)(R\ L)$*). Since* $AC = CA^{-1}$*,* $BC = CB^{-1}$*, and* $AB = BA$*,* $Aut(\mathcal{M}_{m,n})$ *is isomorphic to the split extension* $(\mathcal{Z}_m \times \mathcal{Z}_n) * \mathcal{Z}_2$ *in which*

$$(i, j, k) * (i', j', k') = (i + (-1)^k i', j + (-1)^k j', k + k'),$$

via the identification $A = (1, 0, 0)$*,* $B = (0, 1, 0)$*, and* $C = (0, 0, 1)$*.*

We see that C *generates the stabilizer of* $v_{0,0}$ *and has order 2. Thus we are looking for homomorphisms* $\phi : Aut(M) \to Sym[2]$ *such that* $\phi(C) = (1\,2)$*. There are only four possibilities:* $A, B \mapsto 1_{[2]}$*;* $A \mapsto 1_{[2]}$*,* $B \mapsto (1\,2)$*;* $A \mapsto (1\,2)$*,* $B \mapsto 1_{[2]}$*; and* $A, B \mapsto (1\,2)$*.*

CASE 1. $A, B \mapsto 1_{[2]}$*. Here* $H = \phi^{-1}(\phi(Aut(M))_2) = \langle A, B \rangle \simeq \mathcal{Z}_m \times \mathcal{Z}_n$*.*

CASE 2. $A \mapsto 1_{[2]}$*,* $B \mapsto (1\,2)$*. Since* B *has order* n*, we must have* n *even for this to be a homomorphism. This time,* $H = \phi^{-1}(\phi(Aut(M))_2) = \langle A, B^2, BC \rangle \simeq (\mathcal{Z}_m \times \mathcal{Z}_{n/2}) * Z_2$*, where* $((i, j), k) * ((i', j'), k') = ((i + (-1)^k i', j + (-1)^k j'), k + k')$*.*

CASE 3. $A \mapsto (1\,2)$*,* $B \mapsto 1_{[2]}$*. Completely analogous to Case 2.*

CASE 4. $A, B \mapsto (1\,2)$*. Here both* m *and* n *must be even. We see* $H = \phi^{-1}(\phi(Aut(M))_2) = \langle A^2, B^2, BC, AC \rangle \simeq ((\mathcal{Z}_{m/2} \times \mathcal{Z}_{n/2}) *_1 \mathcal{Z}_2) *_2 \mathcal{Z}_2$*, where*

$(((i,j),k),\ell)*(((i',j'),k'),\ell') = (((i+(-1)^k((-1)^\ell i'+\ell k'), j+(-1)^k((-1)^\ell j'-\ell k'), k+k'), \ell+\ell')$.

In the event $m = n$, the automorphism E that rotates $90°$ about $v_{0,0}$ replaces C in the above discussion. Here E has order 4, so $Sym[4]$ replaces $Sym[2]$ as the range of the homomorphism. We leave the analysis of this case to the interested reader.

5 Balanced Maps and Antibalanced Maps

Two types of distributions of inverses τ have received particular attention in the study of Cayley maps $M = CM(H, X, \tau)$. A distribution of inverses is *balanced* if the inverse $x_{(i+1)\tau}$ of the successor x_{i+1} of x_i is the successor $x_{i\tau+1}$ of the inverse $x_{i\tau}$ of x_i and *antibalanced* if the inverse $x_{(i+1)\tau}$ of the successor x_{i+1} of x_i is the predecessor $x_{i\tau-1}$ of the inverse $x_{i\tau}$ of x_i. (Thus, when there are no repeated generators, if x, y is a consecutive pair in the sequence of generators, then "balanced" implies x^{-1}, y^{-1} is also a consecutive pair, while "antibalanced" implies y^{-1}, x^{-1} is a consecutive pair.) For $d \leq 2$, these notions coincide. In order for later claims about antibalanced maps to be correct, we shall further require that $d \geq 3$ holds for a map to be antibalanced. If $X = (x_1, x_2, \ldots, x_d)$ has distribution of inverses τ, then M is balanced if $(i+1)\tau \equiv i\tau + 1 \pmod{d}$ for each $i \in [d]$ and antibalanced if $(i+1)\tau \equiv i\tau - 1 \pmod{d}$ for any $i \in [d]$. (The two notions were introduced in [41, 40] and the corresponding Cayley maps were characterized in terms of certain automorphism-like bijections of H onto itself which we shall mention in Section 6.)

Example. *The Cayley map for $\mathcal{M}_{m,n}$ with $\tau = (1\ 3)(2\ 4)$ is balanced. The map of the icosahedron introduced earlier has $\tau = (1\,2)(3\,4)(5)$, which is neither balanced nor antibalanced. Following Proposition 7.7, we will show that no Cayley map isomorphic to the icosahedron is balanced or antibalanced.*

Example. *The Cayley map for $\mathcal{M}_{m,n}$, when $\tau = (1\,3)$, (so at least one of m and n must be even — see Case 3 of the long example in Section 4) is antibalanced.*

It is useful to interpret "balanced" and "antibalanced" in terms of τ and η as elements of $Sym[d]$. Since right multiplication by η takes i to $i+1$, the equation $(i+1)\tau = i\tau + 1$ is equivalent to $i\eta\tau = i\tau\eta$, i.e., the distribution of inverses being balanced is equivalent to $\tau\eta\tau = \eta$. The only elements of $Sym[d]$ conjugating η to itself are powers of η (recall that since $\eta = (1\ 2\ \ldots\ d)$, for any $\sigma \in Sym[d]$, $\sigma^{-1}\eta\sigma = (1\sigma\ 2\sigma\ \ldots\ d\sigma)$). Thus, a map is balanced if and only if $\tau = 1_{[d]}$ or $\tau = \eta^{d/2}$. If one views the symbols $1, 2, \ldots, d$ arranged in a circle, the case $\tau = \eta^{d/2}$ corresponds to a rotation through 180 degrees. In both balanced cases, the subgroup of $Sym[d]$ generated by η and τ is cyclic of order d.

Similarly, a map is antibalanced if and only if $\tau\eta\tau = \eta^{-1}$. Thus, a map is antibalanced if and only if the subgroup of $Sym[d]$ generated by η and τ is dihedral of order $2d$. In terms of a circular arrangement of the symbols $1, 2, \ldots, d$, a map is antibalanced if and only if τ is a reflection across a diameter of this circle.

This gives us our characterization of balanced and antibalanced Cayley maps. We henceforth denote by $\mathcal{D}_d$ the dihedral group $\langle a, b : a^d = b^2 = (ab)^2 = 1\rangle$ of order $2d$.

Theorem 5.1 *Let M be a d-valent map with dart group $G = \langle \rho, \lambda\rangle$ and dart e. Then:*

(1) M is a balanced Cayley map if and only if there is a homomorphism $\phi : G \to \mathcal{Z}_d$ such that $\phi(\rho) = 1$ and $G_e \trianglelefteq \mathrm{Ker}(\phi)$.

(2) If $d \geq 3$, then M is an antibalanced Cayley map if and only if there is a homomorphism $\phi : G \to \mathcal{D}_d$ such that $\phi(\rho) = a$, $\phi(\lambda) = b$, and $G_e \trianglelefteq \phi^{-1}(\{1, ba^k\})$, for some $k \in [d]$.

Proof. This is an immediate consequence of Theorem 4.1 and the interpretation of balanced and antibalanced in terms of the subgroup of $Sym[d]$ generated by η and τ. $\otimes$

From the first part of the previous theorem we have the following corollary.

Corollary 5.2 *Let M be a map with dart group $G = \langle \rho, \lambda \rangle$ and let e be a dart of M. Then M is a balanced Cayley map if and only if there exists a normal subgroup K of G such that $K \cdot \langle \rho \rangle = G$, $K \cap \langle \rho \rangle = \{1_G\}$ and G_e is normal in K.*

Proof. Let M be a balanced Cayley map. According to Part (1) of Theorem 5.1, there exists a homomorphism $\phi : G \to \mathcal{Z}_d$ such that $\phi(\rho) = 1$ and G_e is normal in $\mathrm{Ker}(\phi)$. Let $K = \mathrm{Ker}(\phi)$, so $K \trianglelefteq G$ and $G_e \trianglelefteq K$. It is easy to check that $K \cap \langle \rho \rangle$ is trivial and that the union of the cosets $K, K\rho, \ldots, K\rho^{d-1}$ is G.

Conversely, suppose $K \trianglelefteq G$, $K \cdot \langle \rho \rangle = G$, $K \cap \langle \rho \rangle = \{1_G\}$, and $G_e \trianglelefteq K$. Then the homomorphism $\phi : G \to G/K = \{K, K\rho, \ldots, K\rho^{d-1}\} \simeq \mathcal{Z}_d$, in conjunction with Part (1) of Theorem 5.1, shows that M is a balanced Cayley map. $\otimes$

In the final part of this section we turn our attention to regular Cayley maps. As easy consequences of Theorems 4.1 and 4.3 we have:

Corollary 5.3 *Let M be a regular d-valent map with dart group $G = \langle \rho, \lambda \rangle$. Then, the following three statements are equivalent:*

(1) M is a Cayley map.

(2) There exists a subgroup K of G such that $K \cdot \langle \rho \rangle = G$ and $K \cap \langle \rho \rangle = 1_G$.

(3) There exists a group homomorphism $\phi : G \to Sym[d]$ such that $\phi(\rho) = \eta$.

Proof. By Proposition 3.2, the map M is regular if and only if the stabilizer G_e is trivial. The equivalence (1)$\Leftrightarrow$(2) now follows from Theorem 4.3

and (1)$\Leftrightarrow$(3) from Theorem 4.1. $\otimes$

In the special case of regular Cayley maps that are balanced or antibalanced, we have have the following characterization.

Corollary 5.4 *Let M be a regular d-valent map with dart group $G = \langle \rho, \lambda \rangle$. Then, M is a balanced (antibalanced, if $d \geq 3$) Cayley map if and only if there exists a homomorphism $\phi : G \to Sym[d]$ such that $\phi(\rho) = \eta$ and $\phi(G)$ is a cyclic group of order d (dihedral group of order $2d$). Furthermore, M is a balanced Cayley map if and only if there is a normal subgroup K of G such that $K \cdot \langle \rho \rangle = G$ and $K \cap \langle \rho \rangle = \{1_G\}$.*

Proof. For regular maps G_e is trivial, so the result is an immediate consequence of Corollaries 5.2 and 5.3. $\otimes$

An automorphism-type characterization of balanced regular Cayley maps will also be handy later.

Corollary 5.5 *Let $M = CM(H, X, \tau)$ be a regular Cayley map. Then M is balanced if and only if the group $H_L \simeq H$ of left translations is a normal subgroup of $Aut(M)$.*

Proof. As G_e is trivial, in the isomorphism $G \to Aut(M)$ given by Proposition 3.1, the group H_L corresponds to a subgroup K of the dart group G such that $K \cdot \langle \rho \rangle = G$ and $K \cap \langle \rho \rangle = 1_G$. The result follows from Corollary 5.4. $\otimes$

We conclude this section with a few facts about regular tessellations of simply connected surfaces. For given positive integers d and ℓ the group $T(d, \ell) = \langle x, y : x^d = y^2 = (xy)^\ell = 1 \rangle$ is the (d, ℓ)-*triangle group.* It is a finite group if and only if $1/d + 1/\ell > 1/2$. See [8].

Consider now the regular map $M = Map\langle x, y\rangle$ of valence d and covalence ℓ that corresponds to the right G-space (G, G) for $G = T(d, \ell)$. Due to the fact that in the presentation of $G = T(d, \ell)$ there are no additional independent relations, the supporting surface for the map M is simply connected; it is the sphere, Euclidean plane, or hyperbolic plane, depending on whether $1/d + 1/\ell$ is greater than, equal to, or less than $1/2$, respectively. In all cases this regular map M is the *universal (d, ℓ)-tessellation*; it is (up to isomorphism) uniquely determined by the parameters d and ℓ (see further comments in Section 8). It follows from the above discussion that every d-valent, ℓ-covalent regular map is regularly covered by the universal (d, ℓ)-tessellation. In this sense every regular map has a geometry associated with it, and understanding regular maps is the same as understanding normal subgroups of triangle groups. See [24].

Example. *The $(4, 4)$-tessellation is the integer lattice in the plane. Its dart group (which is isomorphic to its automorphism group) is the triangle group $\langle x, y : x^4 = y^2 = (yx)^4 = 1\rangle$. To obtain the dart group G of $\mathcal{M}_{m,n}$, we have to divide out by the normal closure of $(yx^2)^k$, where $k = \mathrm{lcm}\{m, n\}$. The generic regular map for $\mathcal{M}_{m,n}$ is $\mathcal{M}_{k,k}$.*

We shall see that if a Cayley map is covered by a particular universal tessellation, then the tessellation is also Cayley (Proposition 7.1). Thus, it is of interest to know when a universal tessellation is a Cayley map. Our previous results have made this quite straightforward.

Corollary 5.6 *Let τ be an involution on the set $[d]$. The universal (d, ℓ)-tessellation M is a Cayley map with distribution of inverses τ if and only if the order of $\tau\eta$ divides ℓ. In particular, there exists a τ such that M is a balanced Cayley map if and only if either d divides ℓ, or $d/2$ is odd and $d/2$ divides ℓ. For $d \geq 3$, there exists a τ such that the map M is an antibalanced Cayley map if and only if ℓ is even.*

Proof. The dart group of the universal (d,ℓ)-tessellation M is the (d,ℓ)-triangle group $G = \langle \rho, \lambda : \rho^d = \lambda^2 = (\lambda\rho)^\ell = 1 \rangle$. Let k be the order of the product of the permutations τ and η. Since M is regular, every G_e is trivial. Thus, by Corollary 4.2, M is a Cayley map with τ as its distribution of inverses if and only if there is a homomorphism $\phi : G \to Sym[d]$ such that $\phi(\rho) = \eta$ and $\phi(\lambda) = \tau$. The assignment $\rho \mapsto \eta$ and $\lambda \mapsto \tau$ extends to a homomorphism $G \to Sym[d]$ if and only if the relations given for G are satisfied in $Sym[d]$, which is clearly the case if and only if k is a divisor of ℓ.

To clarify the balanced and the antibalanced cases we use Corollary 5.4, according to which the (regular) map M is balanced if and only if there exists a homomorphism ϕ from G onto $Sym[d]$ that takes ρ to η and λ either to the identity or to η^m if $d = 2m$. Now, a homomorphism ϕ as above that takes λ to the identity (which corresponds to the involutory balanced case) exists if and only if the relation corresponding to $(\lambda\rho)^\ell = 1$ is satisfied when λ and ρ are replaced by the identity and η, that is, if and only if $\eta^\ell = 1$, which is equivalent to the condition that d divides ℓ. Similarly, if $d = 2m$, a homomorphism ϕ as above for which $\phi(\lambda) = \eta^m$ (the other balanced case) exists if and only if $(\eta^m\eta)^\ell = 1$, which happens if and only if $d = 2m$ divides $(1+m)\ell$. If m is even then this happens if and only if $2m$ divides ℓ, since $2m$ and $(1+m)$ are relatively prime. If m is odd, this happens if and only if m divides ℓ, since $(1+m)$ is even.

Finally, the map M is antibalanced if and only if there is a homomorphism from G to the dihedral group $\langle a, b : a^d = b^2 = (ab)^2 = 1 \rangle$ taking ρ to a and λ to b; clearly this is possible if and only if 2 divides ℓ. $\otimes$

As a consequence of the method used in the first part of the preceding proof we have a simple but useful necessary condition for a map of given valence and covalence to be a Cayley map.

Corollary 5.7 *Let $d \geq 3$ and $\ell \geq 2$. If a d-valent and ℓ-covalent map is a Cayley map, then ℓ has a divisor t such that $2 \leq t \leq d$.*

We note that the assertion fails for $d = 2$, as the examples of the odd polygons in the plane show.

Proof. Let a map M of valence $d \geq 3$ and covalence $\ell \geq 2$ with dart group $G = \langle \rho, \lambda \rangle$ be a Cayley map for some distribution of inverses τ. By Corollary 4.2, there exists a homomorphism $\phi : G \to Sym[d]$ such that $\phi(\rho) = \eta$ and $\phi(\lambda) = \tau$.

As $(\lambda\rho)^\ell = 1_G$, a necessary condition for the existence of ϕ is that the order k of $\tau\eta$ be a divisor of ℓ. Since $d \geq 3$, $\phi(\rho)\phi(\lambda)$ is not the identity, so in this product there is at least one cycle of length t such that $2 \leq t \leq d$, and so k (and hence ℓ) is divisible by t. $\otimes$

In Theorem 10.1 we will, among other things, show that the converse holds for universal tessellations: for $d \geq 3$, $\ell \geq 2$, the universal (d, ℓ)-tessellation is a Cayley map if and only if ℓ has a nontrivial divisor not exceeding d. Note that by Corollary 5.7 the dodecahedron is not a Cayley map; in fact, it is known that the underlying graph for the dodecahedron map is not even a Cayley graph. This latter fact also follows from our results in Section 10. Since the icosahedron is a Cayley map, we see that dual maps of regular Cayley maps need not be Cayley maps. Other questions of a similar type will be considered in Section 8 as well.

6 Isomorphisms and automorphisms of Cayley maps

In the previous section, we found necessary and sufficient conditions for a map M to be a Cayley map. In particular, we found conditions for deciding whether a regular map is a Cayley map. In the reverse direction it would be of interest to have necessary and sufficient conditions for a Cayley map to be regular. Although the problems sound very similar, we shall see that the answers are quite different. One reason for this is that in the second problem we have the Cayley map as input, and so we expect conditions in terms of the Cayley group, the generating set and the dart-reversing involution only. The results turn out to be consequences of a detailed investigation of the automorphism groups of Cayley maps. We begin here with addressing the problem of *isomorphism* of Cayley maps first; this will provide a natural setting for the study of the structure of the automorphism groups of Cayley maps later in this section. This section is largely based on the paper by Jajcay and Širáň [20].

Let $M_1 = CM(H_1, X_1, \tau_1)$ and $M_2 = CM(H_2, X_2, \tau_2)$ be two d-valent Cayley maps, with $X_1 = (x_1^{(1)}, x_2^{(1)}, \ldots, x_d^{(1)})$ and $X_2 = (x_1^{(2)}, x_2^{(2)}, \ldots, x_d^{(2)})$. For $j = 1, 2$, let D_j be the dart set of M_j and let ρ_j and λ_j be the rotation and dart-reversing involution of M_j. Recall that a map isomorphism of M_1 and M_2 is a bijection $f : D_1 \to D_2$ such that $f((h,i)\rho_1) = (f(h,i))\rho_2$ and $f((h,i)\lambda_1) = (f(h,i))\lambda_2$ for any dart $(h,i) \in D_1$. (We point out that f may exist also in the case when the groups H_1 and H_2 are not isomorphic, as we saw when determining Cayley groups for the maps $\mathcal{M}_{m,n}$ in Section 4.) In what follows we investigate the properties of such an isomorphism f more fully.

Define the functions $\omega : D_1 \to H_2$ and $m : D_1 \to \mathcal{Z}_d$ by $f(h,i) = (\omega(h,i), m(h,i))$. Since $f((h,i)\rho_1) = f(h,i+1) = (\omega(h,i+1), m(h,i+1))$ and $(f(h,i))\rho_2 = (\omega(h,i), m(h,i))\rho_2 = (\omega(h,i), m(h,i)+1)$, for each dart $(h,i) \in D_1$ we have

$$\omega(h,i+1) = \omega(h,i) \quad \text{and} \quad m(h,i+1) = m(h,i) + 1 \ .$$

Thus, ω is independent of i and we can define $\omega : H_1 \to H_2$ by $\omega(h) = \omega(h,i)$. As f is an isomorphism, ω is a bijection from H_1 onto H_2. Also, we can define $|h|$ by $|h| = m(h,d)$, from which we get $m(h,i) \equiv |h| + i \pmod d$. With this new notation, we have

$$f(h,i) = (\omega(h), |h| + i) \quad \text{for any } (h,i) \in D_1 \ . \tag{1}$$

By vertex-transitivity we may assume that f takes the vertex labelled

1_{H_1} to the vertex labelled 1_{H_2}. Let $t \in [d]$ be such that $f(1_{H_1}, d) = (1_{H_2}, t)$. Then $\omega(1_{H_1}) = 1_{H_2}$ and $|1_{H_1}| = t$.

Similarly, we have $f((h,i)\lambda_1) = f(hx_i^{(1)}, i\tau_1) = (\omega(hx_i^{(1)}), |hx_i^{(1)}| + i\tau_1)$ and $(f(h,i))\lambda_2 = (\omega(h), |h| + i)\lambda_2 = (\omega(h)x^{(2)}_{|h|+i}, (|h| + i)\tau_2)$. Therefore, for each dart (h, i) of M_1 we obtain $\omega(hx_i^{(1)}) = \omega(h)x^{(2)}_{|h|+i}$ and

$$|hx_i^{(1)}| + i\tau_1 = (|h| + i)\tau_2. \tag{2}$$

Motivated by these observations, for Cayley maps $M_1 = CM(H_1, X_1, \tau_1)$ and $M_2 = CM(H_2, X_2, \tau_2)$, a function $|\cdot| : H_1 \to \mathcal{Z}_d$ is a (τ_1, τ_2)-*norm* if

$$|hx_i| + i\tau_1 \equiv (|h| + i)\tau_2 \quad \text{for each } h \in H_1 \quad \text{and } i \in \mathcal{Z}_d.$$

The associated *rotary function* is a bijection $\omega : H_1 \to H_2$ such that for every $h \in H_1$ and $i \in \mathcal{Z}_d$,

$$\omega(1_{H_1}) = 1_{H_2} \quad \text{and} \quad \omega(hx_i^{(1)}) = \omega(h)x^{(2)}_{|h|+i}\ . \tag{3}$$

An important point to note is that these definitions make no reference to map isomorphisms. We can now state the basic result on isomorphisms of Cayley maps.

Theorem 6.1 *The maps $M_1 = CM(H_1, X_1, \tau_1)$ and $M_2 = CM(H_2, X_2, \tau_2)$ of valence d are isomorphic if and only if there exists a (τ_1, τ_2)-norm $|\cdot| :$ $H_1 \to \mathcal{Z}_d$ and an associated rotary function $\omega : H_1 \to H_2$.*

Proof. Necessity follows from the analysis above. For sufficiency, a routine verification shows that the function f defined by $f(h, i) = (\omega(h), |h| + i)$ is an isomorphism $M_1 \to M_2$. $\otimes$

To note an immediate consequence of Theorem 6.1, assume that M_1 contains a pair of repeated generators $x_i^{(1)} = x_j^{(1)}$ for $i \neq j$. Then for any rotary function ω and any $h \in H_1$ we have $\omega(hx_i^{(1)}) = \omega(hx_j^{(1)})$, which implies that

$\omega(h)x^{(2)}_{|h|+i} = \omega(h)x^{(2)}_{|h|+j}$, and so $x^{(2)}_{|h|+i} = x^{(2)}_{|h|+j}$. It follows that any Cayley map M_2 isomorphic to M_1 must also have a pair of repeated generators. (This observation is also clear from the observation that repeated generators is equivalent to parallel darts, whose map homomorphic images are still parallel.)

Example. *From the discussion in Section 4 we have seen that the map $\mathcal{M}_{m,n}$ can be endowed with a Cayley structure in four essentially different ways. To get an idea what the corresponding norms and rotary functions look like, let $m = n$ be even and let $M_1 = CM(H_1, X_1, \tau)$ and $M_2 = CM(H_2, X_2, \tau_2)$ be the Cayley maps referred to as Cases 4 and 5, respectively, in the example following Proposition 4.4. There we had $H_1 = (\mathcal{Z}_{m/2} \times \mathcal{Z}_{m/2}) * \mathcal{Z}_4$, $X_1 = (x^{(1)}_1, x^{(1)}_2, x^{(1)}_3, x^{(1)}_4)$, where $x^{(1)}_1 = ((0,0),1)$, $x^{(1)}_2 = ((0,0),3)$, $x^{(1)}_3 = ((1,0),1)$, $x^{(1)}_4 = (((0,1),3)$ for M_1 with $\tau_1 = (1\,2)(3\,4)$, and $H_2 = \mathcal{Z}_m \times \mathcal{Z}_m$, $X_2 = (x^{(2)}_1, x^{(2)}_2, x^{(2)}_3, x^{(2)}_4)$, $x^{(2)}_1 = (1,0)$, $x^{(2)}_2 = (0,1)$, $x^{(2)}_3 = (-1,0)$, $x^{(2)}_4 = (0,-1)$ for M_2, where $\tau_2 = (1\,3)(2\,4)$.*

To see that the two maps are isomorphic using Theorem 6.1 one can take the (τ_1, τ_2)-norm defined by $|((a,b),c)| = c \in \mathcal{Z}_4$. The associated rotary function $\omega : H_1 \to H_2$ is then given by $\omega((a,b),c) = (-2a,-2b) + \varepsilon_c$ where $\varepsilon_c = (0,0)$, $(1,0)$, $(1,1)$, or $(0,1)$, for $c = 0$, 1, 2, or 3, respectively.

One of the most interesting questions here is under what conditions a map isomorphism of two Cayley maps reduces to a group isomorphism of the corresponding Cayley groups. To the best of our knowledge this question has not been considered before in such generality. One special case was proved in [41], where it is shown that two balanced regular Cayley maps $CM(H, X_1, \tau_1)$ and $CM(H, X_2, \tau_2)$ of valence d on the same group and with $\tau_1 = \tau_2$ are isomorphic if and only if there exists a group automorphism ω of G such that $\omega(x^{(1)}_i) = x^{(2)}_{i+t}$ for all $i \in [d]$ and some fixed t. We generalize this by removing the regularity assumption and allowing the two groups to be different.

Theorem 6.2 *Let $M_1 = CM(H_1, X_1, \tau_1)$ and $M_2 = CM(H_2, X_2, \tau_2)$ be balanced Cayley maps of valence d, with $\tau_1 = \tau_2$. Then M_1 is isomorphic to M_2 if and only if there exists a group isomorphism $\xi : H_1 \to H_2$ such that $\xi(x_i^{(1)}) = x_{i+t}^{(2)}$ for all $i \in [d]$ and a fixed $t \in [d]$.*

It follows that two balanced Cayley maps that have the same "pattern of balance" but different Cayley groups can never be isomorphic.

Proof. Let M_1 and M_2 be isomorphic and let $|\cdot| : H_1 \to \mathcal{Z}_d$ and $\omega : H_1 \to H_2$ be the corresponding (τ_1, τ_2)-norm and rotary function that satisfy (2) and (3). Because the M_i are balanced, $i\tau_1 = i\tau_2 = i + c$, where either $c = 0$ or $c = d/2$. Thus, (2) implies $|hx_i^{(1)}| = |h|$ for all $h \in H_1$ and $x_i^{(1)} \in X_1$, which means that $|h| \equiv t$, for some $t \in [d]$. From (3) we obtain $\omega(hx_i^{(1)}) = \omega(h)x_{i+t}^{(2)}$, which easily implies that the rotary function $\omega : H_1 \to H_2$ is a group homomorphism. Since any rotary function is a bijection, ω is a group isomorphism.

Conversely, suppose $\xi : H_1 \to H_2$ defined by $\xi(hx_i^{(1)}) = \xi(h)x_{t+i}^{(2)}$ is a group isomorphism. Define $|\cdot| : H_1 \to \mathcal{Z}_d$ by $|h| = t$. Then Theorem 6.1 shows that M_1 and M_2 are isomorphic. $\otimes$

We comment that everything that has been done to this point can also be done for homomorphisms rather than isomorphisms. In particular, in Theorem 6.2 there is a map homomorphism from M_1 onto M_2 if and only if there is a corresponding group homomorphism from H_1 onto H_2.

If both M_1 and M_2 are regular, then the computations leading to Theorem 6.1 show that we may take $t = d$; we state this as a separate result.

Corollary 6.3 *Let $M_1 = CM(H_1, X_1, \tau_1)$ and $M_2 = CM(H_2, X_2, \tau_2)$ be regular balanced Cayley maps of valence d, with $\tau_1 = \tau_2$. Then M_1 is isomorphic to M_2 if and only if there exists a group isomorphism $\theta : H_1 \to H_2$ such that $\theta(x_i^{(1)}) = x_i^{(2)}$ for all $i \in [d]$.* $\otimes$

The examples $\mathcal{M}_{m,n}$ – in particular, Cases 1 and 5 discussed after Proposition 4.4 – show that one cannot omit the assumption $\tau_1 = \tau_2$ in Theorem 6.2. Nevertheless there is an important class of Cayley maps where even this assumption may be dropped provided that the Cayley groups are Abelian; the result is a slight generalization of Theorem 2 of [26]. A map is *complete* if its underlying graph is a (simple) complete graph.

Theorem 6.4 *Let $M_1 = CM(H_1, X_1, \tau_1)$ and $M_2 = CM(H_2, X_2, \tau_2)$ be complete Cayley maps of valence d, where H_1 and H_2 are Abelian. Then M_1 is isomorphic to M_2 if and only if for some fixed $t \in [d]$ there is a group isomorphism $\xi : H_1 \to H_2$ such that $\xi(x_i^{(1)}) = x_{t+i}^{(2)}$ for each $i \in [d]$.*

Consequently, any two complete Cayley maps with different Abelian Cayley groups are automatically non-isomorphic.

Proof. (Cf. [26].) Let $f(h,i) = (\omega(h), |h| + i)$ be an isomorphism $M_1 \to M_2$ as given by Theorem 6.1. In particular, from the second part of (3) we see that

$$x_{|h|+i}^{(2)} = \omega(h)^{-1}\omega(hx_i^{(1)}) \tag{4}$$

for each $h \in H_1$ and each $i \in [d]$.

Since $d < |H_1| = d + 1$, the norm $|\cdot| : H_1 \to [d]$ cannot be injective, and so there exist $g^*, h^* \in H_1$ such that $g^* \neq h^*$, $h^* \neq 1_{H_1}$, and $|g^*| = |h^*|$. By (4), for each $i \in [d]$ we then have $\omega(g^*)^{-1}\omega(g^*x_i^{(1)}) = \omega(h^*)^{-1}\omega(h^*x_i^{(1)})$. As M_1 is a complete map, each nonidentity element of H_1 is in X_1 and therefore the last equality can be written in the form $\omega(g^*z) = \omega(g^*)\omega(h^*)^{-1}\omega(h^*z)$ for each $z \in H_1$. Taking $z = h^{*-1}$, we see that $\omega(g^*h^{*-1}) = \omega(g^*)\omega(h^*)^{-1}$, which yields $\omega(g^*z) = \omega(g^*h^{*-1})\omega(h^*z)$ for each $z \in H_1$. Invoking completeness of M_1 again, we have $g^*h^{*-1} = x_k^{(1)}$ for some $k \in [d]$. Denoting h^*z by b, we have $g^*z = x_k^{(1)}b$, and so

$$\omega(x_k^{(1)}b) = \omega(x_k^{(1)})\omega(b) \quad \text{for each } b \in H_1. \tag{5}$$

Now the assumption that both H_1 and H_2 are Abelian allows us to rewrite (5) in the form $\omega(b)^{-1}\omega(bx_k^{(1)}) = \omega(x_k^{(1)})$ for each $b \in H_1$. But by (4) the left side of this last equality is $x_{|b|+k}^{(2)}$, which means that $x_{|b|+k}^{(2)} = \omega(x_k^{(1)})$ for each $b \in H_1$. Since X_2 contains no repeated generators it follows that $|b| = t$ for some fixed t and each $b \in H_1$.

Letting $h = 1_{H_1}$ in (4), we have $\omega(x_i^{(1)}) = x_{t+i}^{(2)}$, which together with (4) implies that $\omega(hx_i^{(1)}) = \omega(h)\omega(x_i^{(1)})$ for each $h \in H_1$ and each $i \in [d]$. It is now easy to see that the bijection ω is an isomorphism of H_1 to H_2 with the property $\omega(x_i^{(1)}) = x_{t+i}^{(2)}$ for a suitable $t \in [d]$ and each $i \in [d]$.

Conversely, if such an isomorphism $\xi : H_1 \to H_2$ exists, then the function $f : M_1 \to M_2$ defined on darts of M_1 by $f(h,i) = (\xi(h), t+i)$ is an isomorphism of the two maps. $\otimes$

We now proceed to investigate the structure of the automorphism group of a Cayley map. Much of the material here relies on the ideas of [17], where the theory was developed for maps whose underlying graph is simple (i.e., without loops, parallel edges or semi-edges). Special attention will be paid to balanced and antibalanced Cayley maps, the regular cases thereof generalizing slightly the results of [41, 40].

The automorphism group $Aut(M)$ of a Cayley map $M = CM(H, X, \tau)$ has two important subgroups: the group $H_L \simeq H$ of left translations and the stabilizer Q of an arbitrary but fixed vertex v, which we might as well assume to be the unit element 1_H of H. By Theorem 4.5, $H_L \cap Q = \{1_M\}$ and $H_L \cdot Q = Aut(M)$. Thus, in a sense, $Aut(M)$ is a product of H_L and Q. Our objective is to describe this product and identify the automorphisms of M from properties of H and τ only, without direct reference to M.

We start by investigating Q. Let e be any dart emanating from 1_H and let t be the smallest positive integer for which there is an automorphism $B \in Q$ such that $B(e) = e\rho^t$. Then t is the *rotary index* of M. It is easy to see (c.f., [17]) that t divides the valence d of M and B generates Q, with order d/t.

The automorphism B is the *basic generator* of Q.

Example. *The map $\mathcal{M}_{m,n}$ has rotary index 2 when $m \neq n$ and rotary index 1 when $m = n$.*

There are two special cases to note immediately.

Observation 6.5 *Let M be a map with rotary index t.*

(1) $t = 1$ if and only if M is regular.

(2) If $t = d$, then $Q = \{1_M\}$ and $Aut(M) = H_L$.

We shall consider the regular case in further detail at the end of this section.

Now we apply the theory developed for isomorphisms of Cayley maps to the basic generator B of Q, interpreting B as an isomorphism $M \to M$ that fixes the vertex 1_H. By Theorem 6.1 we have

$$B(h, i) = (\psi(h), |h| + i)$$

for each $h \in H$ and each $i \in [d]$, where $|\cdot| : H \to \mathcal{Z}_d$ is a τ-*norm* that satisfies the relations (2) for $\tau_2 = \tau_1 = \tau$, and $\psi : H \to H$ is the associated *rotary function* fulfilling (3) with ψ in place of ω. (We will reserve the symbol ψ for rotary functions in the automorphism context while ω will be used in connection with isomorphisms.) Note that the rotary index t is equal to $|1_H|$.

Since $H_L \cap Q = \{1_M\}$ and $H_L \cdot Q = Aut(M)$, each element of $Aut(M)$ is uniquely expressible as a product $A_h B^k$ for some $A_h \in H_L$ and $k \in \mathcal{Z}_{d/t}$. In particular, the automorphism $B^k A_h$ is $A_{h'} B^{k'}$, for some h' and k' depending on both h and k.

Note that since $B^k A_h(1_H, i) = B^k(h, i)$, it would be useful to know more about B^k. As $B(h, i) = (\psi(h), |h| + i)$, for all h and i, a straightforward induction shows $B^k(h, i) = (\psi^k(h), |\psi^{k-1}(h)| + |\psi^{k-2}(h)| + \ldots + |\psi(h)| + |h| + i)$. Set $|h|_k = |\psi^{k-1}(h)| + |\psi^{k-2}(h)| + \ldots + |\psi(h)| + |h|$, so that $B^k(h, i) = (\psi^k(h), |h|_k + i)$. Thus,

$$B^k A_h(1_H, i) = B^k(h, i) = (\psi^k(h), |h|_k + i)$$

and we see that $h' = \psi^k(h)$.

The "extension" of the group H_L by the group Q described above was called a *rotary extension* in [17]. It was proved in [19] that every finite group, with the exceptions of $\mathcal{Z}_3$ and $\mathcal{Z}_2 \times \mathcal{Z}_2$, is isomorphic to $Aut(M)$, for some Cayley map M whose underlying graph is simple. The group $\mathcal{Z}_3$ is the automorphism group for the Cayley map consisting of a single vertex incident with three semi-edges. The group $\mathcal{Z}_2 \times \mathcal{Z}_2$ is the automorphism group for the Cayley map consisting of two vertices joined by two parallel edges and each incident with two non-consecutive semi-edges. Thus we have the following.

Theorem 6.6 *Every finite group is isomorphic to the automorphism group of some Cayley map.* $\otimes$

This theorem suggests that there is unlikely to be an explicit general formula for k' as a function of h and k defined by the relation $B^k A_h = A_{h'} B^{k'}$. However, there are some important special cases.

A Cayley map $M = CM(H, X, \tau)$ is *normal* if $H_L \trianglelefteq Aut(M)$. An analogous notion was introduced for Cayley graphs in [49]. Clearly, M is normal if and only if BA_hB^{-1} is in H_L for every $h \in H$. That is, M is normal if and only if, for every $h, h' \in H$ and $i \in \mathcal{Z}$, $BA_hB^{-1}(h', i) = (h'', i)$ for some $h'' \in H$.

From our previous calculations, it is easy to see that if M has rotary index t, then

$$B^{-1}(1_H, i) = (1_H, -t + i),$$

so that $BA_hB^{-1}(1_H, i) = (\psi(h), |h| - t + i)$. Therefore, M is normal if and only if $|h| = t$, for every $h \in H$. In this case, $BA_hB^{-1}(1_H, i) = (\psi(h), i)$ and, more generally, $B^kA_hB^{-k}(1_H, i) = (\psi^k(h), i) = A_{\psi^k(h)}(1_H, i)$. Since at most one automorphism can map $(1_H, i)$ to $(\psi^k(h), i)$, we must have

$$B^kA_hB^{-k} = A_{\psi^k(h)}.$$

In this case, $h' = \psi^k(h)$ and $k' = k$.

Next we show that if M is normal, then the rotary function ψ is a group automorphism of H. Since $|h| = t$ for all $h \in H$, the general relations $\psi(x_i) = x_{t+i}$ and $\psi(hx_i) = \psi(h)x_{|h|+i}$ are combined to show that $\psi(hx_i) = \psi(h)\psi(x_i)$. Thus, an easy induction on the length of h as a word in the elements of X shows that ψ is a group homomorphism. Since ψ is always a bijection, ψ is an automorphism of H.

As $H_L \cap Q = \{1_M\}$ and $H_L \cdot Q = Aut(M)$, normality of M is equivalent to $Aut(M)$ being a split extension of the group H_L by the stabilizer Q. In this case, $(A_h B^k)(A_{h'} B^{k'}) = A_h(B^k A_{h'} B^{-k})B^{k+k'} = A_{h\psi^k(h')}B^{k+k'}$.

Thus, if M is normal, then $Aut(M)$ is the semi-direct product of H with the cyclic group $\mathcal{Z}_{d/t} \simeq Q$. The elements of $Aut(M)$ are the pairs $(h, k) \in H \times \mathcal{Z}_{d/t}$ and multiplication is given by $(h, k)(h', k') = (h\psi^k(h'), k + k')$.

We saw above that the τ-norm $|\cdot|$ of a normal Cayley map with rotary index t satisfies $|\cdot| \equiv t$. We claim that $|\cdot| \equiv t$ is equivalent to $(i+t)\tau \equiv i\tau + t \pmod{d}$, for every $i \in \mathcal{Z}_d$.

To see the necessity, set $h = 1_H$ in (2) (with τ in place of τ_1 and τ_2) to deduce that $t + i\tau = (t + i)\tau$. For the sufficiency, we know $|1_H| = t$. We proceed to show $|h| = t$ by induction on the length of the word h in terms of generators from X. We have $|hx_i| + i\tau = (|h| + i)\tau$. The hypothesis $(t+i)\tau = t + i\tau$ and the inductive assumption $|h| = t$ imply $|hx_i| = (t + i)\tau - i\tau = t$, as required.

Cayley maps with the property that $(i+t)\tau = i\tau + t \pmod{d}$, for all $i \in \mathcal{Z}_d$, are *t-balanced*; equivalently, a map is t-balanced if $\tau\eta^t\tau = \eta^t$ (recall $\eta = (1\,2\,\ldots\,d)$). Clearly "1-balanced" and "balanced" are synonymous. From the definitions, a map M is a normal 1-balanced Cayley map if and only if M is a regular balanced Cayley map. In [29], Martino and Schultz define a Cayley map M to be *t-antibalanced* if $\tau\eta\tau = \eta^t$. Thus "1-antibalanced" is "balanced" and "(-1)-antibalanced" is "antibalanced". They show that for a regular d-valent Cayley map M with n vertices, if $\gcd(d, n) = 1$ and $Aut(M)$ is solvable, then M is t-antibalanced.

The following connections between these concepts generalize slightly a result in [17].

Theorem 6.7 *Let $M = CM(H, X, \tau)$ be a Cayley map with rotary index t. Then the following three statements are equivalent.*

(1) M is a normal Cayley map.

(2) The τ-norm is identically t.

(3) M is t-balanced.

Any of (1)-(3) implies:

(4) The corresponding rotary function is an automorphism of H.

In any of the cases (1)-(3), $Aut(M)$ is isomorphic to a split extension of H and $\mathcal{Z}_{d/t}$, with multiplication $(h, k)(h', k') = (h\psi^k(h'), k + k')$, and each automorphism of M is a lift of an automorphism of the one-vertex quotient M/H_L.

Moreover, if there are no repeated elements in the generating sequence X, then (4) is equivalent to each of (1)-(3).

Proof. Everything is already done except, in the case when X has no repetitions, that (4) implies (2). Since $\psi(hx_i) = \psi(h)x_{|h|+i}$, $\psi(x_i) = x_{t+i}$, and $\psi(hx_i) = \psi(h)\psi(x_i)$, we deduce that $x_{|h|+i} = x_{t+i}$. Since X has no repeated generators, $|h| + i \equiv t + i \pmod{d}$, and, therefore, $|h| = t$, as required. $\otimes$

If X contains repeated generators, then (4) need not imply (1)-(3), as the following example shows.

Example. *Consider the map $M = CM(H, X, \tau)$, with $H = \mathcal{Z}_2 = \{0, 1\}$, $X = (x_1, x_2, x_3) = (1, 1, 1)$ and $\tau = (1\,2)(3)$. For simplicity, we use i to denote the dart $(0, i)$ and i' to denote the dart $(1, i)$. Then $\rho = (1\,2\,3)(1'\,2'\,3')$ and $\lambda = (1\,2')(2\,1')(3\,3')$. (Thus, M consists of three parallel edges in the sphere.)*

The group of left translations is $H_L = \{1_M, A\}$, for $A = (1\,1')(2\,2')(3\,3')$. Furthermore, the rotary index is 1 (it is easy to check that M is regular) and *the basic generator is $B = (1\,2\,3)(1'\,3'\,2')$. The composition AB (read from right to left) is $AB = (1\,2')(2\,3')(3\,1')$.*

Consequently, $Aut(M) = \langle A, B : A^2 = B^3 = (AB)^2 = 1_M \rangle \simeq Sym[3]$. But $\mathcal{Z}_2 \simeq H_L$ is not a normal subgroup of $Sym[3]$, although $\psi \equiv 1_M$ is an automorphism of H. We note that M is not 1-balanced and $|1_H| = 2$.

We note the following interesting consequence of Theorem 6.7.

Corollary 6.8 *Let $M = CM(H, X, \tau)$ be a complete Cayley map of an Abelian Cayley group H. Then M is normal. If in addition M is regular, then it is balanced.*

Proof. Let B be the basic generator of the stabilizer of the vertex 1_H in the group $Aut(M)$ and let t be the corresponding rotary index. From Theorem 6.4 it follows that the rotary function ψ associated with B is an automorphism of H such that $\psi(x_i) = x_{t+i}$. Therefore M is t-balanced and hence normal, by Theorem 6.7. Moreover, if M is regular then $t = 1$ which means that M is balanced. $\otimes$

The last part of this section deals with regular Cayley maps. In order to state our general result, let $M = CM(H, X, \tau)$ be a Cayley map, with $X = (x_1, x_2, \ldots, x_d)$. A *skew-morphism of H* is a bijection $\zeta : H \to H$ such that $\zeta(1_H) = 1_H$ and there is a τ-norm $|\cdot| : H \to \mathcal{Z}_d$ satisfying $|1_H| = 1$ and $\zeta(hh') = \zeta(h)\zeta^{|h|}(h')$. We emphasize that this definition is not in terms of automorphisms of M.

Our goal for the remainder of this section is to prove the following, together with corollaries treating the special cases of balanced and antibalanced Cayley maps.

Theorem 6.9 *Let $M = CM(H, X, \tau)$, with $X = (x_1, x_2, \ldots, x_d)$. Then M is regular if and only if there is a skew-morphism $\zeta : H \to H$ such that, for each $i \in \mathcal{Z}_d$, $x_{i+1} = \zeta(x_i)$.*

We invite the reader to compare this result, i.e., a characterization of Cayley maps that are regular, with, for example, Corollary 5.3, which characterizes which regular maps are Cayley maps.

Proof. Suppose first that M is regular. By Observation 6.5, this is equivalent to the rotary index t of M being 1, so $|1_H| = t = 1$. Thus, for the basic generator B, $B(1_H, i) = (1_H, i+1)$.

The automorphism $A_{\psi(h)^{-1}} B A_h$ takes $(1_H, i)$ to $(1_H, |h|+i)$. Since the automorphism $B^{|h|}$ also takes $(1_H, i)$ to $(1_H, |h|+i)$, they must be equal, i.e., for every $h' \in H$ and $i \in [d]$, $B^{|h|}(h', i) = A_{\psi(h)^{-1}} B A_h(h', i)$. In particular, $\psi^{|h|}(h') = \psi(h)^{-1}\psi(hh')$, which shows ψ is a skew-morphism.

For the converse, suppose there is a skew-morphism ζ of H such that $\zeta(x_i) = x_{i+1}$, for all $i \in \mathcal{Z}_d$. We show that M is regular.

By definition, there is a τ-norm $|\cdot|$ such that $|1_H| = 1$ and $\zeta(hh') = \zeta(h)\zeta^{|h|}(h')$, for every $h, h' \in H$. We note that $\zeta(x_i) = x_{i+1}$, for all i, implies $\zeta^j(x_i) = x_{i+j}$, for all i, j.

Define the function B on the set of darts by $B(h, i) = (\zeta(h), |h| + i)$. Since ζ is a bijection, so is B; we proceed to show B is an automorphism. The computations are straightforward:

$$B((h,i)\rho) = B(h, i+1) = (\zeta(h), |h| + i + 1) = (\zeta(h), |h| + i)\rho = (B(h,i))\rho$$

and

$$\begin{aligned} B((h,i)\lambda) &= B(hx_i, i\tau) = (\zeta(hx_i), |hx_i| + i\tau) \\ &= (\zeta(h)\zeta^{|h|}(x_i), (|h|+i)\tau) \\ &= (\zeta(h)x_{|h|+i}, (|h|+i)\tau) \qquad \text{by}(2) \\ &= (\zeta(h), |h|+i)\lambda = (B(h,i))\lambda. \end{aligned}$$

Evidently, $B(1_H, i) = (\zeta(1_H), |1_H| + i) = (1_H, i+1)$, so M has rotary index 1. Observation 6.5 implies M is regular. $\otimes$

From Theorem 6.9, we can deduce results in [41, 40] concerning regular balanced and antibalanced Cayley maps.

Corollary 6.10 *Let $M = CM(H, X, \tau)$ be a Cayley map with $X = (x_1, x_2, \ldots, x_d)$. Then M is a balanced regular Cayley map if and only if there exists an automorphism ψ of H such that $\psi(x_i) = x_{i+1}$, for every $i \in \mathcal{Z}_d$. In such a case, the group $Aut(M)$ is isomorphic to a split extension of H by $\mathcal{Z}_d$ (or, equivalently, by the automorphism ψ), where multiplication is given by $(h,k)(h',k') = (h\psi^k(h'), k+k')$. Moreover, every automorphism of M is a lift of an automorphism of M/H_L.*

Proof. The skew-morphism of Theorem 6.9 is an automorphism if and only if $|h| = 1$, for every $h \in H$. But $|\cdot| \equiv 1$ if and only if (by Theorem 6.7) M is 1-balanced. Since 1-balanced is equivalent to balanced, the skew-morphism is an automorphism if and only if M is balanced.

The remaining parts follow easily from Theorem 6.7 and the discussion of regular maps preceding Proposition 3.4. $\otimes$

Corollary 6.11 *Let $M = CM(H, X, \tau)$ be a d-valent Cayley map, with $d \geq 3$ and $X = (x_1, x_2, \ldots, x_d)$. Then M is an antibalanced regular map if and only if there exists a skew-morphism ζ of H such that $\zeta(x_i) = x_{i+1}$, for all $i \in \mathcal{Z}_d$ and the corresponding τ-norm satisfies $|x_i| = d-1$, for all $i \in \mathcal{Z}_d$.*

Proof. By Equation (2) (with $\tau_1 = \tau_2 = \tau$ and $h = 1_{H_1}$), we have $|x_i| = (i+1)\tau - i\tau$. Thus, $|x_i| = d-1$ for all i, if and only if $(i+1)\tau \equiv i\tau + d - 1 \equiv i\tau - 1 \pmod{d}$. $\otimes$

Observe that if $|x_i| = d-1$ for all i, then it is easy to use the relations $|1_H| = 1$ and $|hx_i| = (|h| + i)\tau - i\tau$ to prove that $|g| = (-1)^n$, where n is the number of factors in the product $g = x_{i_1}x_{i_2}\cdots x_{i_n}$. Thus, the map $h \mapsto |h|$ is a homomorphism from H onto the multiplicative group $\mathcal{Z}_2 = \{-1, 1\}$. The associated skew-morphisms are called *antiautomorphisms* in [40].

We conclude this section by discussing two prominent classes of maps that have been among the first examples of regular Cayley maps in the literature (e.g., [7]): the complete maps and the complete bipartite maps with underlying graph $K_{n,n}$. Let us begin with three regular embeddings of the complete bipartite graph.

Let $H_1 = \mathcal{D}_n = \langle a, b : a^n = b^2 = (ab)^2 = 1\rangle$ be the dihedral group of order $2n$, and let $X_1 = (x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)})$, where $x_i^{(1)} = a^i b$ for each $i \in [n]$. The distribution of inverses τ_1 is the identity on $[n]$ – each $x_i^{(1)}$ is an involution – and $M_1 = CM(H_1, X_1, \tau_1)$ is a balanced Cayley map whose underlying graph is the complete bipartite graph $K_{n,n}$. The bijection $\psi_1 : \ (a^i b^j) \mapsto a^{i+j} b^j$ for $i \in [n]$ and $j = 0, 1$, is an automorphism of the group $\mathcal{D}_n$ such that $\psi_1(x_i^{(1)}) = x_{i+1}^{(1)}$ for each i (mod n). It follows from Corollary 6.10 that M_1 is a (balanced) regular Cayley map. It can be checked that the covalence of M_1 is $2n$; more precisely, each face boundary is a Hamiltonian cycle.

Next consider the group $H_2 = \mathcal{Z}_n \times \mathcal{Z}_2$, the generating sequence $X_2 = (x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)})$ where $x_i^{(2)} = (i, 1)$, and the distribution of inverses τ_2 on $[n]$ given by $i\tau_2 = -i$ (with $0 = n$). The map $M_2 = CM(H_2, X_2, \tau_2)$ is again a Cayley map for the graph $K_{n,n}$, and again of covalence $2n$ but this time it is antibalanced. One can see that the function $\psi_2 : \ (i, j) \mapsto (i + j, j)$ is an antiautomorphism of the group H_2 with the property $\psi_2(x_i^{(2)}) = x_{i+1}^{(2)}$, which shows (by Corollary 6.11) that M_2 is a regular Cayley map.

A third way to construct a Cayley map M_3 with $K_{n,n}$ as the underlying graph can be based on the group $H_3 = \mathcal{Z}_{2n}$. Here we take the generating sequence $X_3 = (x_1^{(3)}, x_2^{(3)}, \ldots, x_n^{(3)})$ with $x_i^{(3)} = 2i - 1$, and define the involution τ_3 on $[n]$ by $i\tau_3 = n + 1 - i$. The function ψ_3 defined by $\psi_3(2i - j) = 2i + j$ (mod $2n$), where $j \in \{0, 1\}$, turns out to be an antiautomorphism of M_3 such that $\psi_3(x_i^{(3)}) = x_{i+1}^{(3)}$ for each $i \in [n]$. We thus have one more construction of an antibalanced regular map for $K_{n,n}$, which was originally proposed for further study in the Project 5.6.7 of [7].

As the maps M_1, M_2, and M_3 all have $2n$ vertices, n^2 edges, and n faces, their genus is $\frac{1}{2}(2-2n+n^2-n) = \frac{1}{2}(n-1)(n-2)$. For $n=3$, $K_{3,3}$ is the underlying graph of a regular balanced Cayley map of $Sym[3]$ on the torus and also of a regular antibalanced Cayley map of $\mathcal{Z}_3 \times \mathcal{Z}_2 \simeq \mathcal{Z}_6$, also on the torus.

In fact, the three maps are isomorphic (despite different groups H_1, H_2 and H_3), as can be seen by applying Theorem 6.1. For the isomorphism of M_1 and M_2 a corresponding (τ_1, τ_2)-norm is given by $|a^ib^j|_{1,2} = (-1)^j \cdot 2i$, and for the associated rotary function $\omega_{1,2} : H_1 \to H_2$ we have $\omega_{1,2}(a^ib^j) = ((-1)^{j+1}i, j)$. To see that M_2 is isomorphic to M_3 it is sufficient to take a (τ_2, τ_3)-norm defined by $|(i,j)|_{2,3} = j$ (taking only two values 0 and 1, considered as elements of $\mathcal{Z}_n$); the associated rotary function $\omega_{2,3} : H_2 \to H_3$ is given by $\omega_{2,3}(i,j) = 2i - j$.

We also note that this regular map for $K_{n,n}$ is frequently used to illustrate lifting constructions based on ordinary voltage assignments (see [47] for details).

Unlike the complete bipartite case, for regular complete maps we do have a classification of the regular maps having a complete graph as underlying graph [21]. To use the results of this paper for the classification, we still need a substantial input from the theory of permutation groups and two auxiliary results proved in [21].

Let $M = M(\rho, \lambda)$ be a regular map with dart group G whose underlying graph is K_n. Because of regularity we may identify M with the generic map $Map\langle\rho, \lambda\rangle$ and $Aut(M)$ with G, acting on itself by left multiplication. Then G acts regularly on the set of ordered pairs of distinct vertices of the map (i.e., G is doubly transitive). It follows from a deep theorem of Zassenhaus [50] that in this case the action of G on the vertex set of K_n can be identified with the action of the 1-dimensional affine group $AGL(1,F) = \{z \mapsto az+b : a, b \in F,\ a \neq 0\}$ on the elements z of a field F of order n; hence $n = p^s$ for some prime p. With this action, the group $F^+ = \{z \mapsto z+b : b \in F\}$, isomorphic to the additive group of the field F, is a normal subgroup of

$AGL(1,F)$.

We therefore may identify the vertex set of M with F and the group $G \simeq Aut(M)$ with $AGL(1,F)$ so that Zassenhaus' normal subgroup F^+ acts regularly on the vertex set of the map. But then Theorem 2.2 implies that M is a Cayley map for the Abelian group F^+. Furthermore using the fact that M is complete and invoking Corollary 6.8, we see that M is necessarily a balanced Cayley map. Thus, $M = CM(F^+, X, \tau)$ where $X = F^+ \setminus \{1_{F^+}\}$ and $i\tau = i + (n-1)/2$ or $\tau = 1_{[n-1]}$, depending on whether n is odd or even. In what follows we describe the possible cyclic sequences X.

In the above identification, the generators ρ and λ of G will correspond to a pair of functions $g, h \in AGL(1,F)$ of orders $n-1$ and 2. The order restrictions imply that these functions must have the form $g : z \mapsto \xi z + b$ for a primitive element $\xi \in F$, and $h : z \mapsto -z + c$ (with $c \neq 0$ if n is a power of 2); up to conjugacy in $AGL(1,F)$ we may assume [21] that $b = 0$ and $c = 1$. To reveal X we use part (3) of Proposition 4.4; identifying ρ with g and λ with h we have $x_i = g^i h g^{-i\tau}$. However, one should keep in mind that the generators are considered to be elements of $Aut(M)$ and therefore they will act on the left, and so the above composition is to be read from right to left. The evaluation then gives $x_i : z \mapsto z + \xi^i$ for $i \in [n-1]$, which determines our cyclic sequence $X = X_\xi$.

We have seen that each regular complete map is a balanced Cayley map of the form $CM(F^+, X_\xi, \tau)$ for some primitive element ξ of a finite field F of order $n = p^s$. All primitive elements of F have the form ξ^k for k coprime to $n-1$; we therefore have a total of $\phi(n-1)$ primitive elements in F (where ϕ is Euler's function), and so a potential of $\phi(n-1)$ Cayley maps $CM(F^+, X_\xi, \tau)$. To complete the classification it remains to check for which pairs of primitive elements $\zeta, \eta \in F$ the maps $M_\zeta = CM(F^+, X_\zeta, \tau)$ and $M_\eta = CM(F^+, X_\eta, \tau)$ are isomorphic. By Corollary 6.3 this happens if and only if there exists an automorphism ω of the group F^+ such that for each $i \in [n-1]$, ω sends the function $z \mapsto z + \zeta^i$ onto the function $z \mapsto z + \eta^i$. In particular, this is equivalent to the existence of an automorphism of the additive group of the Galois field $F = GF(p^s)$ which sends ζ^i onto η^i for each i. As was shown in [21], such an automorphism is necessarily a *field automorphism*, that is, it belongs to the Galois group of $GF(p^s)$. It is well known that this group is cyclic of order s and is generated by the Frobenius automorphism $\zeta^i \mapsto (\zeta^i)^p$. Therefore the only way M_ζ and M_ξ can be isomorphic is that $\xi = (\zeta^p)^j$ for some $j \in [s]$. This way each primitive element of F gives rise to s mutually isomorphic maps, and hence the number of pairwise non-isomorphic regular complete maps of order $n = p^s$ is equal to $\phi(n-1)/s$.

7 Homomorphic Images of Cayley Maps

In this section, we show that regular Cayley maps are universal objects in the context of map homomorphisms: for every map M, there is a regular Cayley map $\tilde{M}$ and a map homomorphism $f : \tilde{M} \to M$. If M is finite, then $\tilde{M}$ can also be chosen to be finite. Since a map homomorphism is a graph covering if all vertices have the same valence and there are no semi-edges, we deduce results about regularly covering d-valent graphs by d-valent regular Cayley graphs. It may be helpful to remember throughout this section that map homomorphisms correspond to branched coverings of the underlying surfaces. Thus in all the results of this section, one can replace "map homomorphism" by "branched covering".

For Cayley maps with a given distribution of inverses, we start with the fact that a regular map inherits a Cayley structure from a Cayley homomorph.

Proposition 7.1 *Let $\tilde{M} \to M$ be a homomorphism of d-valent maps, where $\tilde{M}$ is a regular map. If M is a Cayley map with distribution of inverses τ, then so is $\tilde{M}$.*

Proof. Let $G = \langle \lambda, \rho \rangle$ be the dart group of M. According to Corollary 4.2 there is a homomorphism $\phi : G \to Sym[d]$ such that $\phi(\rho) = \eta$ and

$\phi(\lambda) = \tau$. There is also a homomorphism $\tilde{\phi}$ from the dart group $\tilde{G} = \langle\tilde{\rho}, \tilde{\lambda}\rangle$ of $\tilde{M}$ onto G such that $\tilde{\phi}(\tilde{\rho}) = \rho$ and $\phi(\tilde{\lambda}) = \lambda$. Thus $\phi\tilde{\phi} : \tilde{G} \to Sym[d]$ is a homomorphism taking $\tilde{\rho}$ to η and $\tilde{\lambda}$ to τ. Since $\tilde{M}$ is regular, G_e is trivial. By Corollary 4.2 $\tilde{M}$ is a Cayley map with distribution of inverses τ. $\otimes$

Corollary 7.2 *Let $M = M(\rho, \lambda)$ be a Cayley map with distribution of inverses τ. Then its generic regular cover $Map\langle\rho, \lambda\rangle$ is a Cayley map with the same distribution of inverses τ.* $\otimes$

We continue with a refinement of the well known fact that every map is covered by a regular map (cf. Proposition 3.4).

Theorem 7.3 *Let M be a finite map and let d be the least common multiple of all valences in M. Let τ be an involution on the set $[d]$. Then there is a finite regular d-valent Cayley map $\tilde{M}$ with distribution of inverses τ and a map homomorphism $f : \tilde{M} \to M$.*

Proof. Let $M = M(\rho, \lambda)$. Then $Map\langle\rho, \lambda\rangle$ is d-valent, and let ℓ denote its covalence. Let k be the order of the product in $Sym[d]$ of the permutations τ and η. Furthermore, let $\tilde{G} = T(d, k\ell) = \langle x, y : x^d = y^2 = (xy)^{k\ell} = 1\rangle$ be the $(d, k\ell)$-triangle group and let $\tilde{M} = Map\langle x, y\rangle$ be the regular map corresponding to the right $\tilde{G}$-space $(\tilde{G}, \tilde{G})$, so $\tilde{M}$ is the universal $(d, k\ell)$-tessellation.

Since k divides $k\ell$ and $\phi : G \to Sym[d]$ defined by $\phi(x) = \eta$ and $\phi(y) = \tau$ is a homomorphism, Corollary 5.6 implies that the tessellation $\tilde{M}$ is a Cayley map $CM(H, X, \tau)$ for some group H, generating sequence X, and distribution of inverses may be chosen to be τ.

The group homomorphism ϕ induces a map homomorphism $f : \tilde{M} \to Map\langle\rho, \lambda\rangle$. By Proposition 3.4 f is regular, and hence $Map\langle\rho, \lambda\rangle \simeq \tilde{M}/K$ for the subgroup K of $Aut(\tilde{M})$ associated with f. We also have another subgroup of $Aut(\tilde{M})$ at hand, namely, the subgroup $H_L \simeq H$ of left translations of the Cayley map $\tilde{M}$ that acts regularly on the vertex set of $\tilde{M}$.

Let J be the core of the subgroup $K \cap H_L$ in the group $Aut(\tilde{M})$, that is, J is the intersection of all conjugates of $K \cap H_L$ by elements of $Aut(\tilde{M})$; obviously J is normal in $Aut(\tilde{M})$. The group H_L has index d in $Aut(\tilde{M})$, and K has finite index in $Aut(\tilde{M})$ because M is finite. We conclude from elementary group theory that J has finite index in $Aut(\tilde{M})$.

Proposition 3.4 (4) shows that the quotient map $M' = \tilde{M}/J$ is a regular map, with $Aut(M') \simeq Aut(\tilde{M})/J$. If v' is a vertex of M', then v' is the set $J(v) = \{j(v) : j \in J\}$, for some vertex v of M.

Set $H' = H_L/J$. For $h' \in H'$, $h' = Jh$. If $h'(v') = h'(w')$, then $Jh(v) = Jh(w)$, so $h(v) \in Jh(w)$. Since $hJ = Jh$, $v \in J(w)$, since H_L is semiregular on $V(M)$. But then $v' = w'$, so H' is semiregular on $V(M')$. Furthermore, if $v', w' \in V(M')$, there is an $h \in H_L$ such that $h(v) = w$, so $Jh(v') = w'$ and H' acts transitively on M'. Therefore, M' is a Cayley map.

More precisely, M' can be identified with $CM(H', X', \tau)$ where the generating sequence X' arises from $X = (x_1, x_2, \ldots, x_d)$ by replacing x_i with the coset Jx_i; note that $Jx_{i\tau} = (Jx_i)^{-1}$ and so we have the same distribution of inverses τ. It follows that M' is a finite regular Cayley map with τ as its distribution of inverses; it remains only to show that M' covers M.

As J is a normal subgroup of $Aut(\tilde{M})$, and hence of K, we have an epimorphism $\psi : Aut(\tilde{M})/J \to Aut(\tilde{M})/K$. Both $\tilde{M}/J \simeq M'$ and $\tilde{M}/K \simeq Map\langle\rho, \lambda\rangle$ are regular maps, and so their automorphism groups can be identified with their dart groups; hence ψ induces a map homomorphism $M' \to Map\langle\rho, \lambda\rangle$. Composing this homomorphism with the covering $Map\langle\rho, \lambda\rangle \to M$ finishes the proof. $\otimes$

Theorem 7.3 can be extended to simultaneous coverings of maps by Cayley maps. Indeed, let $M_1, M_2, \ldots, M_t$ be a finite collection of finite maps with dart groups $G_j = \langle\rho_j, \lambda_j\rangle$ and let $Map\langle\rho_j, \lambda_j\rangle$ be the corresponding generic regular maps of valence d_j and covalence ℓ_j, $1 \leq j \leq t$. Let d and ℓ be the least common multiple of the d_j's and of the ℓ_j's, respectively. As in the previous proof, the generic map $\tilde{M} = Map\langle x, y\rangle$ associated with the triangle group $T(d, k\ell)$ is a regular Cayley map for some group H, and the regular homomorphisms $\phi_j : \tilde{M} \to M_j$ correspond to subgroups K_j of $Aut(\tilde{M})$. Define J to be the core of the subgroup $K_1 \cap K_2 \cap \ldots \cap K_t \cap H_L$ and continue exactly as in the above proof. Thus, we have the following.

Theorem 7.4 *Let $M_1, M_2, \ldots, M_t$ be a finite collection of finite maps and let d be the least common multiple of all valences appearing in the maps in the collection. Let τ be an involution on the set $[d]$. Then there exists a finite regular d-valent Cayley map M' with distribution of inverses τ and homomorphisms $f_i : M' \to M_i$, $i = 1, 2, \ldots, t$.* $\otimes$

A graph G is *1-transitive* if, for any edge e incident with a vertex v and any other edge e' incident with a vertex v', there is an automorphism A of G such that $A(e) = e'$ and $A(v) = v'$. A consequence of Theorem 7.4 is the following, which refines the result of [2].

Corollary 7.5 *If $G_1, G_2, \ldots, G_k$ are all finite graphs, then there is a finite 1-transitive Cayley graph that simultaneously covers all the G_i.* $\otimes$

An immediate consequence of the above results is:

Corollary 7.6 *Let d be any positive integer. There is a regular d-valent map M such that, for every involution $\tau \in Sym[d]$, M is isomorphic to a Cayley map with distribution of inverses τ.* $\otimes$

Corollary 7.6 shows that some regular maps are "Cayley" in all possible ways. On the other hand, there are regular maps that are Cayley in only one way. It seems difficult in general to say whether a regular map can be Cayley in two different ways. We use the notation $Alt[d]$ to denote the alternating group on d symbols.

Proposition 7.7 *Let $d \geq 2$ be an integer and let $\eta = (1\ 2\ \ldots\ d)$. Suppose $\tau \in Sym[d]$ is an involution such that either $\langle \eta, \tau \rangle = Sym[d]$ or d is odd and $\langle \eta, \tau \rangle = Alt[d]$. Then $Map\langle \eta, \tau \rangle$ is isomorphic to a Cayley map $CM(H, X, \tau')$ if and only for some $i \in [d]$, $\tau' = \eta^i \tau \eta^{-i}$.*

Proof. Since $M = Map\langle \eta, \tau \rangle$ is regular and there is obviously a homomorphism from its dart group to $Sym[d]$ taking $\rho = \eta$ to η and $\lambda = \tau$ to τ, M is a Cayley map with distribution of inverses τ. It follows that M is also a Cayley map with distribution of inverses $\eta^i \tau \eta^{-i}$, for all $i \in [d]$.

For the converse, we first suppose $\langle \eta, \tau \rangle = Sym[d]$. Suppose M is also a Cayley map with distribution of inverses τ'. By Corollary 4.2 there is an epimorphism $\phi : \langle \eta, \tau \rangle \to \langle \eta, \tau' \rangle$ such that $\phi(\eta) = \eta$ and $\phi(\tau) = \tau'$.

The image of ϕ has order at least d and, for $d \geq 3$, the only proper normal subgroup of $Sym[d]$ is $Alt[d]$. Thus, ϕ is an automorphism of $Sym[d]$. For $d \neq 6$, the only automorphisms of $Sym[d]$ are inner, so $\phi(\eta) = \eta$ implies τ' is a conjugate of τ by a power of η.

For $d = 6$, ϕ is an automorphism of $Sym[d]$ that takes η^2, the product of two 3-cycles, to itself. This implies [39] that ϕ is inner, and we are done as above.

We leave the case $d = 2$ for the reader.

In the case of $Alt[d]$, we note that for $d \neq 6$ all automorphisms of $Alt[d]$ are conjugation by an element of $Sym[d]$. The remaining details are left to the reader. $\otimes$

Proposition 7.7 cannot be easily generalized to cases when $\langle \eta, \tau \rangle$ is neither $Sym[d]$ nor $Alt[d]$. For example, if $d = 6$ and $\tau = (1\,4)(2\,5)$, then the normal closure of τ in $\langle \eta, \tau \rangle$ consists of just $1_{[6]}$, τ, $(2\,5)(3\,6)$, and $(3\,6)(4\,1)$. Thus, there is a homomorphism from $\langle \eta, \tau \rangle$ to $Sym[d]$ taking η to η and τ to $1_{[6]})$, which shows $Map\langle \eta, \tau \rangle$ is isomorphic to a balanced Cayley map. (We thank Marston Conder for this example.)

The octahedron is the Cayley map $CM(Sym[3], X, (1\,2))$ and so by Proposition 7.7 all of its "Cayley distributions of inverses" are conjugate to $(1\,2)$. Similarly, the icosahedron is $CM(Alt[4], X, (1\,2)(3\,4))$, so its only possible Cayley distributions of inverses are conjugate to $(1\,2)(3\,4)$. (Note that the octahedron is the smallest example of a regular Cayley map that is not isomorphic to either a balanced or an antibalanced Cayley map.)

Initially, we were interested in discovering regular Cayley maps that could not be represented as either balanced or antibalanced Cayley maps. It is easy to see that a regular balanced d-valent Cayley map has covalence a multiple of either $d/2$ or d. Antibalanced maps have even covalences. This gives a very easy way to show that a Cayley map can be neither balanced nor antibalanced. For example, if $\tau = (1\,2)$, then $\eta\tau = (2\,3\,\ldots\,d)$, in which case for d even, $Map\langle\eta, \tau\rangle$ is not isomorphic to either a balanced or an antibalanced Cayley map. Similarly, if $d \geq 5$ is odd and $\tau = (1\,2)(3\,4)$, then $Map\langle\eta, \tau\rangle$ is neither balanced nor antibalanced.

In general, it may not be easy to exhibit a finite regular Cayley map with given covalence and given distribution of inverses. The problem is that due to a nonconstructive step in the proof of Theorem 7.3, it may be difficult to explicitly describe the subgroups H and K and hence the resulting finite map M'. In what follows we therefore give a purely constructive proof of a strengthening of Corollary 7.6 that also includes the feasible covalences of a regular Cayley map with a given distribution of inverses.

To this end, let τ be any involution on $[d]$ and let M_1 be the one-vertex Cayley map $CM(\langle 1\rangle, X, \tau)$ (so X is necessarily a cyclic sequence all of whose entries are 1). We may identify the dart set of M_1 with $[d]$, the dart-reversing involution with τ, and the rotation with the d-cycle η. The faces of M_1 are the orbits of $\sigma = \tau\eta$. Let the orbit of σ containing i have ℓ_i elements and let ℓ_τ be the order of σ in $Sym[d]$ (so ℓ_τ is the least common multiple of the ℓ_i).

Now let $CM(H, X, \tau)$ be a Cayley map of valence d with dart group $G = \langle\rho, \lambda\rangle$. Cycles of the permutation $\lambda\rho$ give the face boundaries of the map. If we concentrate just on the second coordinates of the darts on an arbitrary face boundary, i.e., if we apply the permutation $\lambda\rho$ on any dart (h, i) and disregard the first coordinate, we obtain a sequence of the form i, $i\tau + 1$, $(i\tau + 1)\tau + 1$, etc.; this sequence is just $i, i\sigma, i\sigma^2, \ldots$. Therefore the length of the $\lambda\rho$-orbit of a dart (h, i), that is, the covalence of the face that appears on the left side of the dart (h, i), is a multiple of ℓ_i.

It is obvious from the above discussion that if $CM(H, X, \tau)$ is regular, then its covalence is necessarily a multiple of ℓ_τ, the order of σ. (We have encountered this quantity in Corollaries 5.6 and 5.7 as well as in the proof of Theorem 7.3 where it was denoted by k.) Conversely, we have:

Theorem 7.8 *Let τ be an arbitrary involution on the set $[d]$ where $d \geq 2$. Then for any given multiple ℓ of ℓ_τ there exists a regular Cayley map M of valence d and covalence ℓ whose distribution of inverses is τ.*

We remark that this theorem is a consequence of Theorem 8.1 in the next section, about the existence of regular branched covering with given branching information. Theorem 8.1, however, is not at all constructive and depends heavily on topological and algebraic results beyond the scope of this paper. The proof of Theorem 7.8 given here is via an explicit construction.

Proof. The result is easy for $d = 2$, as shown by Cayley maps of valence two for cyclic ($\tau = (1\ 2)$) and dihedral ($\tau = 1_{[2]}$) groups. We therefore assume that $d \geq 3$ and distinguish two cases depending on the properties of the involutory permutation τ of the set $[d]$. Let M_1 denote the one-vertex map $CM(\langle 1 \rangle, X, \tau)$.

Case 1: M_1 has at least two faces. Then there is some edge of M_1 separating two faces, so if $j \in [d]$ is one of the two darts assigned to this edge, j and $j\tau$ are in the boundaries of different faces of M_1. In other words, j and $j\tau$ are in different σ-orbits. In particular, $j \neq j\tau$.

Let ℓ be the given multiple of ℓ_τ, say, $\ell = q\ell_\tau$. Let $\ell' = \mathrm{lcm}\{\ell_j, \ell_{j\tau}\}$ and let $k = q\ell_\tau/\ell'$. Furthermore, let $L = \{\ell_i : i \notin \{j, j\tau\}\} \cup \{k\ell_j, k\ell_{j\tau}\}$.

We claim that the least common multiple of the collection L is equal to $\ell = q\ell_\tau$. Indeed, setting $L^- = L \setminus \{k\ell_j, k\ell_{j\tau}\}$ and realizing that $\mathrm{lcm}(L^-)$ divides ℓ_τ, we have $\mathrm{lcm}(L) = \mathrm{lcm}(L^-, k\ell_j, k\ell_{j\tau}) = \mathrm{lcm}(L^-, k\ell') = \mathrm{lcm}(L^-, q\ell_\tau) = q\ell_\tau = \ell$.

Let $M = CM(H, X, \tau)$ be the Cayley map for the (additive) cyclic group $H = \mathcal{Z}_k$ and generating sequence $X = (x_1, x_2, \ldots, x_d)$ where $x_j = 1$, $x_{j\tau} = -1$, and $x_i = 0$ for the remaining $i \in [d]$, with distribution of inverses τ. (The underlying graph of M is just a k-circuit with loops and semi-edges at every vertex.)

The face boundary of M containing $(0, j)$ is the $\lambda\rho$ orbit containing $(0, j)$. Since $j\tau$ never appears as the second coordinate of a dart in this orbit and ℓ_j is the smallest positive integer such that $j\sigma^{\ell_j} = j$, this orbit is

$$\{(0,j), (1, j\sigma), \ldots, (1, j\sigma^{\ell_j - 1}), (1, j), (2, j\sigma), \ldots,$$
$$(k-1, j), (0, j\sigma), \ldots, (0, j\sigma^{\ell_j - 1})\},$$

which has length $k\ell_j$. Similarly, the $\lambda\rho$ orbit containing $(0, j\tau)$ has length $k\ell_{j\tau}$. If (h, i) is in neither of these orbits, then the orbit containing (h, i) is

$$\{(h, i), (h, i\sigma), \ldots, (h, i\sigma^{\ell_i - 1})\},$$

which is of length ℓ_i. Therefore the covalences of M are exactly the numbers appearing in L.

By Corollary 7.2, $Map\langle\rho, \lambda\rangle$ is a regular Cayley map with τ as its distribution of inverses. Moreover, we know that the covalence of $Map\langle\rho, \lambda\rangle$ is the least common multiple of the covalences of M, which is equal to $\mathrm{lcm}(L) = \ell$, as required.

Case 2: M_1 has only one face. In this case, σ has only one orbit. In particular, for each $i \in [d]$, there exists an $m_i \geq 0$ such that $i\sigma^{m_i} = i\tau$. Thus, $\ell_\tau = d$. Let $\ell = qd$ be the given multiple of ℓ_τ. We consider two subcases.

a) There are at least two semi-edges in M_1. That is, there exist distinct symbols $j, j' \in [d]$ such that $j\tau = j$ and $j'\tau = j'$.

Let $M = CM(\mathcal{D}_q, X, \tau)$ be the Cayley map for the dihedral group $\mathcal{D}_q = \langle a, b : a^q = b^2 = (ab)^2 = 1\rangle$ of order $2q$ with generating sequence $X = x_1, x_2, \ldots, x_d$ where $x_j = b$, $x_{j'} = ba$, and $x_i = 1$ for the remaining $i \in [d]$. (The underlying graph of M is just a $(2q)$-circuit with loops and semi-edges at every vertex.)

The dart $(1, j)$ is in the $(\lambda\rho)$-orbit

$$\{(1,j),(b,j\sigma),\ldots,(b,j'),(a,j'\sigma),\ldots,\\ (a,j),(ab,j\sigma),\ \ldots,(a^2,j'\sigma),\ \ldots,(a^q,j'\sigma),\ldots\},$$

which has length qd. The dart (b, j) is in a different $(\lambda\rho)$-orbit, which is also of length qd. These are the only $(\lambda\rho)$-orbits.

Thus, M has exactly two faces, both of covalence $\ell = qd$, so by Corollary 7.2 the generic map $Map\langle\rho, \lambda\rangle$ is the desired regular Cayley map with distribution of inverses τ and covalence ℓ.

b) M_1 has at most one semi-edge. Thus, at most one symbol of $[d]$ is fixed by τ. As $d \geq 3$, there exists a smallest positive integer s such that $j\sigma^s = j\tau$ for some $j \in [d]$, $j \neq j\tau$; in what follows we fix such a j.

Since $j\sigma = j\tau + 1$ we see that $s \geq 2$. Moreover, in the case when $(j\sigma)\tau = j\sigma$ we have $s \geq 3$, because then $j\sigma^2 = (j\sigma)\sigma = (j\sigma)\tau + 1 = j\sigma + 1 = j\tau + 2 \neq j\tau$. Let

$$j' = \begin{cases} j\sigma & \text{if } (j\sigma)\tau \neq j\sigma \\ j\sigma^2 & \text{if } (j\sigma)\tau = j\sigma \end{cases};$$

as τ fixes at most one symbol of $[d]$ we always have $j'\tau \neq j'$.

By the definition of j' and the minimality of s, it now follows that the four distinct symbols $j, j\tau, j', j'\tau$ appear in the cycle $(j, j\sigma, j\sigma^2, \ldots, j\sigma^{d-1})$ in the order $(j, \ldots, j', \ldots, j\tau, \ldots, j'\tau, \ldots)$. This time let M be $CM(\mathcal{D}_{2q}, X, \tau)$ for the dihedral group $\mathcal{D}_{2q} = \langle a, b : a^{2q} = b^2 = (ab)^2 = 1\rangle$ of order $4q$, generating set $X = (x_1, x_2, \ldots, x_d)$ such that $x_j = x_{j\tau} = b$, $x_{j'} = x_{j'\tau} = ba$, and $x_i = 1$ for the remaining $i \in [d]$. (It is not so clear what the underlying graph of M is in this case, except to say it is 4-valent plus loops and semi-edges.)

As in the previous cases, it is not hard to check that the $(\lambda\rho)$-orbits containing the darts $(1, j)$, (a, j), (b, j), and (ba, j) are distinct, all have length $\ell = qd$, and are all the $(\lambda\rho)$-orbits. Hence the map M has exactly four faces, all of length ℓ. Again, by Corollary 7.2, the generic map $Map\langle\rho, \lambda\rangle$ is the required regular Cayley map with covalence ℓ and distribution of inverses τ. $\otimes$

8 Existence of Cayley maps with given quotient information

Map homomorphisms correspond topologically to branched coverings of the underlying surfaces. A continuous function $f : \tilde{\mathcal{S}} \to \mathcal{S}$, where $\tilde{\mathcal{S}}$ and $\mathcal{S}$ are surfaces, is a *covering map* if, for each point $x \in \mathcal{S}$, there is a neighborhood N of x such that, for each $\tilde{x} \in f^{-1}(x)$, if $\tilde{N}$ is the component of $f^{-1}(N)$ containing $\tilde{x}$, then $f : \tilde{N} \to N$ is a homeomorphism. The continuous function $f : \tilde{\mathcal{S}} \to \mathcal{S}$ is a *branched covering* if there is a discrete set $\tilde{B}$ of points in $\tilde{\mathcal{S}}$ such that f is a covering map when restricted to $\tilde{\mathcal{S}} - \tilde{B}$. It is a theorem in topology (cf., [24]) that, for every $b \in \tilde{B}$, there is an integer $r > 0$ such that f wraps (in the sense of winding number) some neighborhood of b r times around $f(b)$. Points in both $\tilde{B}$ and $f(\tilde{B})$ are *branch points* (the term comes from the theory of Riemann surfaces); r is the *branching order* at the branch point and may be different at different branch points.

An *automorphism* of a branched covering $f : \tilde{\mathcal{S}} \to \mathcal{S}$ is a homeomorphism $h : \tilde{\mathcal{S}} \to \tilde{\mathcal{S}}$ such that $f \circ h = f$. The set of automorphisms is obviously a group $Aut(f)$. The branched covering is *regular* if, for every non-branch point $x \in \mathcal{S}$, $Aut(f)$ acts transitively (and hence regularly, because two automorphisms either disagree on every non-branch point or are equal, c.f., Lemma V.6.1 in [32]) on $f^{-1}(x)$. It follows that if f is a regular branched covering and $b_1, b_2 \in \tilde{B}$ are such that $f(b_1) = f(b_2)$, then the integer r is the same for both b_1 and b_2. (In what follows, the reader should recall that in this work surfaces are connected.)

A map homomorphism $f : \tilde{M}(\tilde{\rho}, \tilde{\lambda}) \to M(\rho, \lambda)$ takes cycles of $\tilde{\rho}$ and $\tilde{\lambda}\tilde{\rho}$, namely the vertices and faces of $\tilde{M}$, and wraps them an integral number of times around cycles of ρ and $\lambda\rho$, namely the vertices and faces of M. Thus we can associate to f a branched covering of the underlying surfaces for the maps with branch points at vertices and centers of faces; if f takes an edge $\tilde{e}$ of $\tilde{M}$ to a semiedge in M, then there is also a branch point of order two at the center of the edge $\tilde{e}$. A common situation in this work is that the vertices in both $\tilde{M}$ and M have the same valence. In such a case, the associated branched covering has branch points only inside the faces and in edges taken to semiedges; if M has no semiedges, then the branched covering is actually a covering when restricted to the underlying graphs of the maps. It is not hard to see, conversely, that any branched covering of surfaces is induced by a map homomorphism of maps in the surfaces, although the maps are not at all unique (any map situated appropriately with respect to the branch points will do).

Let $M = CM(H, X, \tau)$ and let $M_1 = M/H_L$ be the one-vertex quotient. Then there is a regular branched covering f from M to M_1, and we can identify the edges of M_1 with the generators in X and the dart-reversing involution λ of M_1 with τ. For each face F of M_1, there is the collection of faces $f^{-1}(F)$ of M, each of which is wrapped by f the same number t of times around F; t is the branching order around a point in the interior of F ($t = 1$ means there is no branching). The one-vertex quotient map M_1 together with the branching orders for each face is the *quotient information for the Cayley map M*. In this section we consider the problem of determining when given quotient information is realizable by some Cayley map.

There is an obvious way of trying to realize given quotient information by using the information to define the group H. We simply let $(x_1, \ldots, x_d)$ be the generating sequence for H, where $x_i = x_j^{-1}$ if and only if $i\tau = j$.

Each face F_j of M_1 corresponds to a word W_j in the generators, obtained by tracing out the boundary of the face (i.e., looking at the cyclic order in each orbit of $\lambda\rho$). In order to have branching order t_j in face F_j, we need $W_j^{t_j} = 1$ in the group H. Thus we should begin with the following presentation for H:

$$H = \langle x_1, \ldots, x_d : \forall i \; x_i x_{i\tau} = 1, \; \forall j \; W_j^{t_j} = 1 \rangle.$$

Call H the *universal group for the given quotient information.* Note that since all relations in this presentation correspond to faces, the resulting Cayley map is simply-connected, that is, the surface for the map is the plane or sphere. There is one problem: how do we know such a group exists, where the true orders of the words W_j are really t_j? This is not always the case, and this problem is in general a nontrivial question in combinatorial group theory.

Example. *Suppose M_1 is a single loop embedded in the sphere. Then $X = (x_1, x_2)$ and $\tau = (1\ 2)$, and there are two faces, one bounded by the x_1-dart and the other by the x_2-dart. Suppose we want the branching order to be 3 in the first face and 5 in the second face. Since in our group H we have both $x_1^3 = 1$ and $x_1^{-5} = 1$, we must have $x_1 = 1$. The true order of x_1 is then neither 3 nor 5, but 1.*

Rather than becoming entangled in group theory, we look at the problem in terms of branched coverings. We are fortunate that this problem has already been solved, but unfortunately the methods are beyond the scope of the current work. In this section, we give a summary of the results we need. Details can be found in [3, 4, 27, 31, 38, 42, 48].

Given the surface $\mathcal{S}_1$ containing the one-vertex map M_1, we want to regularly branch cover the surface by the plane or sphere in such a way that there are prescribed branching orders at specified points (the centers of the faces or the nonvertex ends of semiedges). Does such a regular branched covering exist? If so, then the inverse image of the bouquet of loops and semiedges of the map M_1 is a Cayley graph in the covering surface, since the group of covering transformations acts regularly on the inverse image of the single vertex. In fact, the covering transformations also preserve the cyclic ordering at each vertex, so we have a Cayley map whose group is the covering transformation group and whose quotient is M_1 with the prescribed branching.

The existence of regular branched coverings of this type has been known for some time. To describe the situation, we recall the Riemann-Hurwitz equation for an n-sheeted regular branched covering from a surface $\tilde{\mathcal{S}}$ to a surface $\mathcal{S}$. Suppose there are m branch points in $\mathcal{S}$ of branching orders $t_1, \ldots, t_m$. Take any map in $\mathcal{S}$ such that every branch point is a vertex. Every edge and face in the map gives rise to n edges or faces in $\tilde{\mathcal{S}}$, and so does every vertex not at a branch point. A branch point in $\mathcal{S}$ of branching order t_i, however, gives rise to only n/t_i vertices in $\tilde{\mathcal{S}}$ (and thus is deficient in inverse images by $n(1 - 1/t_i)$). Thus if χ denotes Euler characteristic, we have the *Riemann-Hurwitz Equation*:

$$\chi(\tilde{\mathcal{S}}) = n\left[\chi(\mathcal{S}) - m + \sum_{i=1}^{m}(1/t_i)\right]. \tag{6}$$

The right side is negative, except if: (1) there is no branching and $\mathcal{S}$ is the torus; (2) $\mathcal{S}$ is the sphere and there are two or fewer branch points; (3) $\mathcal{S}$ is the torus and either there three branch points of branching orders $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$, or there are four branch points all of branching order 2; or (4) $\mathcal{S}$ is the sphere and there are three branch points either of branching orders $(2,3,r)$, $2 < r < 6$, or of orders (2,2,s), $2 \leq s$. Notice that in the case where $\mathcal{S}$ is the sphere, there cannot be only one branch point. (For if there is one branch point of branching order t, then $2 \geq \chi(\tilde{\mathcal{S}}) = n + n/t$, which is possible only if $n = t = 1$, but then the covering is the identity and there is no branching.) Finally, if $\mathcal{S}$ is the sphere and there are two branch points, then $2 \geq \chi(\tilde{\mathcal{S}}) = n/t_1 + n/t_2$, but since the branching orders t_1 and t_2 divide the number n of sheets, we must have $t_1 = t_2 = n$. Thus, we see why the example above is one of the exceptions.

We conclude that the Riemann-Hurwitz equation for n-sheeted regular branched coverings allows any quotient information with two exceptions: (1) the quotient surface is the sphere and there is just one branch point with branching order greater than 1; and (2) the quotient surface is the sphere and there are exactly two branch points with branching orders greater than 1 and these two orders are different.

Call the quotient information *allowable* in all cases except for the two exceptions. The important result we need is that allowable quotient information is realizable by a regular branched covering of the quotient surface S by the plane, if the right side of the Riemann-Hurwitz equation is nonpositive, or by the sphere, if the right side is positive.

The proof involves exhibiting a group acting appropriately on either the sphere, plane or hyperbolic plane. The case where the right side is negative is a question about Fuchsian groups, or, more generally, nonEuclidean crystallographic groups. The existence of such groups comes down to the existence of polygons in the hyperbolic plane with prescribed internal angles (whose area is related to the right side of the Riemann-Hurwitz equation). When the right side is zero, the groups are the 17 Euclidean crystallographic groups. When the right side is positive, we get the spherical crystallographic groups for regular prisms or platonic solids.

Thus we conclude that all allowable quotient information can be realized by simply-connected maps. What about finite maps? Again, the work has already been done. What we need to do is to add enough relations to the universal group for given quotient information so that the group becomes finite, but so that the orders of the face-words remain as prescribed. In other words, we want the universal group to have a homomorphism onto a finite group such that each face word W_i still has order t_i in the finite group.

A group G is *residually finite* if, given any finite set of nonidentity elements $g_1, \ldots, g_n$, there is a homomorphism from G to a finite group which takes none of $g_1, \ldots, g_n$ to the identity. From our previous discussion, we know that the universal group for given possible quotient information is a crystallographic group, either hyperbolic, Euclidean, or spherical. It is known that all of these groups are residually finite. Spherical groups are finite and therefore residually finite by default; the finite quotients of Euclidean crystallographic groups are not hard to describe (see [13]); Mal'cev proved that finitely generated subgroups of $SL_n(F)$, where $n \geq 2$ and F is any infinite field, are residually finite and therefore the nonEuclidean crystallographic groups are residually finite [28]. To obtain the desired finite quotient group for a given universal group G, just take the homomorphism from G onto a finite group which takes none of the elements W_i^s, $s < t_i$, to the identity. Since this list is finite and G is residually finite, such a homomorphism exists.

We summarize our discussion with this theorem:

Theorem 8.1 *Any quotient information is realizable by a Cayley map, except if the quotient map M_1 is the sphere and either exactly one branching order is greater than one or exactly two branching orders are greater than one and they are unequal (each semiedge has branching order two). The Cayley map can be chosen to be either simply-connected (that is, in the sphere or plane) or finite.* $\otimes$

Thus, suppose we are given a one-vertex map M_1 with k faces having covalences ℓ_i, $i = 1, 2, \ldots, k$. Suppose further that we are given the list $T = t_1, t_2, \ldots, t_k$ of respective branching orders. Then the map $M^T = CM(H, X, \tau)$ is the *covering map induced by the quotient information* (M_1, T), where

$$H = \langle x_1, \ldots, x_d : \forall\, i\ x_i x_{i\tau} = 1,\ \forall\, j\ W_j^{t_j} = 1 \rangle,$$

$X = (x_1, \ldots, x_d)$ with x_i assigned to the dart $i \in [d]$, and τ is the dart-reversing involution of M_1.

With just the two exceptions, the map M^T is a Cayley map with covalences $\ell_j t_j$, $j = 1, 2, \ldots, k$. This Cayley map is in a simply-connected surface and so is either in the sphere (when H is finite) or the plane (when H is infinite).

In terms of actually constructing Cayley maps, the method described above is impractical. It turns out to be difficult to describe maps with one vertex. It is more convenient to describe their duals, i.e., the maps with one face. In the plane, these are just trees. Since this is the actual construction that we will use in the next section to exhibit Cayley maps, we describe it here.

Dual Quotient Information Method. Let M be a locally finite map with one face in an oriented surface. For each dart e of M, let x_e be the label assigned to e. Each vertex v of M has a cyclic rotation of the darts $(e_1, e_2, \ldots, e_{d_v})$ emanating from it. Let W_v denote the product $x_{e_1} x_{e_2} \cdots x_{e_{d_v}}$. (We may arbitrarity choose any of the d_v possibilities, up to cyclic rotation, for W_v.)

For each vertex v of M, we associate a positive integer t_v and we note that, since M has a single face, $\lambda\rho$ has a single cycle $(1\,2\,\ldots\,n)$, where n is the covalence of M. We set $X = (x_1, x_2, \ldots, x_n)$ and

$$G = \langle x_1, x_2, \ldots, x_n : \forall\, i\;\, x_i x_{i\tau} = 1,\;\, \forall\, v\;\, W_v^{t_v} = 1 \rangle.$$

This gives us the Cayley map $CM(H, X, \tau)$, with valence n and covalences $d_v t_v$. This is precisely the covering map M^T induced by the quotient information (M^*, T), where M^* is the dual of M and $T = (t_v)$ are the branching orders. (There are still the two exceptions described earlier.)

The reader familiar with voltage and current constructions will recognize our original description of the construction as a voltage construction and the dual quotient information method as the corresponding current construction.

9 Existence Criteria for Cayley Maps

We begin this section with necessary and sufficient conditions for the existence of a Cayley map with given valence and covalences. This will then be used to construct planar Cayley maps. In particular, we are interested in planar tilings (see, for example, the book [15]). Such tilings have a long history of motivating mathematical research. For us, they provide a convenient class of maps on which to display the utility of our methods. We will see in the next section exactly which regular and edge-transitive tilings of the plane are Cayley maps.

Theorem 9.1 *Let $d \geq 3$ be an integer and let $\ell_1, \ell_2, \ldots, \ell_k$ be distinct positive integers, all at least 3. There is a d-valent Cayley map whose set of covalences is exactly $\{\ell_1, \ell_2, \ldots, \ell_k\}$ if and only if for each $i \in [k]$ there exist a positive integer r_i and (not necessarily distinct) divisors $d_{i,1}, d_{i,2}, \ldots, d_{i,r_i}$ of ℓ_i, such that $\sum_{i=1}^{k} r_i \leq 1 + \lfloor d/2 \rfloor$ and $d = \sum_{i=1}^{k} \sum_{j=1}^{r_i} d_{i,j}$.*

Proof. *Necessity.* Suppose $M = CM(H, X, \tau)$ is a d-valent Cayley map and distinct covalences $\ell_1, \ell_2, \ldots, \ell_k$. Then there is a homomorphism from M to a d-valent 1-vertex map M_1. In particular, if $X = (x_1, x_2, \ldots, x_d)$, we can label the darts of M_1 with $1, 2, \ldots, d$ so that the rotation of M_1 is $(1, 2, \ldots, d)$ and dart i is the image of all darts (h, i) of M. Then $H = \langle x_1, \ldots, x_d : \forall\, i\ x_i x_{i\tau} = 1_H,\ \forall\, F\ W_F^{t_F} = 1_H,\ \ldots \rangle$, where W_F is the word in the x_i's obtained by tracing the boundary of the face F of M_1.

Let r_i be the number of faces of M_1 each covered by a face of M having covalence ℓ_i and let $d_{i,1}, d_{i,2}, \ldots, d_{i,r_i}$ be the covalences of these faces of M_1. Clearly, for each $i \in [k]$ and $j \in [r_i]$, there is a unique face F of M_1 such that $d_{i,j} t_F = \ell_i$. Thus, each $d_{i,j}$ is a divisor of ℓ_i. Furthermore, $d = \sum_{i=1}^{k} \sum_{j=1}^{r_i} d_{i,j}$, since every dart of M_1 is counted in exactly one face of M_1.

For maps with loops, parallel edges and semi-edges, Euler's formula still holds, as long as we do not count semi-edges. That is $v - e + f = \chi$ holds with v, e, and f, respectively the number of vertices, edges (including loops and parallel edges, but not semi-edges), and faces. Since, for the map M_1, $v = 1$, $f = r$, $e \leq d/2$, and $\chi \leq 2$, we have $2 \geq \chi = v - e + f \geq 1 - (d/2) + r$, as required.

Sufficiency. Suppose we have positive integers r_i and $d_{i,j}$ satisfying the inequality

$$\sum_{i=1}^{k} r_i \leq 1 + \left\lfloor \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{r_i} d_{ij} \right\rfloor .$$

We shall exhibit an appropriate tree embedded in the sphere upon which to use the dual quotient information method.

The number of vertices in the tree is to be $r = \sum_{i=1}^{k} r_i$. For each $i \in [k]$ and $j \in [r_i]$, we have one vertex $v_{i,j}$ which is to have valence $d_{i,j}$. Set $d = \sum_{i=1}^{k} \sum_{j=1}^{r_i} d_{i,j}$. Suppose first that every $d_{i,j} = 1$. Then $d = r$ so $d \leq 1 + \lfloor d/2 \rfloor$, which implies that $d \leq 2$, contradicting the hypothesis $d \geq 3$.

In the remaining case, some $d_{i,j} > 1$. Arbitrarily form a path using exactly the vertices $v_{i,j}$ for which $d_{i,j} > 1$.

For each (i, j) for which $d_{i,j} = 1$, we add an edge to the tree joining $v_{i,j}$ to a vertex of the path in such a way as to let no vertex $v_{i',j'}$ in the path have valence $> d_{i',j'}$.

To see that this is always possible, suppose there are u vertices $v_{i,j}$ for which $d_{i,j} = 1$, so $u < r$. In this case, we need that $d - u - 2(r - u - 1) \geq u$, since the sum of the valences available to vertices in the path is $d - u$ and we have placed $r - u - 1$ edges into the tree already. Since $r \leq 1 + \lfloor d/2 \rfloor$, we have $2r \leq 2 + d$, as required.

Now, a vertex $v_{i,j}$ on the original path may not yet have valence $d_{i,j}$. In this case, we add enough semi-edges incident with $v_{i,j}$ to bring its valence up to $d_{i,j}$. This gives the tree T.

Note that the sum of the valences of T is the covalence of T, so the covalence of T is d, as required.

The parameter $t_{i,j}$ associated with the vertex $v_{i,j}$ is the ratio $\ell_i/d_{i,j}$. It is easy to see that, since $d \geq 3$ and all covalences are at least 3, there are at least three branch points, so Theorem 8.1 implies that the constructed map has the desired covalences. $\otimes$

There are a handful of cases not covered by Theorem 9.1. In the event $d \leq 2$, the only possible Cayley maps are:

(1) $d = 1$: M is either a semi-edge with one vertex or an edge joining distinct vertices of degree 1;

(2) $d = 2$: M is either two semi-edges incident with a single vertex, or an edge joining distinct vertices, each incident with one semi-edge, or a circuit.

For $d \geq 3$, covalence 1 can occur only if there is an $i \in [d]$ such that $i\tau = i+1$ and $x_i = 1_H$. In other words, we have a loop with trivial generator and the ends are consecutive in the rotation. Deleting these generators (alternatively, removing the loops) gives a simpler map, but the covalences have changed, so we cannot easily give an arithmetic condition as in Theorem 9.1.

Covalence 2 can occur only if there exists an i such that $x_i = x_{i+1}$ and $(i+1)\tau + 1 = i\tau$. In such a case, the repeated generators x_i, x_{i+1} can be merged into one (or the parallel edges can be merged into one). Repeating this operation as often as necessary removes all covalence 2 faces without changing the lengths of the remaining faces. Thus, Theorem 9.1 applies fairly directly to this situation.

We note that Theorem 9.1 is a refinement of Corollary 5.7, which is the case $k = 1$.

It would be very interesting to provide more detail about the cyclic sequence of covalences. For example, the reader may verify that there is no Cayley map with the cyclic sequence $(8, 8, 10, 10, 8, 10)$ of covalences around each vertex. However, as we shall see in the next section, there is one with $(8, 10, 8, 10, 8, 10)$. (The latter sequence yields an edge-transitive map [14], while the former does not.) Thus, it is not just the numbers or even their multiplicities, but the *order* of the numbers that determines whether or not a given covalence sequence can be achieved in a Cayley map.

A map is *simply connected* if every cycle is contractible. Since the only simply connected surfaces are the sphere and the plane, a map is simply connected if and only if the supporting surface of the map is the sphere or the plane. We shall also refer to such maps as *planar*, with the understanding that they might be in the sphere (if they are finite). We note that if M is generated by the dual quotient information method, then M is simply connected and therefore is planar.

In the following result, we obtain an analogue of Theorem 9.1 for a planar map whose underlying graph is a Cayley graph.

Theorem 9.2 *Let M be a 3-connected planar map with finite covalences whose underlying graph is the Cayley graph with Cayley group H and generating sequence X. Let $d = |X|$ be the valence of M and let $\ell_1, \ell_2, \ldots, \ell_k$ be the distinct covalences that occur in M. Then either M is a Cayley map for H and some cyclic orientation of X or for each $i \in [k]$ there exist divisors $d_{i,1}, \ldots, d_{i,r_i}$ of ℓ_i such that $\sum_{i=1}^{k} \sum_{j=1}^{r_i} d_{i,j} = 2d$ and $\sum_{i=1}^{k} r_i \leq d$.*

Proof. Since M is 3-connected and planar, every graph automorphism extends to a map homeomorphism (i.e., either a map automorphism or a map automorphism composed with a reflection of the plane or sphere). In particular, the left translations by elements of H are examples of such homeomorphisms. For each $h \in H$, left multiplication by h either preserves the orientation of the surface or reverses it. Let K be the subgroup of H consisting of those h that preserve orientation. If $K = H$, then M is a Cayley map. Otherwise, K has precisely two cosets in H and the map $M_2 = M/K$ has just two vertices, both of valence d, with an orientation-reversing homeomorphism that interchanges them.

Let r_i be the number of faces of M_2 covered by faces of M having covalence ℓ_i and let $d_{i,1}, \ldots, d_{i,r_i}$ be the covalences of these faces of M_2. Clearly, each $d_{i,j}$ is a divisor of ℓ_i and $\sum_{i=1}^{k} \sum_{j=1}^{r_i} d_{i,j} = 2d$, since every dart of M_2 occurs in exactly one face of M_2.

To see that $\sum_{i=1}^{k} r_i \leq d$, note that M_2 is a map in a surface with Euler characteristic $\chi \leq 2$, two vertices, at most d edges (again we ignore semi-edges) and $r = \sum_{i=1}^{k} r_i$ faces. Thus, $2 \geq 2 - d + r$, or, equivalently, $r \leq d$. $\otimes$

We have the following analogue of Corollary 5.7.

Corollary 9.3 *Let M be a 3-connected planar map whose underlying graph is the Cayley graph with Cayley group H and generating sequence X. Let $d = |X|$ be the valence of M and let M have constant covalence $\ell < \infty$. Then ℓ has a prime divisor at most d.*

Proof. If M is a Cayley map, then we apply Corollary 5.7. Otherwise, by Theorem 9.2, there are divisors $d_1, d_2, \ldots, d_r$ of ℓ such that $\sum d_i = 2d$ and $r \leq d$. Since $2d > d$, some $d_i > 1$. Thus, $2 \leq d_i \leq 2d$. It will suffice to show that if $d_i > d$, then d_i is even, which will in turn imply that ℓ is even and 2 is the required divisor of ℓ.

Let F be a face of M_2 of covalence d_i. Note that since there are only $2d$ darts, F is the only face that has covalence d_i.

Since $d_i > d$, there is a pair of darts of F, one at each vertex of M_2, that are equivalent under the orientation-reversing map homeomorphism ϕ of M_2 that interchanges the two vertices. Hence the boundary of F is mapped to itself by ϕ, and so it uses the same number of darts incident with each vertex of M_2. Thus, d_i is even. $\otimes$

10 Classifying Planar Tilings as Cayley Maps and Cayley Graphs

In this section, we begin the classification of planar tilings that are Cayley maps or have a Cayley graph as their underlying graph. The book by Grünbaum and Shephard [15] on planar tilings and the memoir by Graver and Watkins [12] on edge-transitive planar maps help provide motivation, along with the fact that all maps with bounded valences and covalences have such tilings as covers.

We start by classifying the regular tilings. Note that a simply connected map of valence d and covalence ℓ must be the universal (d, ℓ) tessellation of the plane or sphere.

Theorem 10.1 *Let d and ℓ be positive integers and let M be the universal (d, ℓ)-tessellation. The following are equivalent.*

(1) M is a Cayley map.

(2) The underlying graph of M is a Cayley graph.

(3) Either $d \leq 2$ or ℓ has a prime divisor p such that $p \leq d$.

Proof. We begin by dealing with the degenerate cases $d \leq 2$ or $\ell \leq 2$. For $d = 1$, we have either one vertex incident with one semi-edge or two vertices each incident with an edge joining them. For $d = 2$, we have either a 1-vertex map incident with two semi-edges or a circuit. In all these cases the map is a Cayley map.

If $\ell \leq 2$ and $d \geq 3$, then $\ell = 2$ and M is a 2-vertex map in the sphere with d parallel edges between the two vertices. This is a Cayley map. Thus, the theorem holds in all these degenerate cases, so in the following we may assume $d \geq 3$ and $\ell \geq 3$.

(1) $\Rightarrow$ (2). This is trivial.

(2) $\Rightarrow$ (3). Let $T(d, \ell) = \langle \rho, \lambda \rangle$ be the (d, ℓ)-triangle group, so $M = Map\langle \rho, \lambda \rangle$. Suppose the underlying graph Γ of M is a Cayley graph. By Corollary 9.3, ℓ has a prime divisor p such that $p \leq d$.

(3) $\Rightarrow$ (1): For $d \geq 3$, we use the dual quotient information method (which applies by Theorem 9.1 since $d \geq 3$ and $\ell \geq 3$). The first step is to find a suitable tree in the sphere. Write $d = qp + r$, with $0 \leq r < p$, and let T be any tree in the sphere obtained as follows. Start with a path

$(v_1, e_2, v_2, e_3, \ldots, e_q, v_q)$. To v_1 add $p-1$ semi-edges, to each of $v_2, \ldots, v_{q-1}$ add $p-2$ semi-edges, and to v_q add r pendant edges and $p-1-r$ semi-edges.

The tree T has $q+r$ vertices, of which q have degree p and r have degree 1. Set $t_v = \ell/p$ for each p-valent vertex v of T and set $t_v = \ell$ for each 1-valent vertex v of T.

The covalence of T is the sum of its valences, and so is $qp+r=d$. Thus, the Cayley M map constructed by the dual quotient information method has all covalences ℓ and valence d. This map is simply connected and therefore is the universal (d,ℓ)-tessellation. $\otimes$

We would like to extend Theorem 10.1 to include more general planar tilings. A natural class consists of the edge-transitive ones, which are completely classified in [12]. If a 3-connected planar map is edge-transitive, then it has at most two valences and at most two covalences. In the case of two different covalences, the two covalences of the incident faces must alternate about each vertex. Therefore, the valences are even. If the map is a Cayley map or has a Cayley graph as an underlying graph, then there is only one valence. Just as the universal (d,ℓ)-tessellation of the plane or sphere is the unique simply connected map of valence d and covalence ℓ, there is a universal tessellation of the plane or sphere that is the unique simply connected map of valence $d=2c$ and covalences ℓ_1 and ℓ_2 alternating around each vertex. We denote that universal tessellation by $M(2c;\ell_1,\ell_2)$.

The goal of the remainder of this section is to prove the following result.

Theorem 10.2 *Let $c \geq 2$ be an integer, let ℓ_1 and ℓ_2 be distinct positive integers, each at least 3, and let $M = M(2c;\ell_1,\ell_2)$.*

(1) A necessary and sufficient condition for M to be a Cayley map is that at least one of the following hold:

(a) ℓ_1 or ℓ_2 is a multiple of c;

(b) each of ℓ_1 and ℓ_2 has a prime divisor at most c and at least one of ℓ_1 and ℓ_2 is even;

(c) c is odd and each of ℓ_1 and ℓ_2 has a prime divisor at most c.

(2) The underlying graph of M is a Cayley graph but not a Cayley map if and only if c is even and each of ℓ_1 and ℓ_2 is odd and has a prime divisor at most c.

Proof. *Necessity.* Suppose $M = CM(H, X, \tau)$. We claim that there is no i such that $i\tau = i$. For if such an i exists, the vertex $1_H \in H$ is incident with a face F_1 containing the dart $(1_H, i)$. Also incident with F_1 are the darts $(x_{(i-1)\tau}, (i-1)\tau)$ and $(x_i, i+1)$. Now $(1_H, i-1)$, $(1_H, i)$, $(1_H, i+1)$ are consecutive in the rotation of darts about 1_H, so F_1 has covalence ℓ_1, say, while the face F_2 containing the darts (x_i, i) and $(1_H, i+1)$ has covalence ℓ_2. Since M is a Cayley map, F_2 is taken by the isomorphism $h \mapsto x_i h$ to F_1, so F_1 has covalence ℓ_2 also. This contradicts the assumption that $\ell_1 \neq \ell_2$.

By Theorem 9.1, there are numbers r_1 and r_2 and divisors $d_{i,j}$ of ℓ_i, $i \in [2]$, $j \in [r_i]$. In this case, $d = 2c$, so $r_1 + r_2 \leq 1 + c$. It is clear that, for a fixed $k \in [d]$, all the darts of M of the form (h, k) are in faces of the same covalence and that the darts of the form $(h, k+1)$ are all in faces of the other covalence. Therefore, for $i = 1, 2$, $\sum_{j=1}^{r_i} d_{i,j} = c$. If every $d_{1,j}$ is 1, then we have $r_1 = c$ and $r_2 = 1$. Thus, $d_{2,1} = c$ and Condition (1)(a) is satisfied.

Otherwise we can assume $d_{1,1} \geq 2$ and $d_{2,1} \geq 2$. It follows that each of ℓ_1 and ℓ_2 has a prime divisor at most c. If either ℓ_1 or ℓ_2 is even, then Condition (1)(b) holds. If both ℓ_1 and ℓ_2 are odd, we show that c is also odd.

Let M_1 be the one-vertex $(2c)$-valent map $CM(\{1_H\}, X', \tau)$. It follows from the preceding paragraphs that M_1 consists of c loops incident with the single vertex. Furthermore, since every covalence of M_1 is odd, the number f of faces is even. By Euler's formula, $1 - c + f$ is even, so c must be odd, as claimed.

Finally, suppose that the graph underlying $M(2c;\ell_1,\ell_2)$ is a Cayley graph and Condition (1) does not pertain. By the foregoing argument, $M(2c;\ell_1,\ell_2)$ is not a Cayley map, and so, by Theorem 9.2 there are divisors $d_{1,1},\ldots,d_{1,r_1}$ of ℓ_1 and $d_{2,1},\ldots,d_{2,r_2}$ of ℓ_2 such that $r_1+r_2\le 2c$ and $\sum_{i=1}^{2}\sum_{j=1}^{r_i} d_{i,j}=4c$. More precisely, half the darts of M_1 are in faces covered by ℓ_1-covalent faces of M, while the other half are in faces covered by ℓ_2-covalent faces of M. Thus for $i=1,2$, $\sum_{j=1}^{r_i} d_{i,j}=2c$.

Since both r_1 and r_2 are positive integers, we must have $d_{1,j_1}>1$ and $d_{2,j_2}>1$, for some j_1,j_2. Thus, each of ℓ_1 and ℓ_2 has the required prime divisor. The only possibility, since Condition (1) does not pertain, is that c is even and both ℓ_1 and ℓ_2 are odd.

Sufficiency. For the sufficiency of (1) yielding a Cayley map, we use the dual quotient information method. In each case, since the constructed map is simply connected, it must be the universal tessellation $M(2c;\ell_1,\ell_2)$.

(a) Suppose $c|\ell_1$. Let T be the star $K_{1,c}$. The c-valent vertex gets parameter ℓ_1/c, while all the 1-valent vertices get parameter ℓ_2.

(b) Suppose $2|\ell_1$ and $p|\ell_2$, with $p\le c$. Write $c=qp+r$, with $0\le r<p$. Let T be the tree obtained as follows. Start with a path on $2q-1$ vertices. By adding pendant edges, alternately make the vertices have degrees p and 2, so the terminal vertices of the path have degree p. This creates a tree with pq edges. Now to r of the neighbors of a vertex with $p-1$ leaf neighbors, add a single new edge with a new neighbor.

The tree is bipartite. One of the parts has all vertices with valences p or 1 and the other part has all vertices with valences 2 or 1. The corresponding parameters, for the first set, are ℓ_2/p and ℓ_2, respectively, while for the second set they are $\ell_1/2$ and ℓ_1, respectively.

(c) We can assume without loss of generality that $p_1\le p_2$, where, for $i=1,2$, $p_i|\ell_i$. If $c=p_2$, then we are in Case (a), so we may assume $c>p_2$.

We begin by writing $c=\alpha(p_2+1)+\beta$, with α and β integers such that $0\le\beta<p_2+1$. As c and p_2 are both odd (or else we are in Case (b)), β is

odd, so $0 < \beta \le p_2$. Also $c > p_2$, whence $\alpha > 0$. Let $\beta+1 = (\gamma+1)(p_1-1)-\delta$, with γ and δ integers such that $0 \le \delta < p_1 - 1$. Since β and p_1 are both odd, δ is even. Therefore, $\delta \le p_1 - 3$.

We construct a graph Γ as follows. Start with a path $(u_1, v_1, u_2, v_2, \ldots, u_\alpha, v_\alpha)$ and replace each edge $u_i v_i$, $1 \le i < \alpha$, with $p_1 - 2$ parallel edges. Replace $u_\alpha v_\alpha$ with $\delta+1$ parallel edges. To each vertex u_i add enough pendant edges to give u_i degree p_2. To each v_i, $1 \le i < \alpha$, add a single pendant edge, so these v_i have degree p_1. To v_α add $p_1 - (1 + \delta)$ pendant edges.

Finally, to each of γ of the degree 1 vertices adjacent to u_α (other than v_α if $\delta = 0$), we add $p_1 - 1$ pendant edges. This is our graph. Its only cycles are of length two and between any pair of adjacent vertices there is an odd number of parallel edges. It is easy to see that such a graph has a one-face embedding in an orientable surface. (For example, embed it in the sphere in a natural way. That the number of faces is odd is an easy exercise. For two consecutive parallel edges, add a handle that uses only those two edges. This increases the genus by one and decreases the number of faces by two. Repeat until there is only one face.)

The construction requires that $\gamma + \delta + 1 \le p_2 - 1$. To see that this inequality holds, we have the following chain of inequalities.

$$
\begin{aligned}
& (p_2 - p_1)(p_1 - 2) &\geq\ & 0 \\
\Longleftrightarrow\ & p_1p_2 - p_1^2 - 2p_2 + 2p_1 &\geq\ & 0 \\
\Longleftrightarrow\ & (p_1 - 1)(p_2 - 1) &\geq\ & p_2 + 1 + p_1(p_1 - 3) \\
\Longrightarrow\ & (p_1 - 1)(p_2 - 1) &\geq\ & \beta + 1 + p_1\delta \\
\Longleftrightarrow\ & (p_2 - 1) &\geq\ & \frac{\beta + 1}{p_1 - 1} + \frac{(\delta p_1)}{(p_1 - 1)} \\
& &=\ & \frac{(\beta + 1)}{(p_1 - 1)} + \frac{\delta}{(p_1 - 1)} + \delta + 1 - 1 \\
& &=\ & \gamma + \delta + 1.
\end{aligned}
$$

We obtain the required parameters as in (b). Note that Γ is bipartite, with one class of vertices having all valences 1 or p_1 and the other class having all valences 1 or p_2. The corresponding parameters are ℓ_1, ℓ_1/p_1, ℓ_2 and ℓ_2/p_2.

(2) Suppose c is even, ℓ_1 and ℓ_2 are both odd and, for $i = 1, 2$, there is a prime divisor p_i of ℓ_i such that $p_i \leq c$. We construct our graph as follows.

Use the construction of Case 1 (c) on the parameters $c + 1$, p_1 and p_2. This produces a graph Γ_1 that has $c + 1$ edges. It is easy to see (by checking the distinct cases $\alpha = 1$ and $\alpha > 1$) that both u_1 and v_1 are incident with pendant edges. Obtain the graph Γ by deleting one pendant edge from each of u_1 and v_1 and adding another edge between u_1 and v_1.

The graph Γ has an orientable embedding with two faces and, by adding a crosscap in any edge that separates these two faces, we obtain a nonorientable embedding with one face. (See the remarks below.) Note that the covalence is $2c$. The graph is bipartite, with all vertices in one part having valences p_1 or 1, and all vertices in the other part having valences p_2 or 1. The same dual quotient information method produces a map with all valences being $d = 2c$

and all covalences being either ℓ_1 or ℓ_2. This map is 3-connected and planar for all the same reasons the maps in the oriented construction have these properties. By the foregoing "necessity" argument, it is not a Cayley map. On the other hand, the dual of the 1-face map is a 1-vertex map. Ignoring the embedding, we have an assignment of group elements to its darts that produces the graph underlying M, showing that this graph is a Cayley graph. $\otimes$

As an example, the map $M(12; 9, 15)$ is not a Cayley map, but its underlying graph is a Cayley graph.

The sudden appearance of a crosscap in the proof of Part (2) of this theorem should not be surprising. If the underlying graph of the planar map is a Cayley graph $C(H, X, \tau)$, by 3-connectivity the action of H on the graph extends to the plane. If this action always preserves orientation, then the map is a Cayley map. Thus, in Part (2), the action must include some homeomorphisms that would reverse orientation of the oriented surface. Therefore the quotient surface is nonorientable. Orientation-reversing loops in the quotient map correspond to generators whose action on the plane reverses orientation. To be complete, we note that Theorem 8.1 on realizability of quotient information generalizes to nonorientable surfaces and that the resulting map is in a simply connected surface, which in particular is orientable, exactly the situation in Theorem 10.2.

It is important to note that Theorems 10.1 and 10.2 are connected via Corollary 4.6 to questions about factoring long cycles in symmetric groups. By Theorem 4.6, we know that the (d, ℓ)-tessellation is a Cayley map if and only if there is a homomorphism ϕ from its automorphism group $T(d, \ell) = \langle x, y : x^d = y^2 = (xy)^\ell = 1 \rangle$ to $Sym[d]$ such that $\phi(x) = \eta$. For $d \geq 3$, the image $\phi(xy) \in Sym[d]$ must contain a non-trivial cycle of length p, with $p \leq d$. Thus, the order of $\phi(xy)$, and, consequently ℓ, must have a prime divisor at most d. (Compare with Corollary 5.7.)

Conversely, suppose we know that if $p \le d$, then there is a permutation of order p (corresponding to xy) whose product with some involution (corresponding to y) is the cycle η. Since $T(d,\ell)$ has so few relations, this is all that is needed to obtain the required homomorphism.

Thus, Theorem 10.1, Parts (1) and (3), are equivalent to the statement: given $2 \le p \le d$, the cycle $(1\ 2\ \ldots\ d)$ can be written in $Sym[d]$ as a product of two permuations, one of order p and the other of order either 1 or 2. This statement is a theorem of G.A. Miller [35]. We conclude that Theorem 10.1 provides a new proof of Miller's Theorem, and, conversely, Miller's Theorem provides an alternative proof of Theorem 10.1.

Similarly, the $(2c;\ell_1,\ell_2)$-tessellation has $T(2c;\ell_1,\ell_2) = \langle x, y : x^{2c} = y^{\ell_1} = (xy)^{\ell_2} = 1\rangle$ for its automorphism group. Thus, it is a Cayley map if and only if, in the symmetric group on $2c$ symbols, we can factor the cycle $(1\ 2\ \ldots\ 2c)$ into permutations of appropriate orders, as described in Theorem 10.2, Conditions 1(a)-(c). We don't know of any such result in the literature on permutation groups.

We conclude this section with the following example.

Example. *Consider the map $M = M(4;3,\ell)$, where $\ell \ge 4$. If ℓ is odd, then we can show in two ways that the underlying graph Γ of M is not even a Cayley graph. One way is immediate by using Theorem 10.2. The other way is to note that Γ is the derived graph (line graph) of the universal tessellation $M' = Map\langle\rho,\lambda\rangle$ whose dart group is the triangle group $T(3,\ell) = \langle\rho,\lambda\rangle$. Since M' is odd-valent, non-bipartite, and both vertex- and edge-transitive, it follows by Corollary 3.2 of [45] that Γ is not a Cayley graph.*

On the other hand, suppose that ℓ is even. The edge-set of every circuit in M' is the symmetric difference of the edge-sets of a finite collection of its face boundaries. Hence M' is bipartite, and the result in [45] does not apply. We see, however, that in this case M satisfies Condition 1(b) of Theorem 10.2 and hence is a Cayley map. In fact, M is the antibalanced Cayley map $CM(H,X,\tau)$, where $\tau = (1\ 4)(2\ 3)$ and $H \simeq \langle x_1, x_2 : x_1^3 = x_2^3 = (x_1x_2^{-1})^{\ell/2} =$

$1\rangle$.

11 Final Words

In this last section we propose a number of questions which we feel merit further research.

1. For which groups H is there a regular Cayley map $CM(H, X, \tau)$?

   This question seems to be interesting even for Abelian groups. It is easy to provide such maps for cyclic groups, but typically they have parallel edges. So we refine this question as follows.

2. For which groups H is there a regular Cayley map $CM(H, X, \tau)$ without parallel edges?

   Except for trivial maps, regular maps cannot have either loops or semi-edges. This question seems to be interesting even for cyclic groups.

   The existence of a regular map of valence $d \geq 3$ and covalence $\ell \geq 3$ in a surface of genus $g > 1$ is equivalent to the existence of a group of order $2d\ell(2g-2)/(d\ell - 2d - 2\ell)$ having a presentation of the form $\langle x, y : x^d = y^2 = (yx)^\ell = 1, \ldots \rangle$. For example, this is where the list of regular maps of genus 2 in [8] comes from. When a map is generated this way by purely algebraic methods, it may have multiple edges. Detecting multiple edges algebraically in the dart group is easy.

   Proposition 11.1 *Let M be a regular map with dart group $G = \langle \rho, \lambda \rangle$. Then M has multiple edges if and only if $\langle \rho \rangle$ contains a nontrivial normal subgroup of G. Also, M has loops or semi-edges if and only if ρ generates G, that is the map has only one vertex.*

   Proof. In a not necessarily regular map, the dart e is parallel to the dart e' if and only if, for some $i, j \in [d] \setminus \{d\}$, $e\rho^i = e'$ and $e\lambda\rho^j = e'\lambda$.

That is, if and only if, for some i, j, $e\rho^i\lambda = e\lambda\rho^j$. If M is regular, then this implies $\lambda\rho^i = \rho^j\lambda$. Thus, the subgroup $\langle\rho^i, \rho^j\rangle$ of $\langle\rho\rangle$ is invariant under conjugation by λ; equivalently, it is a normal subgroup of G.

Similarly, if e is the dart of a semi-edge or one of the two darts of a loop, then there is an i such that $e\rho^i = e\lambda$. For a regular map, this is equivalent to $\lambda \in \langle\rho\rangle$, which in turn is equivalent to $G = \langle\rho\rangle$. In particular, G has just one vertex. $\otimes$

The following variants might be easier.

3. For which groups H is there a regular balanced Cayley map (without parallel edges)?

4. For which groups H are the only regular Cayley maps balanced (without parallel edges)?

5. Which regular maps of small genus are Cayley maps?

6. Which small (or otherwise easily understood) groups have regular Cayley maps?

7. Let $\tau \in Sym[d]$ be an involution and let $Map(\tau)$ denote the generic regular cover of the one-vertex Cayley map $CM(\{1\}, X, \tau)$. For which involutions $\tau, \tau' \in Sym[d]$ is there a homomorphism/isomorphism $f : Map(\tau) \to Map(\tau')$?

   This seems to be a question about subgroups of $Sym[d]$ and homomorphisms/isomorphisms taking η to η.

8. With τ as in the preceding question, how many non-isomorphic maps are there whose dart group is $\langle\eta, \tau\rangle \leq Sym[d]$?

9. Which sequences of covalences are possible in a Cayley map?

   We have results in this work about *sets* of covalences.

10. How can Theorems 6.2 and 6.4 be generalized?

    Interesting examples that show how they do not generalize would be quite welcome.

11. If M is regular, can the rotary function be (nicely) expressed in terms of ρ and λ?

12. Under what circumstances can we conclude that if $\tilde{M}$ is a Cayley map and $f : \tilde{M} \to M$ is a map homomorphism, then M is a Cayley map?

    One example occurs in the proof of Theorem 7.3, which implicitly uses the following very easy fact:

    Proposition 11.2 *Let $M = CM(H, X, \tau)$ be regular and let $K \leq H$ be such that K_L is normal in $Aut(M)$. Then M/K_L is a regular Cayley map with distribution of inverses τ.*

13. How far can the theory of Cayley maps be developed in the orientable case when we allow that at each vertex either ρ or ρ^{-1} gives the rotation? Another way of asking the question is: what happens when we allow orientation-reversing automorphisms.

14. What is the theory of Cayley maps in nonorientable surfaces?

    Presumably here we mean that a map M is a Cayley map if there is a subgroup of $Aut(M)$ that acts regularly on the vertex set.

15. In [16] are applications of regular balanced Cayley maps to interconnection networks. Can that theory be generalized to all regular Cayley maps?

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