Arrangements, Circular Arrangements and the Crossing Number of $C_7 \times C_n$

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Abstract

Motivated by the problem of determining the crossing number of the Cartesian product $C_m \times C_n$ of two cycles, we introduce the notion of an (m,n)-arrangement, which is a set $\{S, T, C_1, C_2, \ldots, C_n\}$ of closed curves and a set $\{P_1, P_2, \ldots, P_m\}$ of paths in the plane, such that S and T are disjoint and in the same face of $C_1 \cup C_2 \cup \ldots \cup C_n$, each P_i joins a point on S to a point on T, and each P_i has a vertex $v_{i,j}$ on C_j so that in traversing P_i from S to T, the $v_{i,j}$ occur in the order $v_{i,1}, v_{i,2}, \ldots, v_{i,n}$. The main result is that every (m,n)-arrangement has at least $(m-2)n$ crossings. This is used to show that "(m,n)-circular arrangements" (no S and T and the P_i are closed curves) which can be broken up into disjoint arrangements have $(m-2)n$ crossings. This last fact implies that the crossing number of $C_7 \times C_n$ is $5n$, in agreement with the general conjecture that the crossing number of $C_m \times C_n$ is $(m-2)n$, for $3 \leq m \leq n$.

1 Introduction

In 1973, Harary, Kainen and Schwenk [6] conjectured that the crossing number of the Cartesian product $C_m \times C_n$ of an m–cycle with an n–cycle is

$(m-2)n$, assuming $m \leq n$. To date, this has only been proved for $n \leq 6$ and for $m = n = 7$ [8, 5, 4, 10, 7, 2, 9, 3]. Typically, the proof goes as follows: prove the result for $C_m \times C_m$ and then use induction on n.

This work is part of the first author's Ph.D. dissertation [1] written under the supervision of the second author. It is motivated by the observation that the hard part in the inductive step is proved by breaking up the drawing of $C_m \times C_n$ into pieces, looking at "consecutive" n–cycles and the parts of the m–cycles running between them. We formalize and generalize this concept by introducing arrangements, which will be used later in the paper to prove, among other things, that $cr(C_7 \times C_n) = 5n$, in agreement with the conjecture.

A *path* is a continuous function $f : [0,1] \to \mathbf{R}^2$ and a *cycle* is a path f such that $f(0) = f(1)$.

An (m,n)-*arrangement* is a quintuple $(S, T, \mathcal{C}, \mathcal{P}, \psi)$, such that:

1. $\mathcal{C} = (C_1, C_2, \ldots, C_n)$ is a sequence of cycles;

2. $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ is a set of paths;

3. S and T are cycles and $S[0,1] \cup T[0,1]$ is contained in one component of $\mathbf{R}^2 \setminus \bigcup_{i=1}^{n} C_i[0,1]$;

4. $\psi : (\{S,T\} \cup \mathcal{C}) \times \mathcal{P} \to [0,1] \times [0,1]$ such that:

    - for each $(C,P) \in (\{S,T\} \cup \mathcal{C}) \times \mathcal{P}$, if $\psi(C,P) = (s,t)$, then $C(s) = P(t)$;

    - for each $P \in \mathcal{P}$, if $\psi(C_i, P) = (s_i, t_i)$, then $0 < t_1 < t_2 < \ldots < t_n < 1$;

    - for each $P \in \mathcal{P}$, there exists $s \in [0,1]$ such that $\psi(S,P) = (s,0)$; and

    - for each $P \in \mathcal{P}$, there exists $t \in [0,1]$ such that $\psi(T,P) = (s,1)$.

Also, if $\psi(C,P) = (s,t)$, define $\psi_{\mathcal{P}}(C,P) = t$ and $\psi_{\mathcal{C}}(C,P) = s$. The definition of ψ can be extended to $\psi : \mathcal{P} \times (\{S,T\} \cup \mathcal{C}) \to [0,1] \times [0,1]$ by $\psi(P,C) = (t,s)$ if $\psi(C,P) = (s,t)$.

The intent of this definition is to capture the idea of two cycles, labelled S and T, together with a sequence of cycles $(C_1, \ldots, C_n)$ between them. Each path in $\{P_1, \ldots, P_m\}$ starts at S, hits each of the C_i in turn, and ends at T. The points of intersection between $\mathcal{C}$ and $\mathcal{P}$ are represented by ψ. An example of an arrangement is shown in Figure 1.

INSERT FIGURE 1 HERE.

Figure 1

A *vertex* in the arrangement $(S,T,\mathcal{C},\mathcal{P},\psi)$ is a quadruple (C,P,s,t) such that $C \in \mathcal{C} \cup \{S,T\}$, $P \in \mathcal{P}$, $s,t \in [0,1]$ satisfy $\psi(C,P) = (s,t)$. Define $\nu : (\mathcal{C} \cup \{S,T\}) \times \mathcal{P} \to \mathbf{R}^2$ by $\nu(C,P) = C(s)$ where $\psi(C,P) = (s,t)$.

If P is a path in $\mathcal{P}$, with $s,t \in [0,1]$ and $s < t$, then $P[s,t]$ is the segment of P given by $\{P(x) : s \leq x \leq t\}$. Also, if $X, Y \in \mathcal{C} \cup \{S,T\}$, let $P[X,Y] = P[\psi_{\mathcal{P}}(X,P), \psi_{\mathcal{P}}(Y,P)]$ (i.e., the segment of P between the vertices of P on X and Y).

A *crossing* in the arrangement $(S,T,\mathcal{C},\mathcal{P},\psi)$ is a quadruple (X,Y,s,t) such that:

1. $X, Y \in \{S,T\} \cup \mathcal{C} \cup \mathcal{P}$; $s,t \in [0,1]$; $X(s) = Y(t)$;

2. either $X \neq Y$ or $s \neq t$; and

3. if $\psi(X,Y)$ is defined, then $(s,t) \neq \psi(X,Y)$.

The *crossing number* $cr(\mathcal{A})$ of an (m,n)-arrangement $\mathcal{A}$ is the cardinality of the set of crossings of $\mathcal{A}$.

If (X,Y,s,t) is a crossing, then so is (Y,X,t,s), but we shall regard these as the same crossing, and only count it once. Furthermore, if $X \in \{S,T\} \cup \mathcal{C}$, then $(X,X,0,1)$ and $(X,X,1,0)$ are not considered to be crossings. More generally, we shall always consider, for such X, the pairs $(X,0)$ and $(X,1)$ to be the same.

Thus, a crossing is a non-vertex intersection of a cycle or a path with a cycle or a path.

The main result of this work is the following.

Theorem 1 *Let $\mathcal{A} = (S,T,\mathcal{C},\mathcal{P},\psi)$ be an (m,n)–arrangement such that $S \cap T = \emptyset$. Then $cr(\mathcal{A}) \geq (m-2)n$.*

2 Preliminary Results

In this section, two elementary preliminaries are presented.

For an (m,n)-arrangement $\mathcal{A} = (S,T,\mathcal{C},\mathcal{P},\psi)$ and a point $z \in \mathbf{R}^2$, let $\mathcal{A}(z) = \{(X,s) : X \in \{S,T\} \cup \mathcal{C} \cup \mathcal{P},\ s \in [0,1],\ X(s) = z\}$. The arrangement $\mathcal{A}$ is in *general position* if, for every $z \in \mathbf{R}^2$, $|\mathcal{A}(z)| \leq 2$.

The following is a standard exercise, and is proved in [1].

Theorem 2 *For every (m,n)-arrangement $\mathcal{A}$, there is an (m,n)-arrangement $\mathcal{A}'$ in general position having finitely many crossings such that $cr(\mathcal{A}') \leq cr(\mathcal{A})$.*

Figure 1 shows an (m,n)-arrangement in general position with finitely many crossings.

An *optimal* (m,n)-arrangement is an (m,n)-arrangement $\mathcal{A}$ such that for any other (m,n)-arrangement $\mathcal{A}'$, $cr(\mathcal{A}) \leq cr(\mathcal{A}')$.

Corollary 3 *There is an optimal (m,n)-arrangement that is in general position. Furthermore, every optimal (m,n)-arrangement $\mathcal{A}$ has the following properties:*

- *$\mathcal{A}$ has only simple cycles (i.e., a cycle does not cross itself);*

- *the paths of $\mathcal{A}$ have distinct endpoints;*

- *if x is a tangential point of intersection in $\mathcal{A}$, then x is a vertex;*

- *for any path P and any $i \in \{1, \ldots, n\}$, the portion of the path P between C_{i-1} and C_i, does not cross with the portion of the path P between C_i and C_{i+1}; and*

- *for any path P and any $i \in \{1, \ldots, n\}$, the portion of the path P between C_{i-1} and C_i does not cross itself.*

The main purpose of this corollary is to note that the kinds of crossings listed above are easily removed, without affecting the other crossings in the arrangement. So we will sometimes note "a crossing can be removed, as in Corollary 3." In particular, in order to prove Theorem 1, it suffices to consider arrangements that, for example, have no tangential intersections that are not vertices and have only simple cycles.

3 Fundamental Theorem of Arrangements

Let $(S, T, \mathcal{C}, \mathcal{P}, \psi)$ be an arrangement. For $X, Y \in \mathcal{C} \cup \mathcal{P} \cup \{S, T\}$, let $X \cap Y = \{(s, t) \in [0, 1] \times [0, 1] : X(s) = Y(t)\}$. Thus, if $X \in \mathcal{C} \cup \{S, T\}$ and $Y \in \mathcal{P}$, then $\psi(X, Y) \in X \cap Y$, so $X \cap Y \neq \emptyset$.

Theorem 4 The Fundamental Theorem of Arrangements

Let $(S, T, (C_1), \mathcal{P}, \psi)$ be a $(3, 1)$-arrangement in general position with $S \cap T = \emptyset$. Then at least one of the following occurs:

1. *there is a path $P \in \mathcal{P}$ and a crossing of the form (C_1, P, s, t);*

2. *there is a path $P \in \mathcal{P}$ and a crossing of the form (S, P, s, t), with $t > \psi_{\mathcal{P}}(C_1, P)$;*

3. *there is a path $P \in \mathcal{P}$ and a crossing of the form (T, P, s, t), with $t < \psi_{\mathcal{P}}(C_1, P)$;*

4. *there are distinct paths $P, Q \in \mathcal{P}$ and a crossing of the form (P, Q, s, t) with $s < \psi_{\mathcal{P}}(C_1, P)$ and $t > \psi_{\mathcal{P}}(C_1, Q)$.*

Proof: For each $P \in \mathcal{P}$, let p_S denote the largest $s < \psi_{\mathcal{P}}(C_1, P)$ such that $P(s) \in S$. Similarly, let p_T denote the smallest $t > \psi_{\mathcal{P}}(C_1, P)$ such that $P(t) \in T$. Set $\mathcal{P}' = \{P : [p_S, p_T] \to \mathbf{R}^2 : P \in \mathcal{P}\}$. Now consider the arrangement $(S, T, (C_1), \mathcal{P}', \psi)$ where each $P \in \mathcal{P}'$ is appropriately reparameterized.

If there is a $P \in \mathcal{P}'$ such that $|P \cap C_1| > 1$, then we have (1).

If there is a $P \in \mathcal{P}'$ such that $|P \cap S| > 1$, then, by construction, any point in $P \cap S$ that is not the origin of P necessarily occurs after $\psi_{\mathcal{P}}(C_1, P)$ and we have (2).

Similarly, if there is a $P \in \mathcal{P}'$ such that $|P \cap T| > 1$, then we have (3).

Thus, we can assume that, for every $P \in \mathcal{P}'$, $P \cap S$, $P \cap C_1$ and $P \cap T$ all have a unique point in them. It follows that neither S nor T separates the other from C_1. Therefore, one face of S contains $C_1 \cup T$ and the other is disjoint from every path in $\mathcal{P}$. The analogous statement holds with S and T interchanged.

Pick a point s (respectively, t) in the empty face of S (resp., T) and join s (resp., t) by paths, pairwise disjoint except for the common end s (resp., t) to the ends of the paths in $\mathcal{P}'$ on S (resp., T). For each $P \in \mathcal{P}'$, let $v_P = P(\psi_{\mathcal{P}}(C_1, P))$.

If, for every $P, Q \in \mathcal{P}'$, there are no crossings of P and Q, then we have an embedding of the graph $K_5 - st$ in the plane, where the vertices are s, t, and the three points v_P, $P \in \mathcal{P}'$. The cycle C_1 joins the three vertices v_P.

In every embedding of $K_5 - st$ in the plane, the cycle C_1 separates s from t, which, in the embedding in question, implies that C_1 separates S from T, contradicting (3) in the definition of an arrangement.

Therefore, there is some $P, Q \in \mathcal{P}'$ such that $P \cap Q \neq \emptyset$. As in Corollary 3, we may assume that no path crosses itself. If these crossings are only of the form $P[S, C_1] \cap Q[S, C_1]$ or $P[C_1, T] \cap Q[C_1, T]$, then these crossings can be eliminated as in Corollary 3, returning us to the previous case. Therefore, there is at least one of the form $P[S, C_1] \cap Q[C_1, T]$. **Q.E.D.**

Corollary 5 *Let $\mathcal{A} = (S, T, (C), \mathcal{P}, \psi)$ be an $(m, 1)$-arrangement such that $S \cap T = \emptyset$. Then $cr(\mathcal{A}) \geq (m - 2)$.*

Proof: If $m \leq 2$, then the theorem is trivial since $m - 2 \leq 0$. So assume $m \geq 3$.

Let $\mathcal{Q}$ be the set of paths P such that $|P \cap C| = 1$, and $\mathcal{Q}' = \mathcal{P} \setminus \mathcal{Q}$.

By the Fundamental Theorem of Arrangements, the set $\mathcal{Q}$ can be partitioned into three sets:

1. $\mathcal{Q}_a = \{P \in \mathcal{Q}$: there is a crossing of the form (S, P, s, t), with $t > \psi_{\mathcal{P}}(C_1, P)\}$;

2. $\mathcal{Q}_b = \{P \in \mathcal{Q} \setminus \mathcal{Q}_a$: there is a crossing of the form (T, P, s, t), with $t < \psi_{\mathcal{P}}(C_1, P)\}$; and

3. $\mathcal{Q}_c = \mathcal{Q} \setminus (\mathcal{Q}_a \cup \mathcal{Q}_b)$

Each member of $\mathcal{Q}_a \cup \mathcal{Q}_b \cup \mathcal{Q}'$ contributes a crossing of P with either S or T or C.

If $|\mathcal{Q}_c| \leq 2$, then $|\mathcal{Q}_a| + |\mathcal{Q}_b| + |\mathcal{Q}'| \geq m - 2$ and we are done.

If $|\mathcal{Q}_c| > 2$, then the Fundamental Theorem of Arrangements applied to $(S, T, (C), \mathcal{Q}_c, \psi)$ implies that every three paths in $\mathcal{Q}_c$ contributes a crossing.

Each crossing occurs between a pair of paths, and this pair can occur in at most $|\mathcal{Q}_c| - 2$ triples. Thus, there are at least $\frac{\binom{|\mathcal{Q}_c|}{3}}{|\mathcal{Q}_c|-2}$ crossings.

Therefore, $cr(\mathcal{A}) \geq |\mathcal{Q}'| + |\mathcal{Q}_a| + |\mathcal{Q}_b| + \frac{\binom{|\mathcal{Q}_c|}{3}}{|\mathcal{Q}_c|-2}$. It remains to show that

$$\frac{\binom{|\mathcal{Q}_c|}{3}}{|\mathcal{Q}_c|-2} = \frac{|\mathcal{Q}_c|(|\mathcal{Q}_c|-1)}{6} \geq (|\mathcal{Q}_c|-2).$$

This inequality is equivalent to

$$(|\mathcal{Q}_c|-3)(|\mathcal{Q}_c|-4) \geq 0,$$

which is satisfied for all integers $|\mathcal{Q}_c|$. As $m = |\mathcal{Q}'| + |\mathcal{Q}_a| + |\mathcal{Q}_b| + |\mathcal{Q}_c|$, $cr(\mathcal{A}) \geq (m-2)$. **Q.E.D.**

4 (m,n)-Arrangements

In this section we prove Theorem 1, i.e., every (m,n)–arrangement has at least $(m-2)n$ crossings. This is shown by assigning $m-2$ crossings to each of the n m-cycles in the arrangement, no two crossings being assigned to more than one cycle.

Let $\mathcal{A} = (S, T, \mathcal{C}, \mathcal{P}, \psi)$ be such an arrangement. Then, for each $C \in \mathcal{C}$, define E_C as the region of $\mathbf{R}^2 \setminus C$ containing S and T. Define I_C as the component of $\mathbf{R}^2 \setminus C$ not containing S and T. Also define $E_{\mathcal{C}} = \cap_{i=1,\ldots,n} E_{C_i}$ and $I_{\mathcal{C}} = \cup_{i=1,\ldots,n} I_{C_i}$.

A path $P \in \mathcal{P}$ and a cycle $C \in \mathcal{C}$ are *tangent* if $|P \cap C| = 1$ and they are *semi-tangent* if there exists an open interval P' of P containing $\nu(C,P)$ such that P' is entirely in E_C (except for $\nu(C,P)$) and $|P[C,T] \cap C| > 1$.

For convenience, we will sometimes use $C_0 = S$ and $C_{n+1} = T$.

For each $j = 1, 2, \ldots, n$, we partition $\mathcal{P}$ into sets as follows:

1. $st(j) = \{P \in \mathcal{P} : P \text{ is semitangent to } C_j\}$;

2. $cross(j) = \{P \in \mathcal{P} \setminus st(j) : |P \cap C_j| > 1\}$;

3. for $0 \leq i < j$, $tan(i,j) = \{P \in \mathcal{P} : |P \cap C_j| = 1,\ \nu(C_j, P) \in E_{C_i},\ \forall k$ such that $i < k < j,\ \nu(C_j, P) \in I_{C_k}\}$

- $aft(i,j) = \{P \in tan(i,j) : P[C_j, T] \cap C_i \neq \emptyset\}$;
- $last(i,j) = \{P \in tan(i,j) \setminus aft(i,j) : P[C_i, C_j] \cap T \neq \emptyset\}$;
- $rest(i,j) = tan(i,j) \setminus (aft(i,j) \cup last(i,j))$.

Lemma 6 *Let $P \in \mathcal{P}$ and $i \in \{1, \ldots, n\}$. If $|P \cap C_i| = 1$, then, for all $j \in \{0, \ldots, n+1\} \setminus \{i\}, \nu(P, C_j) \in E_{C_i}$.*

Proof: This is because $S \cup T \subseteq E_{C_i}$. If some $\nu(P, C_j) \in I_{C_i}$ then P has to enter and leave I_{C_i} and therefore $|P \cap C_i| \geq 2$. **Q.E.D.**

Lemma 7 *Let $j \in \{1, \ldots, n\}$. Then $\mathcal{P} = st(j) \cup cross(j) \cup \bigcup_{0 \leq i < j} tan(i,j)$. Moreover, the sets $st(j)$, $cross(j)$, $tan(i,j)$ ($i = 0, 1, \ldots, j-1$) are pairwise disjoint.*

Proof: Let $P \in \mathcal{P}$. If $P \notin st(j) \cup cross(j)$, then P is tangent to C_j. By definition, $C_0 = S$ is disjoint from C_j. Thus, $\nu(P, S) \in E_{C_j}$, so either $P \in \bigcup_{1 \leq i < j} tan(i,j)$ or $P \in tan(0,j)$. Thus $\mathcal{P} = st(j) \cup cross(j) \cup \bigcup_{0 \leq i < j} tan(i,j)$.

These sets are pairwise disjoint: $st(j)$ and $cross(j)$ are disjoint by the definition of $cross(j)$; $tan(i,j)$ is disjoint from $st(j) \cup cross(j)$ since a path in $tan(i,j)$ has only one intersection with C_j whereas a path in $st(j)$ or $cross(j)$ has at least two intersections with C_j; and finally if $0 \leq i < i' < j$ and $P \in tan(i,j)$, then by definition $\nu(P, C_j) \in E_{C_i}$ and $\nu(P, C_j) \in I_{C_{i'}}$. Thus, $P \notin tan(i', j)$. **Q.E.D.**

For each $j = 1, 2, \ldots, n$ and $0 \leq i < j$, we define the sets $X(j)$ and $X(i,j)$ of crossings as follows:

1. if $P \in st(j)$, put into $X(j)$ the second crossing of $P[C_j, T]$ with C_j;
2. if $P \in cross(j)$, put into $X(j)$ the first crossing of P with C_j;
3. if $P \in aft(i,j)$, put into $X(i,j)$ the first crossing of $P[C_j, T]$ with C_i;

4. if $P \in last(i,j)$, put into $X(i,j)$ the last crossing of $P[C_i, C_j]$ with T;

5. if $P, Q \in rest(i,j)$ are such that $P[C_x, C_{x+1}]$ and $Q[C_y, C_{y+1}]$ cross, with $i \leq x < j \leq y$, then put into $X(i,j)$ the first crossing of $Q[C_j, T]$ with $P[C_i, C_j]$.

Lemma 8 *The sets $X(j)$, $X(i,j)$ are pairwise disjoint. That is:*

1. *if $i \neq j$, then $X(i) \cap X(j) = \emptyset$;*

2. *for all i, j, k, $X(i) \cap X(j,k) = \emptyset$; and*

3. *if $(i,j) \neq (k,\ell)$, then $X(i,j) \cap X(k,\ell) = \emptyset$.*

In order to prove Lemma 8, the following simple fact will be used.

Lemma 9 *If $P \in tan(i,j)$ and $P[C_j, T]$ crosses C_i, then either $P \in st(i)$ or there is a crossing of $P[S, C_j]$ with C_i.*

Proof: If $P \notin st(i)$, then it is because there is a point $x \in [0,1]$ near $\psi_{\mathcal{P}}(C_i, P)$ such that $P(x)$ is in I_{C_i}. Since $\nu(P, C_j) \in E_{C_i}$, $P[x, \psi_{\mathcal{P}}(C_j, P)]$ crosses C_i. **Q.E.D.**

Proof of Lemma 8: To prove (1), the crossings in $X(i)$ are all of paths with C_i, while those in $X(j)$ are with C_j, so if $i \neq j$, $X(i) \cap X(j) = \emptyset$.

To prove (2), suppose $x \in X(i) \cap X(j,k)$. By definition of $X(i)$ x is a crossing of some path P with C_i for $i \in \{1, \ldots, n\}$. Since $x \in X(j,k)$, it follows that $P \in aft(j,k)$ (if $P \in last(j,k)$, then x is a crossing between a path and $C_{n+1} = T$, while $P \in rest(j,k)$ implies x is a crossing between two paths, both contradictions). Thus $i = j$ and, since $P \in aft(j,k)$, x is the first crossing of $P[C_k, T]$ with C_j.

If $P \in st(j)$, then x is the second crossing of $P[C_j, T]$ with C_j. But $\nu(P, C_k) \in E_{C_j}$ implies there is an even number of crossings of $P[C_j, C_k]$

with C_j, so either the second crossing of $P[C_j, T]$ with C_j occurs before C_k or there are no crossings before C_k and the second crossing of $P[C_k, T]$ with C_j is not the same as the first crossing of $P[C_k, T]$ with C_j.

If $P \in cross(j)$, then x is the first crossing of P with C_j. Since $P \notin st(j)$, and since there is a crossing of $P[C_k, T]$ with C_j, by Lemma 9 there must be a crossing of $P[S, C_k]$ with C_j. Such a crossing is in $X(j)$ and is not x, again a contradiction. Therefore, (2) holds.

For (3), suppose $x \in X(i, j) \cap X(k, \ell)$. It is clear that x arises from a crossing involving a path $P \in tan(i, j) \cap tan(k, \ell)$. Without loss of generality, we can assume $j \leq \ell$.

If $j = \ell$, then $P \in tan(i, j) \cap tan(k, j)$, which implies $i = k$, a contradiction. Therefore, $j < \ell$. Since P is tangent to both C_j and C_ℓ, Lemma 6 implies $\nu(C_\ell, P) \in E_{C_j}$ and, therefore, $j \leq k$ (else $P \in tan(j, \ell)$ and not $tan(k, \ell)$). Thus, we have $i < j \leq k < \ell$.

If x is a crossing of a path P with C_i, then $i = k$ is the only possibility, an immediate contradiction.

If x is a crossing between P and C_{n+1}, then x occurs on $P[C_i, C_j]$; x is the last crossing of C_{n+1} by $P[C_i, C_j]$. Similarly x occurs on $P[C_k, C_\ell]$. Obviously, for both of these to occur, the intervals $P[C_i, C_j]$ and $P[C_k, C_\ell]$ must overlap, a contradiction, since $i < j \leq k < \ell$.

The last possibility is that x is a crossing between two paths P and Q. So suppose x is a crossing on $P[C_r, C_{r+1}]$ and $Q[C_s, C_{s+1}]$. We can assume $r < s$. As this is in $X(i, j) \cap X(k, \ell)$, $i \leq r < j \leq s$ and $k \leq r < \ell \leq s$. But then $r < j \leq k \leq r$, a contradiction. **Q.E.D.**

Before showing that the crossing number of an (m, n)-arrangement is at least $(m-2)n$, one final result must be established. An *intersecting arrangement* is a quintuple $(S, T, (C), \mathcal{P}, \psi)$ such that:

1. C is a cycle;

2. $C \cap S \neq \emptyset$;

3. $\mathcal{P} = \{P_1, \ldots, P_m\}$ is a set of paths;

4. S and T are cycles and $T[0,1]$ is contained in one component of $\mathbf{R}^2 \setminus (C[0,1] \cup S[0,1])$;

5. $\psi : (\{S, T, C\}) \times \mathcal{P} \to [0,1] \times [0,1]$ such that:

   - for each $(X, P) \in \{S, T, C\} \times \mathcal{P}$, if $\psi(X, P) = (s, t)$, then $X(s) = P(t)$;
   - for each $P \in \mathcal{P}$, if $\psi(C, P) = (s, t)$, then $0 < t < 1$;
   - for each $P \in \mathcal{P}$, $\psi(S, P) = (s, 0)$; and
   - for each $P \in \mathcal{P}$, $\psi(T, P) = (s, 1)$.

6. If $X \in \{S, C\}$ and $P \in \mathcal{P}$ with $\psi(X, P) = (s, t)$, then there exists an open neighbourhood U of t in $[0,1]$ such that $P(u)$ is on the same face of X as T for all $u \in U \setminus \{t\}$.

Essentially, this is an $(m, 1)$-arrangement, but permitting S and C to intersect. The final condition ensures that every vertex on S is in the exterior of C and every vertex on C is in the exterior of S. Furthermore, we will also assume, without loss of generality, that intersecting arrangements are in general position. We have the following result.

Theorem 10 *If $\mathcal{A} = (S, T, (C), \mathcal{P}, \psi)$ is an intersecting arrangement, then $cr(\mathcal{A}) \geq |\mathcal{P}|$.*

Proof: Let $m = |\mathcal{P}|$ and let J be the boundary of the component of $\mathbf{R}^2 \setminus (S([0,1]) \cup C([0,1]))$ containing T. Since S and C intersect, some segment of S must lie on J. Denote by S_i, $i = 1, \ldots, k$, the disjoint segments of S on J. For each S_i, let $\mathcal{P}_i = \{P \in \mathcal{P} : \nu(S, P) \in S_i\}$ and $m_i = |\mathcal{P}_i|$.

For each S_i, find any arc R_i inside J, disjoint from C except for its ends, so that $D_i = S_i \cup R_i$ is a cycle.

For each i, $\mathcal{A}_i = (D_i, T, (C), \mathcal{P}_i, \psi)$ is an $(m_i, 1)$-arrangement. By Corollary 5 it has at least $m_i - 2$ crossings. Summed over all i gives $\sum_{i=1}^{k}(m_i - 2) = (m - 2k)$ crossings. In $\mathcal{A}$ the two endpoints of each S_i are in C, for an additional $2k$ crossing. Thus, there is a total of m crossings. **Q.E.D.**

We comment that the kinds of crosssings counted in Theorem 10 are of the type that we have been counting all along, namely a crosssing of a path with C, a crossing of a path with S (respectively, T) involving the portion of the path after (resp., before) C, or a crossing of two paths, the part before C of one crossing the part after C of the other. In addition, however, here we are also counting the crossings of S with C.

Lemma 11 *For each j, $|X(j)| + \sum_{0 \le i < j} (|X(i,j)| + |C_i \cap C_j|) \ge m - 2$.*

Proof: By Lemma 7, for every j, $\mathcal{P}$ is the disjoint union $st(j) \cup cross(j) \cup \bigcup_{0 \le i < j} tan(i,j)$. Thus $|st(j)| + |cross(j)| + \sum_{0 \le i < j}(|tan(i,j)|) = m$.

Since every $P \in st(j) \cup cross(j)$ contributes one crossing to $X(j)$, it follows that $|X(j)| = |st(j)| + |cross(j)|$.

Let $a = max\{k < j : C_k \cap C_j = \emptyset\}$. (Note that $C_0 \cap C_j = \emptyset$, so a exists and $0 \le a < j$.) Now $\nu(C_j, P) \in E_{C_a}$, so $tan(i,j) = \emptyset$ for $i < a$. Moreover, $C_a \cap C_j = \emptyset$ and $C_i \cap C_j \ne \emptyset$ for $k < i < j$.

Let $\mathcal{P}'_i = tan(i,j)$ for $i \in \{a, \ldots, j-1\}$. Since C_a is disjoint from C_j, $(C_a, T, \{C_j\}, \mathcal{P}'_a[C_a, T])$ is a $(|P'_a|, 1)$-arrangement. (If C_j separates T and C_a, then all m paths cross C_j, so $\mathcal{P} = cross(j) \cup st(j)$.) By Corollary 5, $|X(a,j)| \ge |P'_a| - 2$.

Now consider $i \in \{a+1, \ldots, j-1\}$. In this instance, $C_i \cap C_j \ne \emptyset$. By Theorem 10 applied to the intersecting arrangement $(C_i, T, (C_j), \mathcal{P}'_i[C_i, C_{n+1}])$, $|X(i,j)| + |C_i \cap C_j| \ge |P'_i|$.

Thus, summing over all $i < j$,

$$|X(j)| + \sum_{0 \le i < j} (|X(i,j)| + |C_i \cap C_j|) \ge m - 2. \quad \textbf{Q.E.D.}$$

We are now prepared to prove Theorem 1.

Proof of Theorem 1: By Lemma 11, for each $j \in \{1, \ldots, n\}$,

$$|X(j)| + \sum_{0 \le i < j} (|X(i,j)| + |C_i \cap C_j|) \ge m - 2.$$

By Lemma 8, none of the crossings in $X(j)$ or $X(i,j)$ is counted more than once. Finally, crossings between C_i and C_j are counted only for the larger of i and j. Thus, over all j, the crossings counted are distinct. Since there are n cycles, there are at least $(m-2)n$ crossings. **Q.E.D.**

5 Comments

In the proof of Theorem 1, in counting the crossings, we looked at intersecting arrangements in which the curve corresponding to S was some C_i, while the curve corresponding to T was in fact always T. Obviously, we could turn everything around and make the proof work with T varying and S static. It is natural to wish that both S and T could vary and a proof of this would be very helpful (see the comments in Section 11).

In order to make this precise, a *generalized* (m,n)*–arrangement* is a triple $\mathcal{A} = (\mathcal{C}, \mathcal{P}, \psi)$ in which: (1) $\mathcal{C} = (C_0, C_1, \ldots, C_{n+1})$ is a sequence of cycles; (2) $\mathcal{P}$ is a set of m paths, each starting on C_0, ending on C_{n+1} and ψ assigns to each path–cycle pair (P, C) a vertex so that the vertices on P occur in the order $\psi(C_0, P), \psi(C_1, P), \ldots, \psi(C_{n+1}, P)$; and (3) for each $i = 1, 2, \ldots, n$, there exist $b(i)$ and $a(i)$ such that $0 \le b(i) < i < a(i) \le n+1$ and $C_{b(i)}, C_i, C_{a(i)}$ are pairwise disjoint and both $C_{b(i)}$ and $C_{a(i)}$ are in the same face of C_i.

Conjecture 1 *If $\mathcal{A}$ is a generalized (m,n)–arrangement, then $cr(\mathcal{A}) \ge (m-2)n$.*

The proof we have given does not seem to extend to this conjecture.

It would also be of interest to investigate what happens to arrangements in which $S \cap T \neq \emptyset$. We have not seriously investigated such arrangements and imagine that there are still $(m-2)n$ crossings.

6 Circular Arrangements

We now consider circular arrangements, which are generalizations of plane drawings of $C_m \times C_n$.

An (m, n)-*circular arrangement* is a triple $(\mathcal{C}, \mathcal{P}, \psi)$ such that:

1. $\mathcal{C} = (C_1, \ldots, C_n)$ is a cyclic sequence of cycles;
2. $\mathcal{P} = \{P_1, \ldots, P_m\}$ is a set of cycles;
3. $\psi : \mathcal{C} \times \mathcal{P} \to [0,1] \times [0,1]$;
4. for each $(C, P) \in \mathcal{C} \times \mathcal{P}$, if $\psi(C, P) = (s, t)$, then $C(s) = P(t)$;
5. for all $P \in \mathcal{P}$, there exists $t \in [0,1]$ such that $\psi(C_1, P) = (0, t)$;
6. for all $P \in \mathcal{P}, C_i \in \mathcal{C}, C_j \in \mathcal{C}$, if $\psi(C_i, P) = (s_i, t_i)$ and $\psi(C_j, P) = (s_j, t_j)$, then $i < j$ if and only if $t_i < t_j$.

Vertices and crossings for circular arrangements are defined as for arrangements.

As with arrangements, it suffices to consider only circular arrangements in general position.

Note that $\mathcal{C} = (C_1, \ldots, C_n)$ imposes a particular cyclic ordering on the sequence of cycles. For $i, j \in \{1, \ldots, n\}$, let $[C_i, C_j]_{\mathcal{C}}$ denote the subsequence $(C_i, C_{i+1}, \ldots, C_j)$. Note that if $j < i$, this still makes sense as the sequence $(C_i, C_{i+1}, \ldots, C_n, C_1, \ldots, C_j)$. Denote by $\langle C_i, C_j \rangle_{\mathcal{C}}$ the sequence $[C_{i+1}, C_{j-1}]_{\mathcal{C}}$ where the indices are read modulo n.

A *partitioned* (m, n)-*circular arrangement with* z *parts* is an (m, n)-circular arrangement $(\mathcal{C}, \mathcal{P}, \psi)$ together with z arrangements $\mathcal{A}_i = (S_i, T_i, \mathcal{C}_i, \mathcal{P}_i, \psi)$, $i = 1, 2, \ldots, z$ such that:

- for each i, if $\mathcal{C}_i = (C_{i_1}, \ldots, C_{i_r})$, then $(S_i, C_{i_1}, \ldots, C_{i_r}, T_i)$ is a subsequence of $\mathcal{C}$ and every cycle in $\mathcal{C}$ appears in exactly one $\mathcal{C}_i$;

- for each i, $\mathcal{P}_i = \{P[S_i, T_i] : P \in \mathcal{P}\}$;
- for $i, j \in \{1, \ldots, z\}$, if $i \neq j$, then either $\langle S_i, T_i \rangle_{\mathcal{C}} \cap \mathcal{C}_j = \emptyset$ or $\langle S_j, T_j \rangle_{\mathcal{C}} \cap \mathcal{C}_i = \emptyset$;
- If $\{S_i, T_i\} \cap \{S_j, T_j\} = \emptyset$, then (S_i, T_j, S_j, T_i) is not a cyclic subsequence of $\mathcal{C}$.

If we let $n_i = |\mathcal{C}_i|$, then it is easy to confirm that $\mathcal{A}_i = (S_i, T_i, \mathcal{C}_i, \mathcal{P}_i, \psi)$ is an (m, n_i)-arrangement. This is implied by the first two conditions. The third and fourth conditions are required to limit the amount of overlap between any two arrangements. As will be seen later, this is required to ensure no double counting of crossings.

INSERT FIGURE 2 HERE.

Figure 2

An example of an $(m, 7)$-circular arrangement is shown in Figure 2. In this figure, the paths have been omitted, showing only the cycles. This circular arrangement may be partitioned into six parts:

$\mathcal{A}_1 = (C_7, C_3, (C_1), \mathcal{P}, \psi)$
$\mathcal{A}_2 = (C_1, C_3, (C_2), \mathcal{P}, \psi)$
$\mathcal{A}_3 = (C_2, C_4, (C_3), \mathcal{P}, \psi)$
$\mathcal{A}_4 = (C_3, C_7, (C_4, C_5), \mathcal{P}, \psi)$
$\mathcal{A}_5 = (C_5, C_7, (C_6), \mathcal{P}, \psi)$
$\mathcal{A}_6 = (C_6, C_1, (C_7), \mathcal{P}, \psi)$

It is not difficult to check that all of the partition conditions are satisfied for this circular arrangement.

7 Partitioned Circular Arrangements

The purpose of this section is to prove the following, which is the heart of this part of the work.

Theorem 12 *If $\mathcal{A}$ is a partitioned (m,n)-circular arrangement, then*

$$cr(\mathcal{A}) \geq (m-2)n.$$

Let $\mathcal{A}$ be a partitioned (m,n)-circular arrangement with parts $\mathcal{A}_1, \ldots, \mathcal{A}_z$. For $j \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots, z\}$, define $st_k(j) = \emptyset$ if $C_j \notin \mathcal{C}_k$ and to be the set of all paths $P \in \mathcal{P}$ for which the segment of $P[S_k, T_k]$ is semi-tangent to C_j when considering only the arrangement $\mathcal{A}_k$, i.e., there is an open interval around $\nu(C_j, P)$ contained in E_{C_j} and the segment $P[C_j, T_k]$ crosses C_j. Similar definitions are given for $cross_k(j)$, $tan_k(i,j)$, $aft_k(i,j)$, $last_k(i,j)$, and $rest_k(i,j)$. The sets of crossings $X_k(j)$ and $X_k(i,j)$ are defined in exactly the same way they were for arrangements:

1. if $P \in st_k(j)$, put into $X_k(j)$ the second crossing of $P[C_j, T_k]$ with C_j;
2. if $P \in cross_k(j)$, put into $X_k(j)$ the first crossing of $P[S_k, T_k]$ with C_j;
3. if $P \in aft_k(i,j)$, put into $X_k(i,j)$ the first crossing of $P[C_j, T_k]$ with C_i;
4. if $P \in last_k(i,j)$, put into $X_k(i,j)$ the last crossing of $P[C_i, C_j]$ with T;
5. if $P, Q \in rest_k(i,j)$ are such that $P[C_x, C_{x+1}]$ and $Q[C_y, C_{y+1}]$ cross, with $i \leq x < j \leq y$, then put into $X_k(i,j)$ the first crossing of $Q[C_j, T_k]$ with $P[C_i, C_j]$.

For an (m,n)-circular arrangement with z parts, for all $i, j \in \{1, \ldots, n\}$, let $X(j) = \cup_{k \in \{1,\ldots,z\}} X_k(j)$ and $X(i,j) = \cup_{k \in \{1,\ldots,z\}} X_k(i,j)$.

Lemma 13 *Let $\mathcal{A}$ be a partitioned (m,n)-circular arrangement with z parts, and $a, b \in \{1, \ldots, z\}$. Then:*

1. *if $i \neq j$, then $X_a(i) \cap X_b(j) = \emptyset$; and*

2. *for all* i, j, k, $X_a(i) \cap X_b(j,k) = \emptyset$;

3. *if* $(i,j,a) \neq (k,\ell,b)$, *then* $X_a(i,j) \cap X_b(k,\ell) = \emptyset$.

Proof: By Lemma 8 we can assume $a \neq b$.

For (1) assume $C_i \in \mathcal{C}_a$. The crossings in $X_a(i)$ are all crossings of paths with C_i, while those in $X_b(j)$ are with C_j, so if $i \neq j$, then $X_a(i) \cap X_b(j) = \emptyset$.

For (2), suppose $C_i \in \mathcal{C}_a$, $C_j \in \mathcal{C}_b$, $C_k \in \mathcal{C}_b$, and $x \in X_a(i) \cap X_b(j,k)$. Since C_i is a cycle in $\mathcal{A}_a$, it is not a cycle in $\mathcal{C}_b$. Thus, $C_i = S_b$ or $C_i = T_b$. Let P be the path such that x is a crossing between P and C_i.

- If $C_i = S_b$, then $P \in aft_b(j,k)$ and x is the first crossing of $P[C_k, T_b]$ with C_i. In particular, this crossing is from E_{C_i} to I_{C_i}.

  - If $P \in st_a(i)$, then x is the second crossing of $P[C_i, T_a]$ with $C_i = C_j$. This implies x is a crossing from I_{C_i} to E_{C_i}, a contradiction.
  - If $P \in cross_a(i)$, then x is the first crossing of $P[S_a, T_a]$ with C_i. Since $C_i = S_b$, it follows that $S_b \in \langle S_a, T_a \rangle_{\mathcal{C}}$ and, by the definition of partitioned circular arrangements, $T_b \notin \langle S_a, S_b \rangle_{\mathcal{C}}$. Thus, $P[S_a, C_i]$ is disjoint from $P[S_b, T_b]$, so the crossing occurs on $P[C_i, T_a]$. This implies that the crossing is from I_{C_i} to E_{C_i}, a contradiction.

- If $C_i = T_b$ then it follows that $P \in last_b(j,k)$ and x is the last crossing of $P[S_b, C_k]$ with C_i. In particular, this crossing is from I_{C_i} to E_{C_i}.

  - $P \in st_a(i)$ implies x is a crossing of $P[C_i, T_a]$ with C_i. Thus, $T_a \in \langle S_b, T_b \rangle_{\mathcal{C}}$, otherwise the two intervals are disjoint, so the crossing occurs between $P[S_b, T_a]$ and C_i. But $C_i = T_b$ and C_i a cycle in $\mathcal{A}_a$ implies $S_a \in \langle T_a, T_b \rangle_{\mathcal{C}}$. This gives the subsequence (S_b, T_a, S_a, T_b), a contradiction.

- $P \in cross_a(i)$ implies x is the first crossing of $P[S_a, T_a]$ with C_i. But since x is also a crossing from I_{C_i} to E_{C_i}, x must be a crossing of $P[C_i, T_a]$ with C_i. Since $C_i = T_b$, it must follow that $T_a \in \langle S_b, T_b \rangle_{\mathcal{C}}$, as otherwise the segments are disjoint. But $C_i = T_b$ now implies that $S_a \in \langle T_a, T_b \rangle_{\mathcal{C}}$. This gives the subsequence (S_b, T_a, S_a, T_b), a contradiction.

For (3), suppose $x \in X_a(i,j) \cap X_b(k,\ell)$. If x is a crossing of a path with a cycle C, then either $C \notin \mathcal{C}_a$ or $C \notin \mathcal{C}_b$. Without loss of generality, we assume $C \notin \mathcal{C}_b$. As $x \in X_b(k,\ell)$, $C \in \{S_b, T_b\}$. Therefore, there are three possibilities to consider: x is a crossing of a path P with S_b, x is a crossing of P with T_b, or x is a crossing of two paths.

- x is a crossing of S_b. This implies $P \in aft_b(k,\ell)$ and $C_k = S_b$. In particular, x is the first crossing of $P[C_\ell, T_b]$ with C_k, and is a crossing from E_{C_k} to I_{C_k}.

  - $P \in aft_a(i,j)$ implies x is a crossing of $P[C_j, T_a]$ with C_i. Thus, both C_j and C_ℓ occur on the segment of P between $C_i = C_k$ and x, and both T_a and T_b occur after x. This implies that $C_j \in \langle S_b, T_b \rangle_{\mathcal{C}}$ and $C_\ell \in \langle S_a, T_a \rangle_{\mathcal{C}}$, a contradiction to the definition of a partitioned circular arrangement.

  - $P \in last_a(i,j)$ implies x is the last crossing of $P[S_a, C_j]$ with T_a, and hence is a crossing from I_{T_a} to E_{T_a}, a contradiction.

- x is a crossing of T_b. This implies $P \in last_b(k,\ell)$. In particular, x is the last crossing of $P[C_k, C_\ell]$ with T_b, and is a crossing from I_{T_b} to E_{T_b}.

  - $P \in aft_a(i,j)$ implies x is the first crossing of $P[C_j, T_a]$ with C_i, and hence is a crossing from E_{C_i} to I_{C_i}, a contradiction.

  - $P \in last_a(i,j)$ implies $T_a = T_b$ and x is the last crossing of $P[S_a, C_j]$ with T_a. Thus, both C_j and C_ℓ occur on the segment of P from x to $T_a = T_b$. This implies that $C_j \in \langle S_b, T_b \rangle_{\mathcal{C}}$ and $C_\ell \in \langle S_a, T_a \rangle_{\mathcal{C}}$, a contradiction to the definition of a partitioned circular arrangement.

- Finally, assume x is a crossing of paths $P, Q \in rest_a(i,j) \cap rest_b(k,\ell)$. We can assume that x is a crossing of P between C_u and C_{u+1}, and of Q between C_v and C_{v+1}. We can further assume without loss of generality that $C_u, C_{u+1} \in [C_i, C_j]_{\mathcal{C}}$ and $C_v, C_{v+1} \in [C_j, T_a]_{\mathcal{C}}$. Now there are two possibilities:
  - $C_u, C_{u+1} \in [C_k, C_\ell]_{\mathcal{C}}$ and $C_v, C_{v+1} \in [C_\ell, T_b]_{\mathcal{C}}$. In this case, $C_j \in [C_{u+1}, T_b]_{\mathcal{C}}$, so $\mathcal{C}_a \cap \langle S_b, T_b \rangle_{\mathcal{C}} \neq \emptyset$. Similarly, $C_\ell \in [C_{u+1}, T_a]_{\mathcal{C}}$, so $\mathcal{C}_b \cap \langle S_a, T_a \rangle_{\mathcal{C}} \neq \emptyset$, a contradiction.
  - $C_u, C_{u+1} \in [C_\ell, T_b]_{\mathcal{C}}$ and $C_v, C_{v+1} \in [C_k, C_\ell]_{\mathcal{C}}$. As $T_b \in [C_{u+1}, C_v]_{\mathcal{C}}$, it follows that $T_b \in \langle S_a, T_a \rangle_{\mathcal{C}}$. Also, $S_b \in [T_b, C_v]_{\mathcal{C}}$, so $S_b \in \langle T_b, T_a \rangle_{\mathcal{C}}$, yielding the sequence (S_a, T_b, S_b, T_a), a contradiction to the definition of a partitioned circular arrangement. **Q.E.D.**

The following lemma is now proven in the same way as its counterpart Lemma 11.

Lemma 14 *For each j, $|X(j)| + \sum_{0 \leq i < j} (|X(i,j)| + |C_i \cap C_j|) \geq m - 2$.*

Theorem 12 follows immediately from this lemma. The following corollaries are direct consequences of Theorem 12 and will be significantly improved in the next section.

A circular arrangement $(\mathcal{C}, \mathcal{P}, \psi)$ is *non-separating* if there do not exist $C, C', C'' \in \mathcal{C}$ such that C' and C'' are in different components of $\mathbf{R}^2 \setminus C$.

Corollary 15 *Suppose $\mathcal{A}$ is a non-separating (m,n)-circular arrangement. If there are three cycles C, C', C'' of $\mathcal{A}$ such that each is disjoint from all the other $n-1$ cycles of $\mathcal{A}$, then $cr(\mathcal{A}) \geq (m-2)n$.*

Proof: Assume without loss of generality that in the cyclic ordering of $\mathcal{C}$, the three cycles appear in the order (C, C', C''). Consider the arrangements $\mathcal{A}_1 = (C, C'', [C, C'] \setminus \{C\}, \mathcal{P}, \psi)$, $\mathcal{A}_2 = (C', C, [C', C''] \setminus \{C'\}, \mathcal{P}, \psi)$, and $\mathcal{A}_3 = (C'', C', [C'', C] \setminus \{C''\}, \mathcal{P}, \psi)$. It is routine to verify that these arrangements partition $\mathcal{A}$. **Q.E.D.**

Corollary 16 *Let $\mathcal{A}$ be a non-separating (m, n)-circular arrangement with $n \geq 3$. If no two cycles of $\mathcal{A}$ intersect, then $cr(\mathcal{A}) \geq (m-2)n$.*

8 Circular Arrangements With At Least Three Components

Let $\mathcal{A} = (\mathcal{C}, \mathcal{P}, \psi)$ be an (m, n)-circular arrangement. A *pre-component* H of $\mathcal{C}$ is a set of cycles such that if $C \in H$ and $C' \notin H$, then $C \cap C' = \emptyset$. A *component* of $\mathcal{C}$ is a (set-wise) minimal pre-component of $\mathcal{C}$. The main point of this section is to prove that if $\mathcal{A}$ is a non-separating (m, n)-circular arrangement with at least three components, then $cr(\mathcal{A}) \geq (m-2)n$.

Let $\mathcal{A}$ be an (m, n)-circular arrangement with at least three components. The *standard sub-arrangements* of $\mathcal{A}$ are the arrangements defined as follows: for each $i \in \{1, \ldots, n\}$ such that C_i is in component H and C_{i-1} is in component H' with $H \neq H'$, let $C_{i'}$ be the cycle such that $[C_i, C_{i'}]_{\mathcal{C}} \subseteq H$ and $C_{i'+1} \notin H$. Let C_j be the cycle such that $[C_i, C_{j-1}]_{\mathcal{C}} \subseteq H \cup H'$ and $C_j \notin H \cup H'$. Now define $\mathcal{A}_i = (C_{i-1}, C_j, [C_i, C_{i'}]_{\mathcal{C}}, \mathcal{P}, \psi)$.

Theorem 17 *Let $\mathcal{A}$ be a non-separating (m, n)-circular arrangement with at least three components, and let the $\mathcal{A}_i = (S_i, T_i, \mathcal{C}_i, \mathcal{P}_i, \psi)$ be the standard sub-arrangements.*

1. *If there do not exist i, j such that (S_i, T_j, S_j, T_i) is a subsequence of $\mathcal{C}$, then the standard sub-arrangements partition $\mathcal{A}$ and $cr(\mathcal{A}) \geq (m-2)n$;*

2. *If there exists i, j such that (S_i, T_j, S_j, T_i) is a subsequence of $\mathcal{C}$, then $\mathcal{A}$ has exactly three components H_1, H_2, H_3 such that:*

   - $S_i, S_j \in H_1$; $C_i \in H_2$; *and* $C_j \in H_3$;
   - $T_j = C_i$ *and* $T_i = C_j$;
   - $\langle S_i, S_j \rangle_{\mathcal{C}} \subseteq (H_1 \cup H_2)$;
   - $\langle S_j, S_i \rangle_{\mathcal{C}} \subseteq (H_1 \cup H_3)$;

In particular, if $\mathcal{A}$ has at least four components, then $cr(\mathcal{A}) \geq (m-2)n$.

Proof: Every $C_i \in \mathcal{C}$ is in some standard sub-arrangement $\mathcal{A}_i$ since either C_i and C_{i-1} are in different components (so $C_i \in \mathcal{C}_i$) or they are in the same component (so C_i is in the same arrangement as C_{i-1}). Furthermore, for every $\mathcal{A}_i$, by definition S_i, T_i, and $\mathcal{C}_i$ are in different components, and are therefore disjoint.

Suppose (1) holds, i.e., there are no i, j such that (S_i, T_j, S_j, T_i) is a subsequence of $\mathcal{C}$. If $\mathcal{C}_i \cap \langle S_j, T_j \rangle_{\mathcal{C}} \neq \emptyset$, then, since $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ and S_j immediately precedes C_j in $\mathcal{C}$, $C_i \in \langle S_j, T_j \rangle_{\mathcal{C}}$ and in particular $S_i \neq S_j$ and $S_i \neq T_j$ so $S_i \in \langle S_j, T_j \rangle_{\mathcal{C}}$. Symmetrically, if $\mathcal{C}_j \cap \langle S_i, T_i \rangle_{\mathcal{C}} \neq \emptyset$, then $S_j \neq S_i$ and $S_j \neq T_i$ and $S_j \in \langle S_i, T_i \rangle_{\mathcal{C}}$. Since (S_i, S_j, T_i) and (S_j, S_i, T_j) are subsequences of $\mathcal{C}$, it follows that (S_i, T_j, S_j, T_i) is a subsequence of $\mathcal{C}$, a contradiction. Therefore, $\mathcal{A}$ with the standard sub-arrangements is a partitioned circular arrangement and $cr(\mathcal{A}) \geq (m-2)n$ by Theorem 12.

Now suppose that (S_i, T_j, S_j, T_i) occurs. Then $\mathcal{C} \subseteq \langle S_i, T_i \rangle_{\mathcal{C}} \cup \langle S_j, T_j \rangle_{\mathcal{C}}$. Let H_1, H_2, H_3 be the distinct components such that $S_i \in H_1$, $\mathcal{C}_i \subseteq H_2$, and $T_i \in H_3$. By definition of T_i, $\langle S_i, T_i \rangle_{\mathcal{C}} \subseteq H_1 \cup H_2$. In particular, $S_j \in H_1 \cup H_2$ and $T_j \in H_1 \cup H_2$. Since S_j and T_j are in different components, one is in H_1 and the other is in H_2.

Let $H' \in \{H_1, H_2\}$ be the component containing S_j and $H'' \in \{H_1, H_2\}$ be the component containing T_j . Since $\mathcal{C}_j$ is in a different component than S_j and T_j, $\mathcal{C}_j \cap (H_1 \cup H_2) = \emptyset$. Since $C_{j-1} = S_j$, either $C_j \in \langle C_i, T_i \rangle_{\mathcal{C}}$ or

$C_j = T_i$. The former contradicts $\mathcal{C}_j \cap (H_1 \cup H_2) = \emptyset$. Therefore, $C_j = T_i$ and, symmetrically, $C_i = T_j$. Consequently, $H'' = H_2$, $H' = H_1$, and $\mathcal{C}_j \subseteq H_3$. If $\mathcal{A}$ has at least four components, then there is a cycle $C \notin (H_1 \cup H_2 \cup H_3)$, contradicting the earlier conclusion that $\mathcal{C} \subseteq \langle S_i, T_i\rangle \cup \langle S_j, T_j\rangle$. Therefore $\mathcal{A}$ has three components. **Q.E.D.**

We now extend the preceding result to the one we really want.

Theorem 18 *Let $\mathcal{A}$ be a non-separating (m, n)-circular arrangement with at least three components. Then $cr(\mathcal{A}) \geq (m-2)n$.*

Proof: By Theorem 17 we can assume there are exactly three components and that for the standard subarrangements, there is some pair i and j for which (S_i, T_j, S_j, T_i) is a subsequence of $\mathcal{C}$, $S_i = C_{i-1}$, $S_j = C_{j-1}$, $S_i, S_j \in H_1$, $C_i = T_j \in H_2$, $C_j = T_i \in H_3$, $\langle S_i, S_j\rangle_{\mathcal{C}} \subseteq (H_1 \cup H_2)$, and $\langle S_j, S_i\rangle_{\mathcal{C}} \subseteq (H_1 \cup H_3)$.

Let S'_i denote the last cycle in $[C_j, C_{i-1}]_{\mathcal{C}}$ that is in H_3. (Since $C_j \in H_3$, this cycle exists.) Let T'_i denote the first cycle in $[C_i, C_{j-1}]_{\mathcal{C}}$ that is in H_1. (Since $C_{j-1} \in H_1$, this cycle exists.) Replace the arrangement $(S_i, T_i, \mathcal{C}_i)$ with $(S'_i, T'_i, \mathcal{C}_i)$. Since S'_i, T'_i anc $\mathcal{C}_i$ are all in different components, $(S'_i, T'_i, \mathcal{C}_i)$ is an arrangment. Now (S'_i, S_j, T_j, T'_i) occurs as a subsequence instead of (S_i, T_j, S_j, T_i). We prove that with this one change of subarrangements, the standard subarrangements form a partition of $\mathcal{A}$.

Let $\mathcal{A}' = (S_k, T_k, \mathcal{C}_k)$ be any other standard subarrangement. Then S_k, T_k and $\mathcal{C}_k$ are contained in different components of $\mathcal{A}$. If the first cycle C_k of $\mathcal{C}_k$ is in $\langle C_i, C_j\rangle$, then $C_k \in H_1 \cup H_2$ and $S_k = C_{k-1} \in H_1 \cup H_2$. Thus, $T_k = T_i = C_j \in H_3$. Similarly, if $C_k \in \langle C_j, C_i\rangle$, then $T_k = T_j = C_i$. Therefore, (S'_i, T_k, S_k, T'_i) does not occur.

It follows that no other two standard subarrangements can satisfy the conclusions of Theorem 17 (2). The preceding paragraph also makes clear that the third and fourth conditions of a partitioned circular arrangement hold for $\mathcal{A}'$ and $(S'_i, T'_i, \mathcal{C}_i)$. Therefore, the new arrangements show the circular arrangement to be partitioned. The result now follows from Theorem 12. **Q.E.D.**

9 $(7, n)$-Circular Arrangements

In this section we prove the following fact, which quite surprised us. The surprise is that the "circularity" of the arrangement is in one direction only, while drawings of $C_m \times C_n$ are "circular" in both directions.

Theorem 19 *If $\mathcal{A}$ is an (m, n)-circular arrangement with $m \leq 7$ and $n \geq m$, then $cr(\mathcal{A}) \geq (m-2)n$.*

We recall from [10] that an (m, n)-*mesh* is a pair $(\mathcal{C}_1, \mathcal{C}_2)$ of sets of cycles such that $|\mathcal{C}_1| = m$, $|\mathcal{C}_2| = n$ and, for every $C_1 \in \mathcal{C}_1$, $C_2 \in \mathcal{C}_2$, $C_1 \cap C_2 \neq \emptyset$. A main result in [10] is that, for $3 \leq m \leq n \leq 5$, if $\mathcal{A}$ is an (m, n)-mesh, then $cr(\mathcal{A}) \geq (m-2)n$. In [2, 3] it is proved that if $m \in \{6, 7\}$ and $\mathcal{A}$ is an (m, m)-mesh, then $cr(\mathcal{A}) \geq (m-2)m$.

Proof: If $m \leq 2$, then the result is trivial. Hence we assume $m \geq 3$. Every (m, n)-circular arrangement is an (m, n)-mesh, so the preceding remarks imply that for each $m \in \{3, 4, 5, 6, 7\}$, every (m, m)-circular arrangement $\mathcal{A}$ satisfies $cr(\mathcal{A}) \geq (m-2)m$. Thus, for each such m, we have the base for an induction on n.

Now suppose $n > m$ and let $\mathcal{A} = (\mathcal{C}, \mathcal{P}, \psi)$ be an optimal (m, n)-circular arrangement. If $C \in \mathcal{C}$ separates two other curves in $\mathcal{C}$, then C is crossed by every $P \in P$ and so has at least m crossings. Delete C to get an $(m, n-1)$-circular arrangement $\mathcal{A}'$. Inductively, $cr(\mathcal{A}') \geq (m-2)(n-1)$. Since $cr(\mathcal{A}) \geq cr(\mathcal{A}') + m$, we are done in this case. Thus, we can suppose $\mathcal{A}$ is non-separating.

If $\mathcal{A}$ has at least three components then we are done by Theorem 18, so we assume $\mathcal{A}$ has at most two components. If $n \leq 4$, then $(m, n) = (3, 4)$.

Since every $(3,4)$-mesh has at least four crossings, we are done in this case. Thus, we can assume $n \geq 5$.

Therefore, some component of $\mathcal{A}$ has at least three cycles in it, so some cycle has at least four crossings. Thus, for $m \leq 6$, we are done by induction.

For $m = 7$, some component H has at least four cycles in it. If some cycle has at least five crossings, then we are done by induction. Therefore, we can assume that no cycle in H has five or more crossings.

It follows that some pair of cycles C, C' in H intersect each other and each has four intersections. Since $|C \cap C'|$ is even, it must be exactly two. Let x and y be the two points of the plane in $C \cap C'$ (i.e., $x = C(s) = C'(s')$ and $y = C(t) = C'(t')$ for some $s, s', t, t' \in [0,1]$). There are two arcs A, B in C that have x and y as ends, and there are two arcs A', B' in C' that have x and y as ends.

Because C and C' have exactly four intersections, no element of $\mathcal{P}$ crosses either C or C' and therefore one of A and B is disjoint from all the paths in $\mathcal{P}$ and one of A' and B' is disjoint from all the paths in $\mathcal{P}$. Thus, we can exchange these arcs to make two new cycles $\hat{C}$ and $\hat{C}'$ to replace C and C' and make a new $(7,n)$-circular arrangement $\hat{\mathcal{A}}$ with two fewer crossings than $\mathcal{A}$, a contradiction to the optimality of $\mathcal{A}$. **Q.E.D.**

As a trivial consequence of Theorem 19 we have the following.

Theorem 20 *The crossing number of $C_7 \times C_n$ for $n \geq 7$ is $5n$.*

10 $(8,n)$-Circular Arrangements

So far we have been unable to prove that $cr(C_8 \times C_n) = 6n$. The purpose of this section is to present the inductive step, leaving open only the case $cr(C_8 \times C_8)$. In fact, we present the inductive step for $(8,n)$-circular arrangements.

An (m,n)-circular arrangement is *fully optimal* if it is optimal and, among all optimal (m,n)-circular arrangements, it has as few cycle–cycle crossings as possible.

The inductive step for $(8, n)$-circular arrangements is given in the following result.

Theorem 21 *Let $\mathcal{A}$ be a fully optimal $(8, n)$-circular arrangment with $n > 8$. Then either $cr(\mathcal{A}) \geq 6n$ or some cycle has at least six crossings.*

In order to prove this result, we need one more notion.

Let $\mathcal{A}$ be an (m, n)-circular arrangement and let C, C', C'' be cycles of $\mathcal{A}$ such that C'' is disjoint from both C and C'. For $D \in \{C, C'\}$, let $I(D)$ denote the component of $\mathbf{R}^2 \setminus D$ not containing C''. Suppose $|C \cap C'| = 2$. Let A (respectively, A') denote the arc of C (resp., C') contained in $I(C')$ (resp., $I(C)$). The crossings of $\mathcal{A}$ corresponding to $|C \cap C'|$ are *removable* if at most one path intersects $A \cup A'$ and if there is such a path P, then $|P \cap (A \cup A')| \leq 2$.

The motivation for this definition is that if the crossings are removable, then it is easily seen that there is another (m, n)-circular arrangement $\mathcal{A}'$ such that either $cr(\mathcal{A}') = cr(\mathcal{A}) - 2$ (when no such P exists) or $cr(\mathcal{A}') \leq cr(\mathcal{A})$ and $\mathcal{A}'$ has fewer cycle-cycle crossings than $\mathcal{A}$.

More significantly, if $\mathcal{A} = (\mathcal{C}, \mathcal{P}, \psi)$ is a fully optimal (m, n)-circular arrangement and $\mathcal{C}'$ is any cyclic subsequence of $\mathcal{C}$, then $(\mathcal{C}', \mathcal{P}, \psi)$ has no removable crossings.

Lemma 22 *Let $\mathcal{A} = ((C_0, C_1, C_2, \ldots, C_5), \mathcal{P}, \psi)$ be an $(m, 6)$-circular arrangement with no removable crossings such that $\{C_0\}$ is a component of $\mathcal{A}$. Then either $\mathcal{A}$ has at least three components or $\mathcal{A}$ has a cycle with at least six crossings.*

Proof: Assume no cycle has six crossings and the drawing $\mathcal{A}$ has just two components. Then one component H is $\{C_1, C_2, \ldots, C_5\}$. Since no cycle in H has six crossings, each cycle in H intersects at most two others and no two cycles of H can intersect in more than two points. Thus, the cycles of H can be labelled as C^1, C^2, C^3, C^4, C^5 such that $C^i \cap C^{i+1} \neq \emptyset$ for $1 \leq i \leq 4$. For each $i \in \{1,2,3,4,5\}$, there is a component $I(C^i)$ of $\mathbf{R}^2 \setminus C^i$ that is disjoint from C_0.

Let $i \in \{2,3\}$. Then C^i intersects both C^{i-1} and C^{i+1}, each precisely twice. Let A be the segment of C^i in $I(C^{i+1})$ and let B be the segment of C^{i+1} in $I(C^i)$. If $|\{P : P \cap (A \cup B) \neq \emptyset\}| \in \{0,1\}$, then the crossings corresponding to $C^i \cap C^{i+1}$ are removable, a contradiction. This leaves three possibilities. The first is that there are at least two paths with vertices on B. This gives six crossings of C^i, a contradiction. Second, there are at least two paths with vertices on A. This gives six crossings of C^{i+1}, a contradiction. Third, there is one path with a vertex on A and one path with a vertex on B. Let P_i be the path with a vertex on B (and hence intersects $I(C^i)$) and let P'_{i+1} be the path with a vertex on A.

Next we show that $P_3 = P'_3$. Both P_3 and P'_3 intersect $I(C^3)$ and, therefore, both cross C^3. Thus, if $P_3 \neq P'_3$, C^3 has at least six crossings, a contradiction.

By construction, P_3 must have a vertex on the segment $C^2 \cap I(C^3)$ and a vertex on the segment $C^4 \cap I(C^3)$. Thus, if $C^3 = C_j$, then by traversing the path (in the backwards direction if need be), we get that either $C^4 = C_{j+1}$ and $C^2 = C_{j+2}$ or $C^2 = C_{j+1}$ and $C^4 = C_{j+2}$. For the sake of definiteness, we assume the latter.

Now consider C^2. The path P_2 must have a vertex on $C^3 \cap I(C^2)$, and by the same reasoning as above, if $C^2 = C_i$, we have one of the following situations (it is not necessary for P_2 to have a vertex on $C^1 \cap I(C^2)$, so we get all four possibilities):

1. $C^3 = C_{i-2}$ and $C^1 = C_{i-1}$;

2. $C^3 = C_{i-1}$;

3. $C^3 = C_{i+1}$; or

4. $C^3 = C_{i+2}$ and $C^1 = C_{i+1}$;

Since $C^2 = C_{j+1}$, we have $i = j + 1$ and therefore $C^3 = C_j = C_{i-1}$. Thus, Case 2 holds.

Likewise, considering C^4, we get that if $C^4 = C_k$, then $k = j + 2$, so $C^3 = C_j = C_{k-2}$ and $C^5 = C_{k-1} = C_{j+1} = C_i = C^2$, the final contradiction. **Q.E.D.**

Proof of Theorem 21. If $\mathcal{A} = (\mathcal{C}, \mathcal{P}, \psi)$ has at least three components, then we are done by Theorem 18. If it has one or two components, then some component has at least five cycles. Since no cycle intersects three others, we can find six cycles of $\mathcal{A}$ so that $\mathcal{C}' = (C_{i_0}, C_{i_1}, C_{i_2}, \ldots, C_{i_5})$ is a subsequence of $\mathcal{C}$ and $\mathcal{C}'$ has only two components, one of which is $\{C_{i_0}\}$ and the other of which contains the remaining cycles of $\mathcal{C}'$. Since $\mathcal{A}$ has no removable crossings, neither does $\mathcal{A}' = (\mathcal{C}', \mathcal{P}, \psi)$. By Lemma 22, some cycle of $\mathcal{A}'$ has at least six crossings. **Q.E.D.**

11 Circular Arrangements With Only Small Components

An (m, n)–circular arrangement $\mathcal{A}$ is *k-restricted* if every component has at most k cycles. The main result of [11] can be interpreted as the following.

Theorem 23 *[11] If $\mathcal{A}$ is a 1–restricted (m, n)–circular arrangement with $n \geq m$, then $cr(\mathcal{A}) \geq (m - 2)n$.*

We shall generalize this result as follows.

Theorem 24 *If $\mathcal{A}$ is a 3–restricted (m, n)-circular arrangement with $n \geq m$, then $cr(\mathcal{A}) \geq (m - 2)n$.*

Our proof is based on the much stronger Theorem 18, but we do not see how to do an induction maintaining the property "has at least three components", while we can do the induction on the property "has components of size at most three". We invite the reader to improve Theorem 24, as we cannot yet do the 4–restricted case. (The sticking point is $m = n = 8$.)

Proof: The proof is by induction on $m + n$, with the base being $m \leq 7$, given by Theorem 19. Now suppose $\mathcal{A}$ is a 3–restricted (m, n)–circular arrangement. If $\mathcal{A}$ is not non-separating, then we are done easily by induction. So we can assume $\mathcal{A}$ is non-separating. Since $n \geq 7$, there are at least three components, so we are done by Theorem 18.**Q.E.D.**

12 Concluding Remarks

Obviously, it would be very helpful to know that if $\mathcal{A}$ is an $(8, 8)$-circular arrangement, then $cr(\mathcal{A}) \geq 48$. This seems hard today.

As with the earlier comments concerning arrangements, we believe it should be possible to generalize our results on circular arrangements.

A circular arrangement $(\mathcal{C}, \mathcal{P}, \psi)$ is *2-disjoint* if for every curve $C \in \mathcal{C}$, there exist distinct cycles $C', C'' \in \mathcal{C} \setminus \{C\}$ such that C, C', C'' are pairwise disjoint.

Conjecture 2 *If $\mathcal{A}$ is a non-separating 2-disjoint (m, n)-arrangement, then $cr(\mathcal{A}) \geq (m - 2)n$.*

This would greatly generalize Theorem 12 and would have a tremendous impact on the Harary-Kainen-Schwenk Conjecture. For one thing, it would immediately yield the inductive steps for every m: if $n > m$, then (as usual) either $\mathcal{A}$ is non-separating or we are done by induction. If $C \in \mathcal{C}$ and no cycle in $\mathcal{C}$ is disjoint from C, then C has at least $2(n - 1)$ crossings and we can induct. If C and C' are disjoint but no other cycle in $\mathcal{C}$ is disjoint from both C and C', then one of C and C' has at least $2(n - 2)/2$ crossings. This is always at least $m - 2$, so again we can induct. Therefore, we can assume $\mathcal{A}$ is non-separating and 2-disjoint.

For a second implication, for every m, there is an $N = N(m)$ such that $n \geq N$ implies $cr(C_m \times C_n) = (m-2)n$. To see this, note that, for each $n \geq m+2$, either some m-cycle has at least m crossings or the circular arrangement is separation-free and 2-disjoint. Now suppose $n \geq m^2 + 2m$ and let $\mathcal{A}$ be an (m,n)-arrangement.

If no m-cycle has at least m crossings, then $\mathcal{A}$ is separation-free and 2-disjoint, so by Conjecture 2 we are done. More generally, suppose there are indices $i_1, i_2, \ldots, i_k$ such that, the cycle C_{i_j} has at least m crossings in the arrangement obtained by deleting the cycles $C_{i_1}, C_{i_2}, \ldots, C_{i_{j-1}}$. The original arrangement obviously has at least km crossings, so if $k \geq (m-2)n/m$, then $cr(\mathcal{A}) \geq (m-2)n$.

Therefore, we can suppose $k < (m-2)n/m$ and, after deleting the cycles $C_{i_1}, \ldots, C_{i_k}$, no cycle in the resulting arrangement $\mathcal{A}'$ has at least m crossings. Thus, $\mathcal{A}'$ is separation-free. Since $\mathcal{A}'$ has $n-k \geq m+2$ cycles, $\mathcal{A}'$ is 2-disjoint. Thus, by Conjecture 2, $cr(\mathcal{A}') \geq (m-2)(n-k)$. Since $cr(\mathcal{A}) \geq km + cr(\mathcal{A}')$, we see $cr(\mathcal{A}) \geq (m-2)n$, as required.

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