Crossing Numbers of Sequences of Graphs II: Planar Tiles

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Abstract

We describe a method of creating an infinite family of crossing-critical graphs from a single small planar map, the *tile*, by gluing together many copies of the tile together in a circular fashion. This method yields all known infinite families of k-crossing-critical graphs.

Furthermore, the method yields new infinite families, which extend from $[4, 6)$ to $(3.5, 6)$ the interval of rationals r for which there is, for some k, an infinite sequence of k-crossing-critical graphs all having average degree r.

Keywords: crossing number, planar tile.

1 Introduction

The main point of this work is to provide a general method of constructing infinite families of k-crossing-critical graphs, i.e., graphs G for which the crossing number $cr(G)$ satisfies, for every edge e of G, $cr(G - e) < k$, yet $cr(G) \geq k$. Richter and Thomassen [6] proved that if G is k-crossing-critical, then $cr(G) \leq (5/2)k + 16$. With some additional argument, they show this implies that if k is a positive integer and there is an infinite collection of k-crossing-critical r-regular simple graphs, then $r \in \{4, 5\}$. They gave an example with $r = 4$ and $k = 3$.

Salazar [7] has observed that the same argument as in [6] shows that if k is a positive integer and there is an infinite collection of k-crossing-critical simple graphs, all having the same average degree d, then d is a rational number in the open interval $(3, 6)$. He has given examples to show that every rational in $[4, 6)$ can be achieved as the average crossing number of such a collection.

(We remark that we do not allow vertices of degree 2 in crossing-critical graphs, since subdivision does not affect either the crossing number or criticality. Thus, crossing-critical graphs have minimum degree at least 3.)

Kochol [3] gave the first example of an infinite family of simple, 3-connected graphs, each 2-crossing-critical with average degree 3.2. Hliněný [2] has proved there is a function $f(k)$ such that if G is k-crossing-critical, then G has path-width at most $f(k)$.

These all suggest that in looking for sequences of graphs with bounded crossing numbers, it is worthwhile to look at graphs that are made up from small pieces (the "tiles") glued together in a circular fashion. In [5] this notion was introduced and it was shown how to compute the asymptotic value of the crossing number of the graph obtained by gluing n copies of a tile in a circular fashion. This was applied to graphs such as the Cartesian product $C_m \times C_n$ of two cycles and the generalized Petersen graphs $P(n, k)$.

Oporowski has shown (personal communication) that there is an N so that if G is 2-crossing-critical with at least N vertices, then G is constructed in just the manner described here, except that combinations of 13 tiles are used, rather than just one tile.

In Section 2 we give the precise definitions of a tile T and the circular graph $T^{\otimes}(n)$. The main point is that $cr(T^{\otimes}(n))$ is eventually constant (Section 3). Moreover, we give a simple sufficient condition that in some cases permits an easy computation of the eventual crossing number of $T^{\otimes}(n)$ (Section 4). These results are used to show that if r is a rational number in the interval $(3.5, 4)$, then there is an infinite set K of positive integers k, such that if $k \in K$, then there is an infinite family of connected k-crossing-critical graphs, all having average degree r.

2 Tiles and Sequences of Graphs

A *tile* is a 3-tuple $T = (G, L, R)$, where

- G is a connected graph;
- L is a finite sequence of vertices of $V(G)$, called the *left-wall*;
- R is a finite sequence of vertices of $V(G)$, called the *right-wall*; and
- all the vertices in L and R are distinct.

A *tile drawing* of a tile $T = (G, L, R)$ is a drawing of T in the unit square $S = [0,1] \times [0,1]$ such that: (i) the intersection with the boundary $\{0,1\} \times [0,1] \cup [0,1] \times \{0,1\}$ is precisely $L \cup R$; (ii) the vertices in L occur in the line $x = 0$, with the y-coordinates of $L[1], L[2], ..., L[\,|L|\,]$ decreasing; and (iii) the vertices in R occur in the line $x = 1$, with the y-coordinates of $R[1], R[2], ..., R[\,|R|\,]$ also decreasing. The *tile crossing number* $tcr(T)$ of a tile T is the smallest number of crossings in any tile drawing of T.

A tile $T = (G, L, R)$ is *planar* if $tcr(T) = 0$. A tile is *perfect* if

(i) $G - L$ and $G - R$ are both connected;

(ii) for every left- (right-, respectively) wall vertex v, there is a path to the right- (left-) wall disjoint from the left- (right-) wall except for v;

(iii) for every pair i and j, $i \neq j$, there are two disjoint paths, one joining $L[i]$ and $R[i]$, and the other joining $L[j]$ and $R[j]$.

A tile $T_1 = (G_1, L_1, R_1)$ is *compatible* with a tile $T_2 = (G_2, L_2, R_2)$ if the function $f : R_1 \to L_2$ defined by $f(R_1[i]) = L_2[i]$ is a bijection and

$$\forall\, v_1, v_2 \in R_1, v_1 v_2 \in E(G_1) \Leftrightarrow f(v_1) f(v_2) \in E(G_2).$$

A tile is *self-compatible* if it is compatible with itself.

Let G and H be (disjoint) graphs and let X and Y be sequences of distinct vertices of G and H, respectively, having the same length t. Then $(G \cup H)\{X, Y\}$ is the graph obtained by identifying $X[i]$ with $Y[i]$, for $i = 1, 2, \ldots, t$.

Let $S = (G, L, Q)$ and $T = (H, M, R)$ be compatible tiles. Then the tile ST is $((G \cup H)\ \{Q, M\}, L, R)$. (That is, we simply identify the right wall of S with the left wall of T.) If $T = (G, L, R)$ is a self-compatible tile, then $\circ(T)$ is the graph $G\{L, R\}$ and T^n is defined inductively by $T^1 = T$ and $T^n = T^{n-1}T$. For a tile $T = (G, L, R)$, let $\widetilde{T}$ denote $T = (G, L, R^{\leftarrow})$, where $R^{\leftarrow}$ is the reverse sequence of R. If the appropriate compatibilities hold, then $T^{\bowtie}(m, n)$ denotes $T^m\widetilde{T}T^n$ and $T^{\otimes}(n)$ is $\circ(T^n\widetilde{T})$.

We note the following basic facts about the relationships between the crossing numbers of these constructs.

Lemma 1. *If T_1 is compatible with T_2, then $tcr(T_1T_2) \leq tcr(T_1) + tcr(T_2)$ and if T is self-compatible, then $cr(\circ(T)) \leq tcr(T)$.*

The main results in [5] are: (1) $\lim_{n\to\infty} tcr(T^n)/n$ exists (easy) and this limit is the same as $\lim_{n\to\infty} cr(\circ(T^n))/n$ (not so easy).

3 $cr(T^{\otimes}(n))$ is eventually constant

In this section we prove the following main result.

Theorem 2. *Let T be a perfect planar tile. Then there are integers N, N' such that if $n \geq N'$, then $cr(T^{\otimes}(n)) = tcr(T^{\bowtie}(N, N))$.*

We start with the following simple fact.

Lemma 3. *The sequence $tcr(T^{\bowtie}(n, n))$ is non-increasing and therefore eventually constant.*

Proof: For a planar tile T, Lemma 1 implies $tcr(T^{\bowtie}(n+1, n+1)) \leq tcr(T^{\bowtie}(n, n+1)) \leq tcr(T^{\bowtie}(n, n)) \leq tcr(\widetilde{T})$.□

The hard part of the proof of Theorem 2 is the inequality $tcr(T^{\bowtie}(N, N)) \leq cr(T^{\otimes}(n))$. One ingredient in the argument, a direct consequence of the definition of perfect tile, is the following.

Lemma 4. *Let $T = (G, L, R)$ be a perfect tile. If $n \geq 3$, then, for any copy $T_i = (G_i, L_i, R_i)$ of T in $T^{\otimes}(n)$: (i) if u, v are neighbors in $T_{i-1} - R_{i-1}$ of vertices in $R_{i-1} = L_i$, then there is a path in T_{i-1} joining u and v that is disjoint from T_i; and (ii) if u, v are neighbors in $T_{i+1} - L_{i+1}$ of vertices in $L_{i+1} = R_i$, then there is a path in T_{i+1} joining u and v that is disjoint from T_i.*

Next is the main point in the proof of Theorem 2.

Theorem 5. *Let $T = (G, L, R)$ be a perfect planar tile and let c be a fixed positive integer. If $n \geq 6c+4$ and $\mathcal{D}$ is a drawing of $T^{\otimes}(n)$ having at most c crossings, then there is a tile drawing of some $T^{\bowtie}(\ell, n-1-\ell)$ having exactly $cr(\mathcal{D})$ crossings.*

***Proof*:** Let $c = tcr(\widetilde{T})$. Take $n \geq 6c + 4$. Ignore $\widetilde{T}$ and the copies of T on each side of $\widetilde{T}$; there are $6c + 2$ copies left. Since each crossing can involve edges from two copies of T and the edges can be wall edges, at most $4c$ copies are involved in the crossings. Viewing the tiles involved in the crossing as separators (if the edge involved in the crossing is a wall edge, then the corresponding separator contains the two consecutive copies containing that edge), we have at most $2c$ separators (involving at most $4c$ copies) and at most $2c + 1$ intervals in the remaining $2c + 2$ copies. Hence one interval has at least $(2c + 2)/(2c + 1) > 1$ copies; that is, one interval contains at least 2 consecutive copies of T, both having no edges crossed in $\mathcal{D}$. Let $T_1 = (G_1, L_1, R_1)$ and $T_2 = (G_2, L_2, R_2)$ be two such tiles.

Now consider the tile T_1. Then there is a face F of $\mathcal{D}(T_1)$ that contains the tile $\widetilde{T}$. Since $\mathcal{D}(T_1)$ has no crossing and every vertex of $T^{\otimes}(n)$ outside of T_1 is joined to $\widetilde{T}$ by a path disjoint from T_1, all of $T^{\otimes}(n)$ other than T_1 is drawn in F.

Furthermore, every wall vertex of T_1 must be incident with F. To see this, let $v \in L_1 \cup R_1$, say $v \in R_1$. Since T is perfect, Condition (ii) guarantees that there is a path P in T_2 from v to R_2 which is disjoint from all the other wall vertices of L_2 $(= R_1)$ and hence disjoint from the boundary of F. Thus, P must cross the boundary of F, a contradiction.

Let $W = (v_0, e_1, v_1, e_2, \ldots, e_r, v_r)$ denote the boundary walk of F. We have the following three claims:

Claim 1. *There do not exist $a < b < c < d$ and $p \neq q$ such that $v_a = L_1[p]$, $v_b = L_1[q]$, $v_c = R_1[p]$ and $v_d = R_1[q]$.*

Claim 2. *There do not exist $a < b < c < d < e < f$ and $p < q < r$ such that $v_a = L_1[p]$, $v_b = L_1[r]$, $v_c = L_1[q]$ and $v_d = R_1[q]$, $v_e = R_1[r]$, $v_f = R_1[p]$.*

Claim 3. *Let $i_1 < i_2 < \ldots < i_s$ be the indices i such that v_i (in this position in W) is incident with an edge of $T^{\otimes}(n)$ not in T_1. There do not exist*

$a < b < c < d$ such that v_{i_a}, v_{i_c} ($v_{i_a} \neq v_{i_c}$) are in L_1 and v_{i_b}, v_{i_d} ($v_{i_b} \neq v_{i_d}$) are in R_1.

Proof of Claim 1. Suppose otherwise. By Condition (iii) of the definition of a perfect tile, there exist two disjoint paths in T_1, one joining $L_1[p]$ with $R_1[p]$ and the other joining $L_1[q]$ with $R_1[q]$. These paths are on the same side of W and must cross each other, a contradiction.

Proof of Claim 2. Suppose otherwise. Condition (iii) implies that there are disjoint paths P_q and P_r, where P_q joins $L_1[q]$ and $R_1[q]$, and P_r joins $L_1[r]$ and $R_1[r]$. By the connectivity (and Condition (i)), there is a path P' joining P_q and either $L_1[p]$ or $R_1[p]$. By the planarity of the tile, P' does not cross P_q and so is disjoint from P_r. Note that both P_r and P' are paths of T_1 and so must be drawn on the same side of W. As a consequence, P' must cross P_r which is a contradiction.

Proof of Claim 3. Suppose otherwise. By Lemma 4 there is, in the other tile adjacent to T_1, a path P joining v_{i_a} and v_{i_c} edge-disjoint from L_1 and another path Q in T_2 joining v_{i_b} and v_{i_d} edge-disjoint from L_2. Since P and Q are not in T_1, they must be drawn on the same side of W. Because they are disjoint, they cross each other, which contradicts the fact that T_2 has no crossing.

These claims imply that the vertices of L_1 and R_1 appear in W in the order we expect. The only small point left to observe is that some vertex might occur more than once in W and in more than one case be incident with edges not in T_1. However, all such occurrences of a vertex in $L_1 \cup R_1$ will be consecutive. Thus, we may adjust the drawing slightly so that all the edges incident with that vertex and not in T_1 are incident with the same occurrence of that vertex.

Let γ be a curve obtained by going around the perimeter of F and just going through a vertex in the boundary of F when that vertex is incident with edges of $T^{\otimes}(n)$ not in T_1. Now γ has $\mathcal{D}(T_1)$ on one side, giving a planar drawing of T_1, and the rest of $T^{\otimes}(n)$ on the other side. We delete the inner vertices and the inner edges (i.e., other than wall-vertices and wall-edges) of T_1 from $\mathcal{D}(T^{\otimes}(n))$ and obtain a tile drawing $\mathcal{D}'$ of $T^{\bowtie}(\ell, n-1-\ell)$. Note that all the crossings of $\mathcal{D}(T^{\otimes}(n))$ are in $\mathcal{D}'$, so $cr(\mathcal{D}') = cr(\mathcal{D}(T^{\otimes}(n)))$. □

Now we prove the main theorem in this section.

***Proof of Theorem* 2:** By Lemma 3, there is an integer N such that if $n \geq N$, then $tcr(T^{\bowtie}(n,n)) = tcr(T^{\bowtie}(N,N))$. Let $c = tcr(\tilde{T})$ and suppose $n \geq N' = \max\{2N, 6c+4\}$. Let $\mathcal{D}$ be an optimal drawing of $T^{\otimes}(n)$. By Lemma 1, $cr(\mathcal{D}) \leq tcr(T^{\bowtie}(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)) \leq tcr(T^{\bowtie}(N,N))$.

By Theorem 5, there is, for some positive integer ℓ, a tile drawing $\mathcal{D}'$ of $T^{\bowtie}(\ell, n-1-\ell)$, such that $tcr(\mathcal{D}') = cr(\mathcal{D})$. Letting $a = \max\{\ell, n-1-\ell\}$, we see that $tcr(T^{\bowtie}(N,N)) \leq tcr(T^{\bowtie}(a,a)) \leq tcr(T^{\bowtie}(\ell, n-1-\ell)) \leq cr(T^{\otimes}(n)) = cr(\mathcal{D}) \leq tcr(T^{\bowtie}(N,N))$. Therefore, we must have equality throughout, so if $n \geq N'$, then $cr(T^{\otimes}(n)) = tcr(T^{\bowtie}(N,N))$. $\square$

Similarly, if, for tile $T = (G, L, R)$, we let $T^{\leftrightarrow} = (G, R, L)$ and $T^{\ddagger} = (G, L^{\leftarrow}, R^{\leftarrow})$, we have the following results.

Theorem 6.
Let T be a planar tile. If $TT^{\leftrightarrow}$ and $TT^{\ddagger}$ are perfect and the appropriate compatibilities hold, then:

i) $\exists N, N' \ \forall n \geq N' : cr(\circ((TT^{\leftrightarrow})^n \widetilde{T})) = tcr((TT^{\leftrightarrow})^N \widetilde{T} (TT^{\leftrightarrow})^N)$*; and*

ii) $\exists N, N' \ \forall n \geq N' : cr(\circ((TT^{\ddagger})^n \widetilde{T})) = tcr((TT^{\leftrightarrow})^N \widetilde{T} (TT^{\ddagger})^N)$. $\square$

4 Computation of $cr(T^{\otimes}(n))$

In this section, we give a simple sufficient condition for $cr(T^{\otimes}(n)) = tcr(\widetilde{T})$.

Let $T = (G, L, R)$ be a tile. A set $\mathcal{P}$ of pairs of paths in $T\tilde{T}T = (G', L', R')$ is *pair-disjoint* if:

1. for $P \in \{P, Q\} \in \mathcal{P}$, there exist $i_P, j_P \in \{1, 2, \ldots, |L|\}$ and paths P_1, P_2, and P_3 in T such that: (i) P_1 joins $L[i_P]$ to $R[i_P]$, P_2 joins $L[i_P]$ to $R[n - j_P + 1]$, and P_3 joins $L[j_P]$ to $R[j_P]$; and (ii) $P = P_1 P_2 P_3$ joins $L'[i_P]$ to $R'[j_P]$;

2. for $\{P, Q\} \in \mathcal{P}$, P and Q are totally disjoint, and $i_P < i_Q$ if and only if $j_P > j_Q$; and

3. if $\{P_1, Q_1\}$ and $\{P_2, Q_2\}$ are distinct pairs in $\mathcal{P}$, then either $E(P_1) \cap E(P_2) = \emptyset$ or $E(Q_1) \cap E(Q_2) = \emptyset$.

For a tile T, $pd(T)$ denotes the maximum size of a pair-disjoint set.

Lemma 7. *Let T be a perfect planar tile. For all $m, n \geq 1$, $tcr(T^{\bowtie}(m,n)) \geq pd(T)$.*

Proof: Let $\mathcal{P}$ be a set of pair-disjoint paths in $T\tilde{T}T$. Let P be a path in a pair in $\mathcal{P}$, with $P = P_1P_2P_3$ as in (1) of the definition of pair-disjoint. Let P^* be the path in $T^m\tilde{T}T^n$ consisting of all the copies of P_1 in the first m copies of T, P_2 in $\tilde{T}$, and all the copies of P_3 in the last n copies of T. Then P^* joins the i_P^{th} vertex of the left wall of $T^{\bowtie}(m,n)$ to the j_P^{th} vertex of the right wall of $T^{\bowtie}(m,n)$. By (2), if $\{P,Q\} \in \mathcal{P}$, then P^* and Q^* must cross in any tile drawing of $T^{\bowtie}(m,n)$. By (3), no two of these crossings are the same. $\square$

A consequence is the following:

Theorem 8. *Let T be a perfect planar tile. If $pd(T) = tcr(\widetilde{T})$, then, for n sufficiently large, $cr(T^{\otimes}(n)) = tcr(\widetilde{T})$.*

Proof: Assume that $pd(T) = tcr(\widetilde{T})$. By Theorem 2 there are integers N_0, N such that if $n \geq N_0$, $tcr(T^{\bowtie}(N,N)) = cr(T^{\otimes}(n))$. By Lemma 7 we have $tcr(\widetilde{T}) = pd(T) \leq tcr(T^{\bowtie}(N,N))$. By Lemma 1, $tcr(\widetilde{T}) \geq tcr(T^{\bowtie}(N,N))$. Hence $tcr(\widetilde{T}) = tcr(T^{\bowtie}(N,N)) = cr(T^{\otimes}(n))$. $\square$

Similarly, we compute $pd(TT^*)$, where T^* is either $T^{\leftrightarrow}$ or $T^{\updownarrow}$, only then in this context by considering paths in $TT^*\tilde{T}TT^*$, rather than $T\tilde{T}T$.

Theorem 9. *Let T be a planar tile and let n be sufficiently large.*

i) If $TT^{\leftrightarrow}$ is perfect and $pd(TT^{\leftrightarrow}) = tcr(\widetilde{T})$, then

$$cr(\circ((TT^{\leftrightarrow})^n\widetilde{T})) = tcr(\widetilde{T}).$$

ii) If $TT^{\updownarrow}$ is perfect and $pd(TT^{\updownarrow}) = tcr(\widetilde{T})$, then

$$cr(\circ((TT^{\updownarrow})^n\widetilde{T})) = tcr(\widetilde{T}). \quad \square$$

The preceding theorems can be used to show that if B, K, and M are the tiles shown in Figure 1, then, eventually we have $cr(B^{\otimes}(n)) = 3$, $cr(\circ((KK^{\updownarrow})^n\tilde{K})) = 2$, and $cr(\circ((MM^{\leftrightarrow})^n\tilde{M})) = 6$. The first two can easily be proved without the theory we have developed here, while the last has only been proved using a computer to verify the case $n = 2$ [1] and then a very long induction [4]. Our theory yields the first and third immediately (for

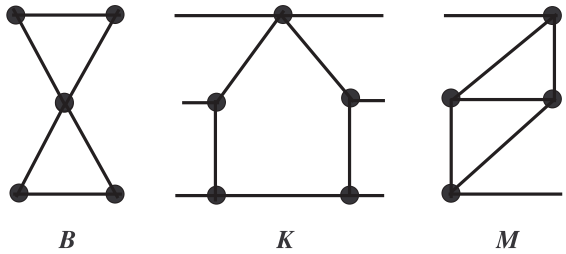


Figure 1: Tiles B, K, and M

n sufficiently large); some small additional argument is required for K, as $pd(K)$ is only 1.

As an aside, the tile M was suggested as a candidate for generating an infinite family of 5-regular 6-crossing-critical graphs. The theory we developed here shows two things: the graphs $\circ((MM^{\leftrightarrow})^n\tilde{M})$ eventually have crossing number 6 but (because not all edges are used in a maximum pair-disjoint set) are not 6-crossing-critical. So we still do not know of an infinite family of 5-regular k-crossing-critical simple graphs.

We conclude with the following.

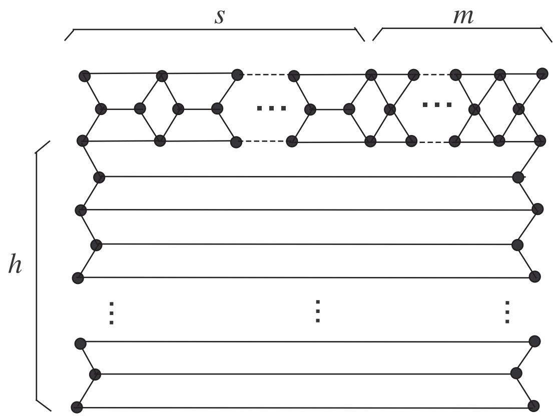
Theorem 10. *Let h be a positive integer and let r be rational in the interval $(3.5,4)$. There is an infinite family of $\binom{2h+3}{2}$-crossing-critical simple graphs with average degree r.*

Proof: Let $T_{(h,s,m)} = (G, L, R)$ be the tile in Figure 2 (ii) consisting of the tile S^sB^m, $s + m \geq 1$, with S and B from Figure 2 (i), and $h \geq 0$ copies of a "long" version of S, stacked on each other with the top one (with the appropriate subdividing of its top edge) glued under S^sB^m.

For $1 \leq i \leq h+2$, let Q_i^1 be the horizontal path joining $L[i]$ and $R[i]$. For $1 \leq i \leq h+1$, let Q_i^2 be the path joining $L[i]$ and $R[i]$ disjoint from Q_{i+1}^1 and edge-disjoint from Q_i^1. Similarly, for $1 \leq i \leq h+1$, let Q_i^3 be the path joining $L[i+1]$ and $R[i+1]$ disjoint from Q_i^1 and edge-disjoint from Q_{i+1}^1.

Let P_i^1 be the path in $T_{(h,s,m)}\widetilde{T}_{(h,s,m)}T_{(h,s,m)}$ consisting of Q_i^1 from the first $T_{(h,s,m)}$, Q_i^1 in the twist, and Q_{h+3-i}^1 from the second $T_{(h,s,m)}$. Let P_i^2 be the path in $T_{(h,s,m)}\widetilde{T}_{(h,s,m)}T_{(h,s,m)}$ consisting of Q_i^2 from the first $T_{(h,s,m)}$, Q_i^2 in the twist, and Q_{h+2-i}^3 from the second $T_{(h,s,m)}$. Finally, let P_i^3 be the path in $T_{(h,s,m)}\widetilde{T}_{(h,s,m)}T_{(h,s,m)}$ consisting of Q_i^3 from the first $T_{(h,s,m)}$, Q_i^3 in the twist, and Q_{h+2-i}^2 from the second $T_{(h,s,m)}$.

(i): Tiles S and B.



(ii): Tile $T_{(h,s,m)}$.

Figure 2: Tiles S, B, and $T_{(h,s,m)}$.

Then the following set is pair-disjoint:

$$\mathcal{P}_h = \left(\bigcup_{i=1}^{h+1} \bigcup_{j=i}^{h+1} \{\{P_i^1, P_{j+1}^1\}, \{P_i^1, P_j^3\}\} \right) \cup$$

$$\left(\bigcup_{i=1}^{h} \bigcup_{j=i+1}^{h+1} \{\{P_i^2, P_j^1\}, \{P_i^2, P_j^2\}\} \right) \cup \left(\bigcup_{i=1}^{h+1} \{\{P_i^2, P_{h+2}^1\}\} \right) .$$

We note that $|\mathcal{P}_h| = \binom{2h+3}{2}$. Therefore, $pd(T_{(h,s,m)}) \geq \binom{2h+3}{2}$. By drawing $\widetilde{T}_{(h,s,m)}$ such that the crossings only occur among the horizontal paths, we have $tcr(\widetilde{T}_{(h,s,m)}) \leq \binom{2h+3}{2}$. Hence, by Lemma 7, $pd(T_{(h,s,m)}) = tcr(\widetilde{T}_{(h,s,m)}) = \binom{2h+3}{2}$. Therefore, for large enough n, $cr(T^{\otimes}_{(h,s,m)}(n)) = \binom{2h+3}{2}$, by Theorem 8.

The proof that $G = T^{\otimes}_{(h,s,m)}(n)$ is crossing-critical is easy, although a little tedious. For each edge e of G, the optimal drawing can be adjusted slightly to provide a drawing of $G - e$ with fewer than $cr(G)$ crossings. There are a number of cases to consider, but the analysis is helped by understanding which pair of paths in $\mathcal{P}$ no longer need to cross.

In $T^{\otimes}_{(h,s,m)}(n)$, there are $(2s+2h)n$ vertices of degree 3, $(2(s+m)+m)n$ vertices of degree 4 and hn vertices of degree 6. Therefore, the average degree of $T^{\otimes}_{(h,s,m)}(n)$ is

$$\frac{(3(2s+2h)+4(2(s+m)+m)+6h)n}{(2s+2h+2(s+m)+m+h)n} = \frac{14s+12h+12m}{4s+3h+3m},$$

which, it is easy to see, is between 3.5 and 4, inclusive.

We would like to have, for each rational between 3.5 and 4 (exclusive), and for each positive integer h, infinitely many pairs (s,m) of positive integers so that $r = (14s+12h+12m)/(4s+3h+3m)$, which is equivalent to $s = (12-3r)(m+h)/(4r-14)$. For each h, infinitely many choices of m make $(12-3r)(m+h)/(4r-14)$ a positive integer, as required. $\square$

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