MATH 650 : Mathematical Modeling

Spring, 2019 - Written Assignment #3

Due by 11:59 p.m. EST on Tuesday, July 30th, 2019

Instructions:

- The problems on this assignment involve concepts, solution methods, and applications of nonhomogeneous linear second order differential equations and systems, and nonlinear systems in the plane, as introduced in Module 7, Sections 3, 4, and 5, and Module 8. Thus it is a *comprehensive* and *summative* assessment of your understanding of this material.
- Feel free to discuss the problems with one another, and to make use of any other resources which help you to work your way through them. However, your submissions must be your own, and must reference any source (human or otherwise) from whom you received help.
- Presentation is important. Please write your solutions in clear sentences which convey your reasoning. (See the Sample Solutions for a handy guide.) Remember that I can't know what you're thinking...I can only know what you tell me. Handwritten solutions are just fine, but they must be well-organized and legible, with the problems in numerical order. (I will send the LaTeX file if requested.)
- A few of the problems use specific Maple files which will be made available. (I'm assuming you can do simple Maple commands such as *plot* and *implicitplot* on your own.)
- **I suggest you spend no more than 16 hours in total on this assignment, including writing up your solutions.** While you might not complete all the problems you'd like to, give it your best try...but don't worry about it beyond that time span. Partially completed problems are acceptable.
- If you wish to do one (but ONLY ONE) of the 'Choice' problems 2c., 5iii., and 6, feel free. It will be marked as a 'bonus' problem.
- In deciding which of problems 10 or 11 you wish to do, please note the following: Problem 10 involves a fair bit of work analyzing slight generalizations of familiar models from the lectures. On the other hand, Problem 11 is a biochemical network that will be new to you (although not new concepts), and is a much more open-ended problem for those of you who enjoy the novelty and a wee challenge.
- All assignments will be submitted electronically in PDF format using the assignment drop box in LEARN.

Available Marks for Each Problem

Compulsory Problems	Marks	Choice Problems	Marks
1	20		
2 a.,b.	13	2 c.	7
3	11		
4	14		
5 i.,ii.	6	5 iii.	6
		6	6
7	14		
8	12		
9	17		
10 or 11	19		
Presentation	9		

	135		

Part I: Applications of Nonhomogeneous Second Order DEs, and Variation of Parameters

Do Problems 1, 2 a., b., and 3 - 5ii; Problems 2 c., 5iii., and 6 are Optional

Undetermined Coefficients

1. Review the examples of Section 3 in Module 7. Then apply the method of undetermined coefficients as needed in each problem.

a. Find the general solution of the DE $y'' + y' = 2 + 3\cos(2t)$.

b. Solve the IVP $y'' + 2y' - 3y = 6\,t\,e^{2t},\ y(0) = 1,\ y'(0) = 0$.

Variation of Parameters

2. Apply the method of variation of parameters for second order DEs in part a. (slides 2-4 of Section 5.1 of Module 7), and for a system of two DEs in part b. (slides 4-6 of Section 5.2 of Module 7).

a. Find the general solution of the DE $y'' + 4y' + 4y = \dfrac{\ln t}{e^{2t}}$ for all real t.

b. Apply the method of variation of parameters to find the general solution of the system $\mathbf{x}' = \mathbf{A}\,\mathbf{x} + \mathbf{g}(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} \sec t \\ 0 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, **given** that $\mathbf{\Phi}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ is the fundamental matrix .

c. The method of variation of parameters can also be applied to linear systems where the matrix $\mathbf{A}$ varies, i.e., systems of the form $\mathbf{x}' = \mathbf{P}(t)\,\mathbf{x} + \mathbf{g}(t)$. **BUT** in this case, while Properties 1 and 2 of the fundamental matrix still hold, Properties 3 and 4 do not, i.e., $\mathbf{\Phi}(t_1 + t_2) \neq \mathbf{\Phi}(t_1)\mathbf{\Phi}(t_2)$, and hence $\mathbf{\Phi}^{-1}(t) \neq \mathbf{\Phi}(-t)$. Thus you must find the inverse matrix $\mathbf{\Phi}^{-1}(t)$ in order to calculate $\mathbf{u}(t) = \int_0^t \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds$.

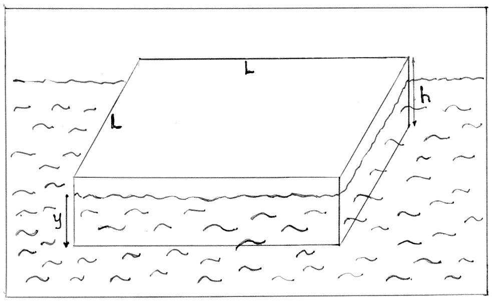
Verify that $\mathbf{\Phi}(t) = \begin{pmatrix} \dfrac{1}{1-t} & \dfrac{te^{-t}}{1-t} \\ 0 & e^{-t} \end{pmatrix}$ is the required fundamental matrix for $\mathbf{x}' = \mathbf{P}(t)\,\mathbf{x}$, and then find a **particular** solution for the system

$$\mathbf{x}' = \mathbf{P}(t)\,\mathbf{x} + \mathbf{g}(t) = \begin{pmatrix} \dfrac{1}{1-t} & 1 \\ 0 & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} (1+t)^2 \\ 1+t \end{pmatrix}.$$

Undamped Vibrations with Constant Forcing

3. Review Archimedes Law of Buoyancy on slide 31 of Example 2.2.3 in Module 2, and slides 3-4 of Section 2.2 in Module 7 on Simple Harmonic Motion.

a. A square wooden raft of side L, height h, and density ρ_{wd}, floats when only half-submerged in water. Apply Archimedes Law to explain why this implies that the density ρ_w of the water is twice that of the wood.



b. When the block is pushed downward briefly and released, both gravity and buoyancy act vertically on it. Assuming these are the only forces, start with Newton's Law of Motion $m\,y'' = G - B$, where y is the vertical displacement (as shown), G is the force of gravity, and B is the buoyant force, and show that the vertical motion of the block satisfies the DE

$$y'' + \omega_0^2\, y = g\,, \quad \text{where } \omega_0^2 = \frac{2g}{h}\,.$$

c. We know that the DE in part b. has $y_h(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$. Find the full solution $y(t)$ with initial conditions $y(0) = h$, and $y'(0) = 0$, and hence show that the block undergoes periodic oscillations (SHM) about its equilibrium. State the amplitude and period of the motion (in terms of g and h), and show that the raft never leaves the water completely (i.e., never becomes airborne).

d. Discuss briefly why this model is not realistic (i.e., what physical aspects of the problem have been ignored), and suggest how the model could be modified to remedy this.

The Pendulum with Periodic Forcing

4. Consider a simple pendulum with mass m on a light rigid rod of length l. The pivot from which it hangs oscillates on a horizontal track with periodic displacement $X(t) = X_0 \cos\omega t$. This problem explores the horizontal displacement $x(t)$ of the pendulum.

The diagram reveals the basic assumptions of the model. For small θ (say $|\theta| \leq 0.1^r \approx 6^\circ$), we have:

- the circular displacement of the pendulum $s = l\theta \approx l \sin\theta = y$, the horizontal displacement from $\theta = 0$, since $\theta \approx \sin\theta$ for small θ;
- balancing the vertical forces implies that $mg = T\cos\theta \approx T$, since $\cos\theta \approx 1$;
- thus $T\sin\theta \approx mg\sin\theta$ is the horizontal force acting on the pendulum.

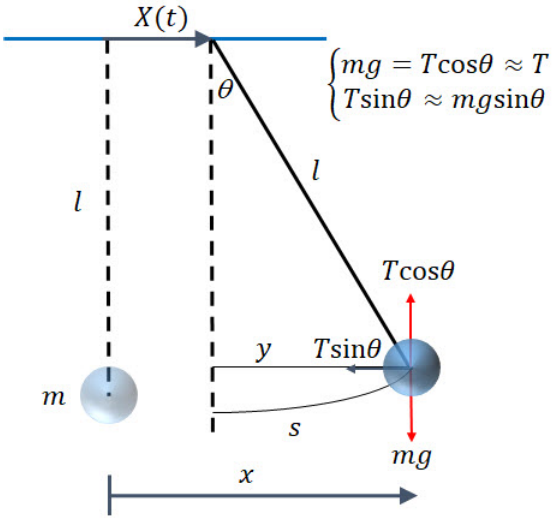


Figure 4.5.3: Forced pendululm

With $x(t) := y(t) + X(t)$ as the total horizontal displacement (as shown), Newtons Law of Motion implies

$$mx'' = -\gamma x' - mg\sin\theta$$

where γ is a positive constant, the damping coefficient.

a. Assuming small angle oscillations of the pendulum, and noting that $\sin\theta = \frac{y}{l}$, show that this DE becomes

$$x'' + 2\omega_0\zeta\, x' + \omega_0^2\, x = f_0 \cos(\omega t)\,,$$

where $\omega_0^2 := \frac{g}{l}$, $\zeta := \frac{\gamma}{2m\omega_0}$, and $f_0 := \omega_0^2 X_0$.

b. Prove that for critical damping, the DE becomes

$$x'' + 2\omega_0\, x' + \omega_0^2\, x = f_0 \cos(\omega t)\,,$$

c. For arbitrary ω_0 and f_0, find the particular solution $x_p(t)$ of the DE in **b.**

d. Show that the steady-state amplitude of $x_p(t)$ is $\mathcal{A}_{ss} = \dfrac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\,\omega_0^2\omega^2}}$.

e. Show that amplitude resonance cannot occur for any $\omega > 0$, and explain why this is desireable in this situation.

[Hint: The denominator of $\mathcal{A}_{ss}$ simplifies greatly!]

Amplitude Resonance and Phase Angle

5. Review Module 7, slides 1-4 of Section 4.3: Further Analysis and Examples (especially the animations on slide 4), where you will find a discussion of amplitude resonance and phase angle for the steady-state solution of the model

$$m\,y'' + \gamma\, y' + k\, y = F\,\cos(\omega t)\,.$$

The *steady-state amplitude* of the solution is $\mathcal{A}_{ss} = y_c \dfrac{1}{\sqrt{(1-r^2)^2 + 4\,\zeta^2\,r^2}} = y_c\,M\,,$ where $r = \dfrac{\omega}{\omega_0}$ and $\zeta = \dfrac{\gamma}{2\sqrt{m\,k}}$ are dimensionless constants, $\omega_0 = \sqrt{\dfrac{k}{m}}$ is the *natural frequency*, $y_c = \dfrac{F}{k}$ the *static deflection*, and M is the *magnification factor.*

i. Complete the questions a), b), and c) regarding the behaviour of the magnification factor M in Exercise 7.4.2 on slide 3 of Section 4.3 of Module 7. (Use the first derivative test to justify that the critical point is a maximum in b).) Then restate each of the properties in terms of dimensionful variables by using the definitions of r and ζ above.

ii. Explain in your own words why amplitude resonance does not occur if ζ is 'large'. HINT: Think about how ζ is related to the damping coefficient γ.

iii. In Example 7.4.1 of Module 7, we looked at the model

$$m\,y'' + \gamma\,y' + k\,y = F\cos(\omega t)$$

for $m = 4\,\text{kg}$, $\gamma = 0.5\,\text{kg/s}$, $k = 4\,\text{N/m}$, and $F = 12\,\text{N}$. The two sets of plots on slide 4 show solutions $y(t)$ plotted versus the scaled forcing function $\frac{F}{k}\cos(\omega t)$ (or $y_c\cos(\omega t)$) for $\omega = 1$, and for $\omega = 2$. Find the values of the parameters ζ and y_c for this problem. Then, considering what you now know about how the steady-state amplitude $\mathcal{A}_{ss}$ and the phase angle ϕ depend on $r = \frac{\omega}{\omega_0}$, explain the differences in both amplitude and phase angle (of the steady state) between the animation on the left for $\omega = 1$, and the one on the right for $\omega = 2$ on slide 4 of Example 7.4.1.

Dimensional Analysis of the Bouncy Ride

6. In the model of Example 7.4.2, the Bouncy Ride, there are five dimensional parameters, m, γ, k, u_0, and ω in the dimensional model, and **three** dimensionless parameters ζ, r, and ϵ in the dimensionless model.

a. Form the dimensional matrix for the five physical quantities, and show that the Pi Theorem predicts the existence of only **two** dimensionless variables, an apparent contradiction.

b. Using the definitions of $\cos\epsilon$ and $\sin\epsilon$ given on slide 2 of Example 7.4.2, show that, in fact, you can rewrite these definitions completely in terms of r and ζ, thus resolving the apparent contradiction.

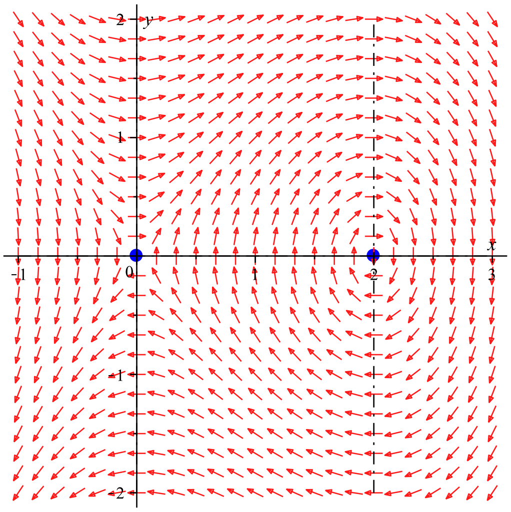
[HINT: The calculation simplifies if you let $w_0^2 = \frac{k}{m}$ in the original definitions.]

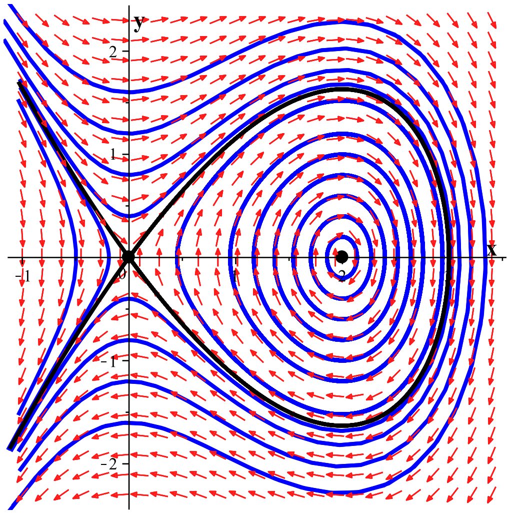
Part II: Nonlinear Systems in the Plane

Do Problems 7, 8, 9, and ONE of Problems 10 or 11

Phase Portrait Sketches

7. Recall the 'Fish' of Portrait 3 in Module 8, Section 1, given by the system $x' = y\,,\ y' = x(2-x)\,$, with direction field and phase portrait as shown below.

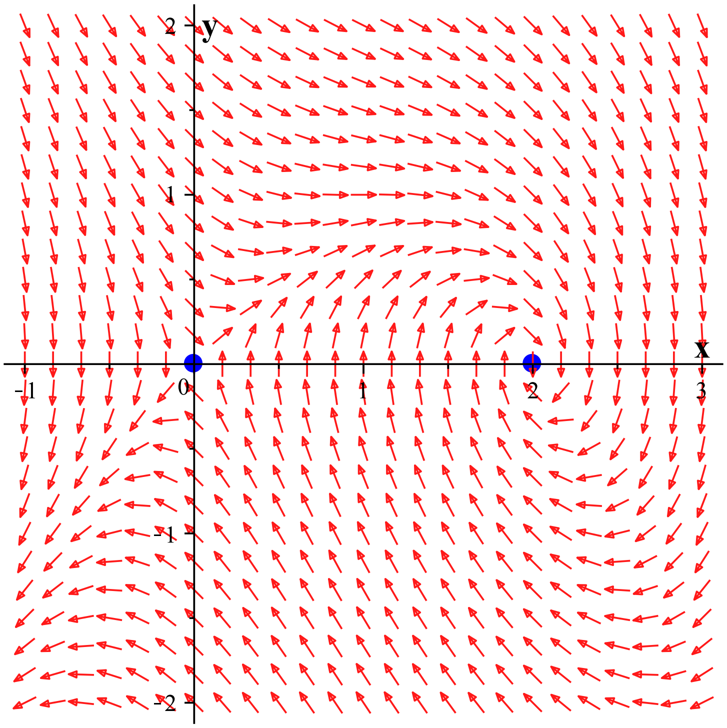




Now suppose we add a 'damping' term to the system, giving

$$x' = F(x,y) = y$$
$$y' = G(x,y) = x(2-x) - y\,.$$

a. Below is a plot of the direction field of the damped system. Find the nullclines and add them carefully to the sketch as dotted lines. Sketch typical solutions and make a conjecture about the type and stability of the equilibrium points.



b. Use Maple to find the Jacobian of F and G at each equilibrium point. Find the eigenvalues at each point and hence confirm your conjecture from part a.

c. There are two special orbits which approach the equilibrium at $\mathbf{0}$, becoming tangent to one of the eigenvectors of the linearized system as they near $\mathbf{0}$. Using the direction field in part a., and the desired eigenvector, sketch these two orbits carefully in black.

[HINT: These two '*separatrix*' orbits divide the plane into points through which orbits approach $(2, 0)$, and points for which they do not.]

d. Describe the *basin of attraction* of the equilibrium at $(2, 0)$ in relation to the two orbits described in part c.

van der Pol Revisited

8. Reconsider the van der Pol system

$$\begin{aligned} x' &= F(x, y) = y \\ y' &= G(x, y) = -x + \mu(1 - x^2)y\,. \end{aligned}$$

In Module 8, we considered the behaviour of this system for $\mu \geq 0$; now let's investigate further.

a. Show that this system has a single, isolated equilibrium point at $(0, 0)$.

b. If the term in x^2 is ignored, we obtain the linearized system near $\mathbf{0}$, namely

$$\begin{aligned} x' &= y \\ y' &= -x + \mu y\,. \end{aligned}$$

Find the eigenvalues of this linear system and, **with no further analysis**, state the type and stability of the equilibrium at $\mathbf{0}$ for each of the cases:

i. $\mu \leq -2$ **ii.** $-2 < \mu < 0$ **iii.** $\mu = 0$ **iv.** $0 < \mu < 2$ **v.** $\mu \geq 2$.

c. According to the Theorem on slide 4 of Lecture 3 A i, for which of these cases does the linearized system **not** reliably predict the behaviour of the nonlinear system? What is the actual behaviour of the nonlinear system in this case?

d. Examine the animation of plots for the van der Pol system for various values of μ. (See the Maple file Math650WA3StudentPlots which accompanies WA3.)

- Observe that for $\mu < 0$, there is a periodic solution. Is this solution stable or unstable? How does this relate to the stability of the equilibrium at the origin?
- How does the shape of this periodic orbit change as μ nears 0?
- What happens to the shape and stability of the periodic orbit as μ passes through 0 and becomes positive?

The parameter value $\mu = 0$ is thus critical in determining the behaviour of this system.

Level Sets and Lyapunov

9. In this problem, we explore an implicit solution and its orbits, then look at a failure of linearization which is resolved by the method of Lyapunov.

a. Consider the system $x' = -x + y + x^2$, $y' = y - 2xy$.

i. Show that the orbits of this system lie within the level sets of the function $\mathcal{F}(x, y) = x^2y - xy + \dfrac{y^2}{2}$. (Review Example 8.2.1 in the lectures if necessary.)

ii. Use Maple's *implicitplot* command to plot the level sets $\mathcal{F}(x, y) = C$ for $C = 0, \pm 0.01, \pm 0.02, \pm 0.03$, with $gridrefine = 3$, and print your plot. [Hint: These fit nicely in a window $x = -0.5..1.5$, $y = -0.5..1.5$.]

iii. Using the direction field information in the DEs for the given system, show the direction of increasing t on each curve in your plot, and indicate the position of each equilibrium point.

iv. Just by observing your plot (no analysis required), state the type, and stability of each equilibrium point of this system. At which equilibrium point does $\mathcal{F}(x, y)$ have an extreme value? How do you know?

b. The system $x' = -y - x(x^2 + y^2)$, $y' = x - y(x^2 + y^2)$ has a single isolated equilibrium point at $(0, 0)$.

i. Ignoring the nonlinear terms in the DE reveals that the linearization of this system near $(0, 0)$ is just $x' = -y$, $y' = x$. Find the type and stability of $(0, 0)$ for this linear approximation.

ii. Show that the function $V(x, y) = x^2 + y^2$ satisfies the conditions for a Lyapunov function on any deleted neighbourhood of $(0, 0)$.

(Review Lyapunov's Theorem on slide 3 of Section 4 of Module 8.)

iii. Find $\dfrac{dV}{dt}$ along the orbits of the given nonlinear system, and hence show that $(0, 0)$ is an asyptotically stable equilibrium.

iv. Using the command for Problem 9b in the Maple file Math650WA3StudentPlots, plot the phase portrait and Lyapunov sets to illustrate your results. Label the resulting diagram appropriately.

This illustrates a failure of the linearization to predict the equilibrium stability of the nonlinear system.

Two-Species Interactions

10. Before beginning this problem, review Examples 8.2.1 and 8.3.1 in your lectures, which introduce models for predator-prey and competition interactions.

a. Recall the Lotka-Volterra equations for the interaction between a prey population $N_1(t)$ and a predator population $N_2(t)$, namely

$$\frac{dN_1}{dt} = r_1 N_1 - aN_1N_2\,, \quad \frac{dN_2}{dt} = -r_2N_2 + bN_1N_2\,.$$

Suppose that N_1 is a species of aphid devouring your roses, and N_2 is a species of ladybugs which loves to munch on these aphids. Disgusted, you spray insecticide on the roses, which kills a fraction of each of these insects per unit time. A suitably revised model would thus be

$$\frac{dN_1}{dt} = r_1 N_1 - aN_1N_2 - f_1N_1\,, \quad \frac{dN_2}{dt} = -r_2N_2 + bN_1N_2 - f_2N_2\,,$$

where f_1 and f_2 are positive fractions.

i. Show that this is just a slightly different version of the Lotka-Volterra equations, and find the new equilibrium point.

ii. Does this practice help eliminate the aphids, or just make matters worse?

b. Recall the competition model discussed in the course lectures, where $N_1(t)$ and $N_2(t)$ are two similar species with underlying logistic growth, but in competition for the same resources, namely

$$\frac{dN_1}{dt} = r_1 N_1\left(1 - \frac{N_1}{K_1}\right) - aN_1N_2\,, \quad \frac{dN_2}{dt} = r_2N_2\left(1 - \frac{N_2}{K_2}\right) - bN_1N_2\,.$$

The equivalent dimensionless model, with variables $x = \dfrac{N_1}{K_1}$, $y = \dfrac{N_2}{K_2}$, and $\tau = r_1 t$, is

$$\frac{dx}{d\tau} = x(1 - x - Ay)\,, \quad \frac{dy}{d\tau} = Ry(1 - y - Bx)\,,$$

where $A = \dfrac{aK_2}{r_1}$, $B = \dfrac{bK_1}{r_2}$, and $R = \dfrac{r_2}{r_1}$ are dimensionless constants.

In Example 8.3.1 we discussed the case of 'weak competition', namely $A < 1$, $B < 1$, for which the competition effects are weaker than the intrinsic growth rates for each species.

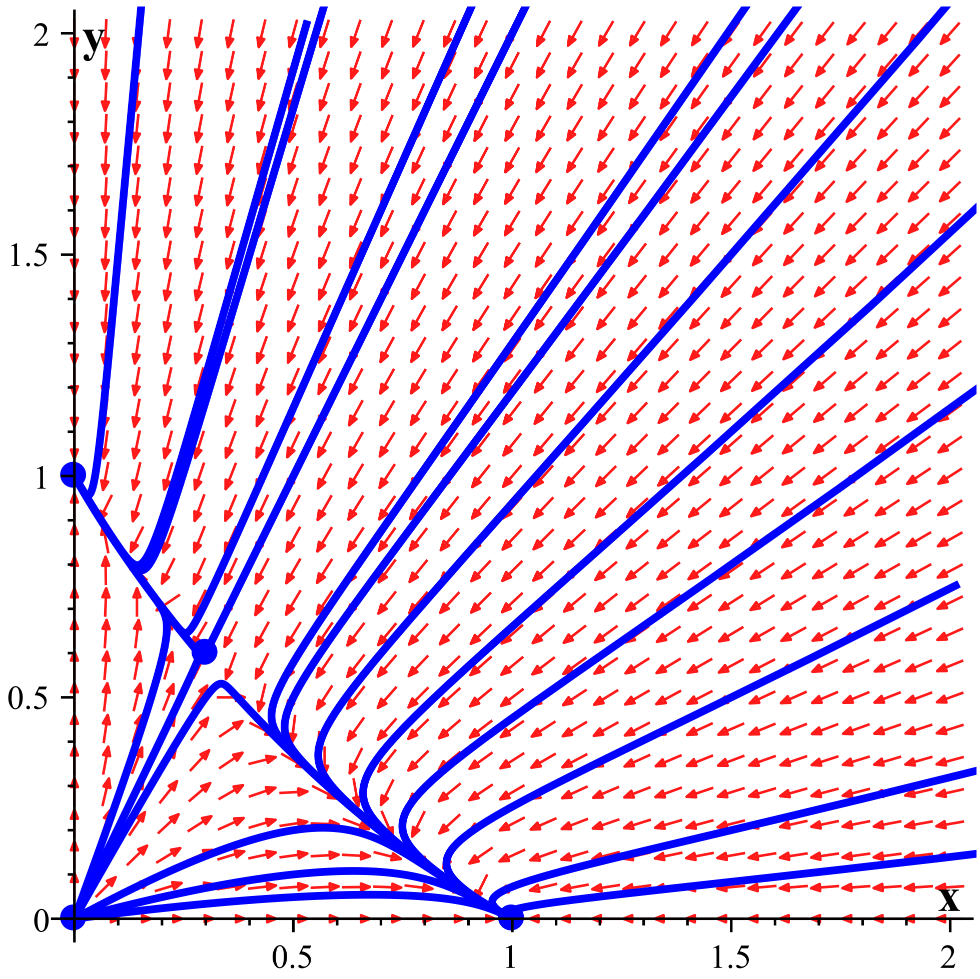
Below are three phase portraits which portray other possibilities. In each case:

1. Sketch the non-axial isoclines on the plot, and find the equilibrium points.
2. Examine the orbits, and explain the outcome for the competing species.

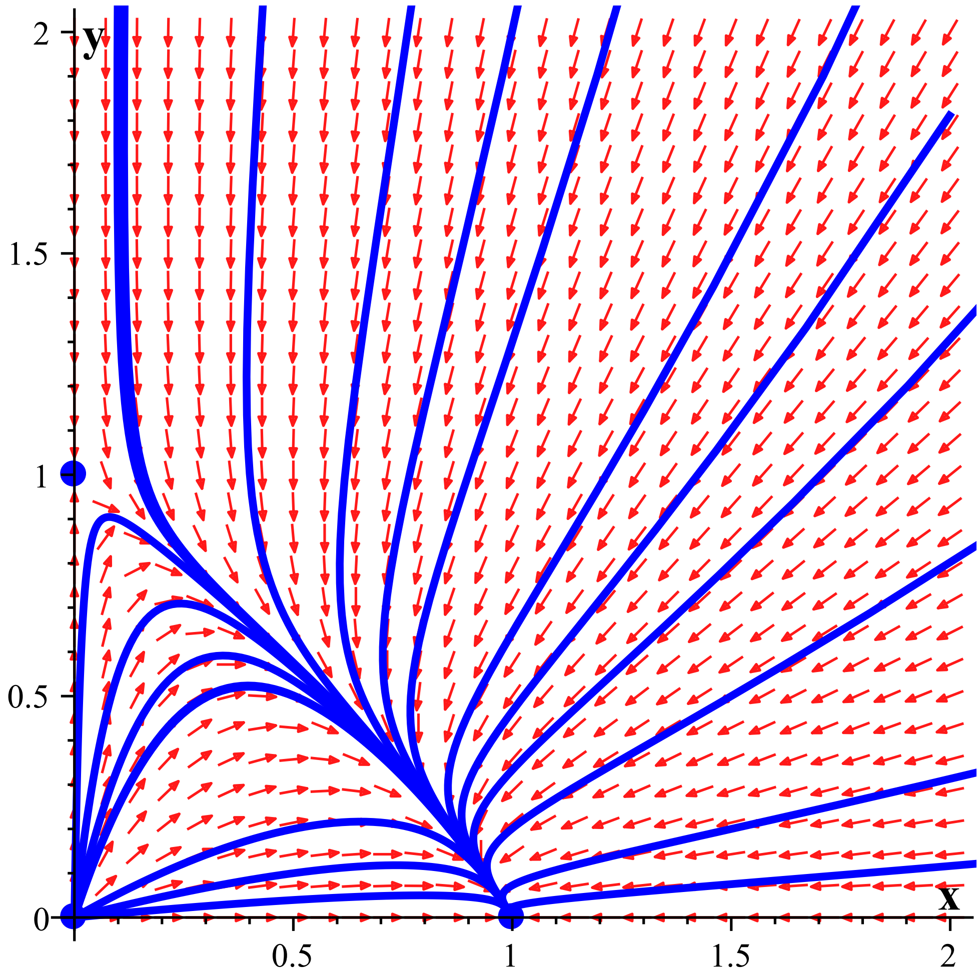
3. Justify what's happening with reference to the size of the dimensionless parameters A and B which govern the system. Pay particular attention to how the maximal competition effects ($a\,K_2$ and $b\,K_1$) relate to the intrinsic growth rates (r_1 and r_2) for each interaction. (Review slide 3 of Example 8.3.1 to see a sample of such reasoning.)

NOTE: $R = 1$ is reasonable because competing species are generally similar.

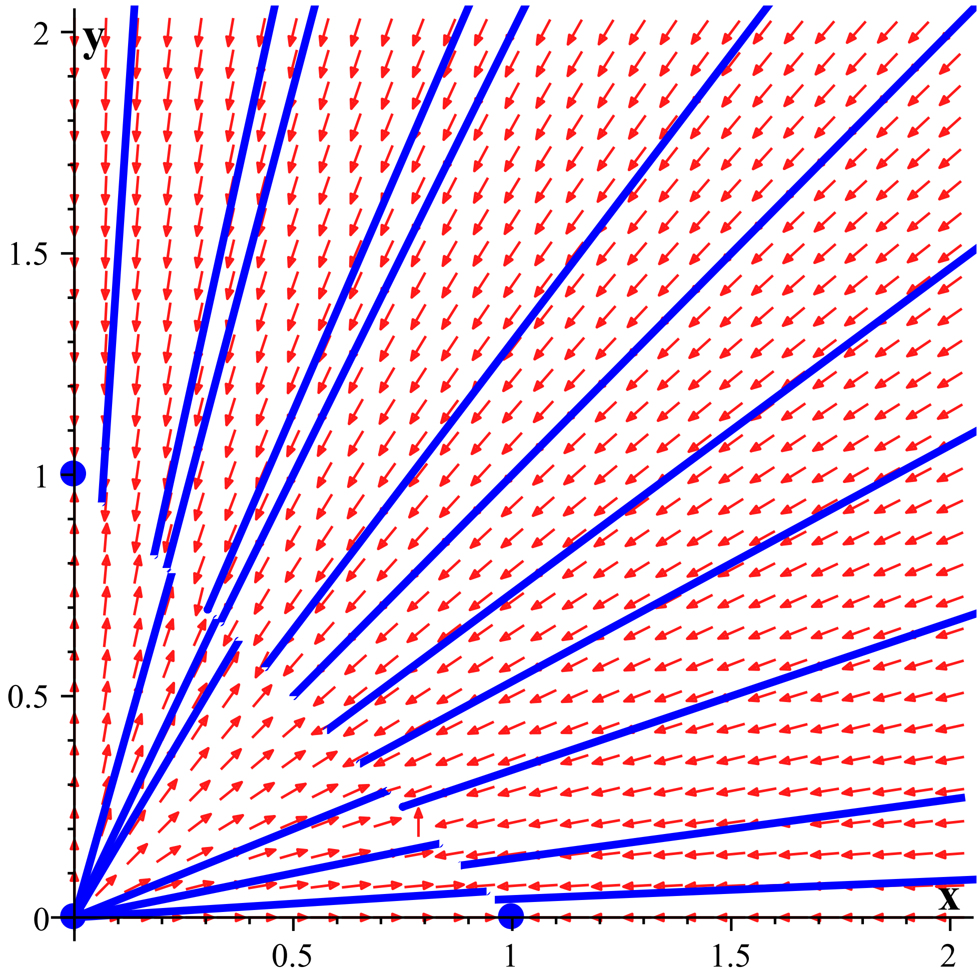
i. For this portrait, $A = 1.2$, $B = 1.4$, $R = 1$.



ii. For this portrait, $A = 0.5$, $B = 1.2$, $R = 1$.



iii. For this portrait, $A = 1$, $B = 1$, $R = 1$.



11. In a biochemical interaction involving two species X and Y, production is governed by the system

$$\frac{dx}{dt} = \frac{20}{1+y^{n_1}} - 5\,x\,, \qquad \frac{dy}{dt} = \frac{20}{1+x^{n_2}} - 5\,y\,,$$

where $x(t)$, $y(t)$ are the concentrations of species X and Y respectively, and n_1, n_2 are positive constants.

a. If species Y is absent, what is the longterm behaviour of species X? Similarly, what is the longterm behaviour of Y if species X is absent? Justify your answers.

b. Explain how the presence of species Y affects the growth rate $\frac{dx}{dt}$ of species X. Similarly, how does the presence of X affect $\frac{dy}{dt}$? Why do you think such a system is called *cooperative inhibition*?

c. Use the Maple routine given for Problem 11 for the following analysis.

i. Show that, for $n_1 = 4\,, n_2 = 1\,$, there is a single equilibrium point $x_e \approx 0.01662$, $y_e \approx 3.93460\,$. Plot the phase portrait, and explore the linearization about the equilibrium. Does it reliably predict the behaviour of the nonlinear system? Give a physical interpretation of the longterm behaviour of the system, and explain why this would be expected for $n_1 > n_2\,$.

ii. Repeat the steps of part **c i.** for $n_1 = 4 = n_2\,$. (You should find three equilibrium points in this case.) Give a detailed description of how the longterm behaviour of the system depends on the initial conditions. What is the *basin of attraction* of the equilibrium point that is near $(4, 0)$? Why do you think this system is called *bistable*?

[Reference for Problem 11: *Mathematical Modeling in Systems Biology*, by Brian P. Ingalls, MIT Press, 2013, pages 95-97.]