

## MATH 650 : Mathematical Modeling

## Spring, 2019 - Written Assignment #2

Due by 11:59 p.m. EST on Tuesday, July 9th, 2019

**Instructions:**

- The problems on this assignment involve concepts, solution methods, and applications of dimensional analysis, systems of two first order differential equations, and homogeneous second order differential equations, as introduced in Modules 5, 6, and up to the end of Section 2 of Module 7. Thus it is a *comprehensive* and *summative* assessment of your understanding of this material.
- Feel free to discuss the problems with one another, and to make use of any other resources which help you to work your way through them. However, your submissions must be your own, and must reference any source (human or otherwise) from whom you received help.
- Presentation is important. Please write your solutions in clear sentences which convey your reasoning. (See the Sample Solutions for a handy guide.) Remember that I can't know what you're thinking...I can only know what you tell me. Handwritten solutions are just fine, but they must be well-organized and legible, with the problems in numerical order. (I will send the LaTeX file if requested.)
- A few of the problems use specific Maple files which will be made available. (I'm assuming you can do simple Maple commands such as *plot* on your own.)
- **I suggest you spend no more than 15 hours in total on this assignment, including writing up your solutions.** While you might not complete all the problems you'd like to, give it your best try...but don't worry about it beyond that time span. Partially completed problems are acceptable.
- There are 17 problems on this assignment, of which you are required to do 11 for a total of 126 marks. While in no way wishing to discourage you from trying as many other problems as you find interesting, in order to limit the size of the submissions, I would ask that you **submit no more than 2 'extra' problems** for marking. (Though I will be happy to discuss any of the problems, any time.)
- All assignments will be submitted electronically in PDF format using the assignment drop box in LEARN.

## Available Marks for Each Problem

Compulsory Problems	Marks	Choice Problems	Marks
1	8		
2	12		
1 of 3 or 4	9	1 of 3 or 4	9
1 of 5 or 6	9	1 of 5 or 6	9
1 of 7 i. or 7 ii.	13	1 of 7 i. or 7 ii.	13
1 of 8 or 9	13	1 of 8 or 9	13
1 of 10 or 11	13	1 of 10 or 11	13
12	18		
13	11		
14	9		
1 of 15 or 17	11	1 of 15 or 17	11
		16	8
Presentation	9		
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	135		

## Part I: Dimensional Analysis

Do **both** problems 1 and 2, plus **one** of 3 or 4, and **one** of 5 or 6

### Mixing Tank

1. Recall the general model of Example 2.2.1 for a substance  $Q$  mixed homogeneously in a tank containing fluid of volume  $V$ , with input concentration  $c_i$ , and input and output flow rates  $r_i$  and  $r_o$ , respectively, namely

$$\frac{dQ}{dt} = r_i c_i - r_o \frac{Q}{V}, \quad Q(0) = Q_0.$$

Consider the simplest case, with constant input concentration  $c_i$ , and constant flow rate  $r_i = r_o = r$  (hence  $V$  is also constant), giving the IVP

$$\frac{dQ}{dt} = r c_i - r \frac{Q}{V}, \quad Q(0) = Q_0.$$

- a. Show that a suitable choice of characteristic time is  $t_c = \frac{V}{r}$ , and give a physical interpretation of  $t_c$ .
- b. Suppose that  $Q$  is measured in units of mass, and  $c_i$  as mass per unit volume. Show that a suitable choice of characteristic mass is  $m_c = c_i V$ , and give a physical interpretation of  $m_c$ .
- c. Show that with dimensionless variables defined by  $y = \frac{Q}{m_c}$  and  $\tau = \frac{t}{t_c}$ , the equivalent dimensionless IVP is

$$\frac{dy}{d\tau} = 1 - y, \quad y(0) = \lambda,$$

where  $\lambda$  is a dimensionless constant which you should find.

### Skydiver Revisited

2. Reconsider the falling sky diver after her parachute opens (see Problem 13, part d. on Written Assignment 1). Suppose that when the 'chute opens at  $t = 0$ , her velocity is  $v_0 > 0$ , giving the model

$$m \frac{dv}{dt} = m g - \gamma v^2, \quad v(0) = v_0.$$

- a. Suppose you have done experiments which confirm that the *terminal velocity*  $v_T$  does NOT depend on the initial velocity  $v_0$ . Form the dimensional matrix  $\mathcal{D}$  for the four physical quantities mass  $m$ , drag coefficient  $\gamma$ , gravitational acceleration  $g$ , and terminal velocity  $v_T$ .

- b. Use the Buckingham Pi theorem to deduce that  $v_T = C \sqrt{\frac{mg}{\gamma}}$ , where  $C$  is a dimensionless constant. Then explain how the DE predicts that  $C = 1$ .
- c. Show that with dimensionless variables defined by  $u = \frac{v}{v_T}$  and  $\tau = \frac{gt}{v_T}$ , the equivalent dimensionless IVP is

$$\frac{du}{d\tau} = 1 - u^2, \quad u(0) = \lambda,$$

where  $\lambda$  is a dimensionless constant which you should find.

### A Buoyant Idea!

3. Suppose you've never heard of Archimedes, but you've experimented with various fluids and objects, and ascertained that whether an object floats or sinks depends on its weight, and on the fluid. You've also observed that whereas your stainless steel measuring cup floats nicely in the dish water, a steel ball bearing of the same mass sinks, so buoyancy must also depend on how much fluid is displaced. You thus conjecture that the buoyant force  $B$  depends on the object's mass  $m$ , the volume  $V$  of fluid displaced, the density  $\rho$  of the fluid, and gravitational acceleration  $g$ .
- Explain why the gravitational acceleration should be included.
  - Form the dimensional matrix  $\mathcal{D}$  of the five physical quantities  $B$ ,  $m$ ,  $V$ ,  $\rho$ , and  $g$ .
  - Use dimensional analysis and the Pi Theorem to show that  $B = V\rho g f\left(\frac{m}{\rho V}\right)$ .
  - What does Archimedes' Law of Buoyancy (see Example 2.2.3 in your lectures) tell you about the function  $f$ ?

### A Flawed Idea

4. In 1862, William Thompson (a.k.a. Lord Kelvin) proposed that the age  $t^*$  of the Earth depended on four quantities:
- the temperature  $u_0$  of the Earth's crust at depth  $y$  at the time  $t = 0$  when the Earth's crust solidified;
  - the surface temperature  $u_s$  of the Earth's crust (which he assumed to be constant);
  - the thermal conductivity  $\kappa$  of the rock ( $[\kappa] = \mathcal{L}^2\mathcal{T}^{-1}$ ); and
  - the gradient  $u_y = \frac{\partial u}{\partial y}$  of the crust temperature near the surface.
- Form the dimensional matrix  $\mathcal{D}$  of the physical quantities  $t^*$ ,  $u_0$ ,  $u_s$ ,  $\kappa$ , and  $u_y$ .

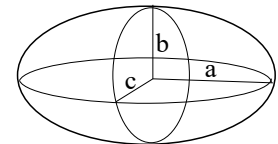
- b.** Use dimensional analysis and the Pi Theorem to show that  $t^* = \frac{u_s^2}{\kappa u_y^2} f\left(\frac{u_0}{u_s}\right)$
- c.** Lord Kelvin actually solved some nasty DEs to prove that, under his assumptions,  $t^* = \frac{(u_0 - u_s)^2}{\pi \kappa u_y^2}$ . If correct, what does this imply about the function  $f\left(\frac{u_0}{u_s}\right)$ ?

COMMENT: Thompson estimated the age of the Earth to be about 98 million years, with a range of 20 to 400 million years. Once the effect of radioactivity in igneous rocks was discovered, his error in using a heat conduction model was corrected.

[Reference for Problem 4: *Topics in Mathematical Modeling*, by K.K.Tung, Princeton University Press, 2007, pages 243-250.]

## Gravity-free Drag

- 5.** In a spacecraft, objects are essentially free of the force of gravity. Thus it seems reasonable to assume that the drag force  $f_d$  on a smooth object of cross sectional area  $A$  depends only on the density  $\rho$  of the air inside the craft, the speed  $v$  of the object, and  $A$ .
- a.** Form the dimensional matrix for the four physical quantities  $f_d$ ,  $A$ ,  $\rho$ , and  $v$ .
- b.** Apply dimensional analysis to show that  $f_d$  is proportional to  $\rho A v^2$ .
- c.** Two astronauts are playing catch with a toy football in an orbiting spacecraft. One throws a spiral pass at speed  $v_1$ , presenting cross sectional area  $A_1 = \pi b c$ . The other returns the ball with a sideways lob at speed  $v_2 = \frac{1}{2} v_1$ , presenting cross sectional area  $A_2 = \pi a b$ . If  $a = 10$  cm and  $c = 5$  cm, what is the ratio of  $f_{d_1}$  to  $f_{d_2}$ ?



Toy football  
(an ellipsoid)

## Escape Velocity

- 6.** Review Example 4 (pages 59-61 of Edition 2, or 63-65 of Edition 1 of your text) which discussed *escape velocity*, i.e., the necessary initial velocity an object needs to escape the Earth's gravitational field.
- a.** Form the dimensional matrix for the four physical quantities, the mass  $m$  of the object, the Earth's radius  $R$ , the gravitational acceleration  $g$ , and the escape velocity  $v_e$ .
- b.** Show that dimensional analysis gives the escape velocity correctly, up to a dimensionless constant  $C$ .
- c.** Suppose Planet X has radius 10 times that of Earth, and gravitational acceleration 2.5 times Earth's. How does the escape velocity for Planet X compare to that for Earth?

## Part II: Linear Systems in the Plane

Do problem 7 i. OR 7 ii, plus one of 8 or 9, and one of 10 or 11

### Phase Portrait Sketches (Do **only one** of 7 i. OR 7 ii)

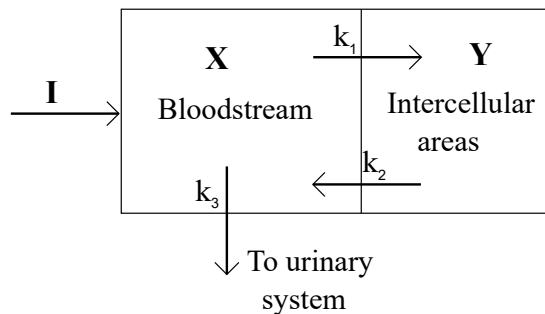
7. Consider the system  $\mathbf{x}' = \mathbf{A}\mathbf{x}$  for each matrix (i)  $\mathbf{A} = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix}$ , (ii)  $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 5 & -3 \end{pmatrix}$ .

- a. Find the eigenvalues, eigenvectors (if any), and state the general solution.
- b. Identify the type and stability of the equilibrium  $\mathbf{x} = \mathbf{0}$  in each case.
- c. Make a **hand-drawn sketch** of the phase portrait for each system by following the steps below and labelling your diagrams carefully.
  - Find the nullclines and sketch them as dashed lines in the plane. Analyze the signs of  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$  in each of the four regions between the nullclines, and show the orbit direction by an arrow on your sketch.
  - Add the eigenvector directions (if any) to your sketch, in blue if they are stable solutions or red if unstable.
  - Now that you have a framework for your phase portrait, and you have a good idea of the direction field, try to sketch several typical solutions. Verify your samples by doing a phase portrait in Maple if you like, but submit only hand drawn orbits for this question.
- d. Find the equivalent second order DE for each system.

### Following a Tracer

8. Tracers which can be detected by x-rays are often used to detect flow patterns in the body. One such tracer is *inulin*, which is injected into the blood, transfers back and forth between the bloodstream and intercellular areas, and also moves from the bloodstream into the urinary system.

Consider a compartment model for this process, as depicted in the diagram, where  $X$  and  $Y$  are the amounts of inulin in the bloodstream and the intercellular areas, respectively, and the positive constants  $k_i$  measure the proportions leaving/entering each compartment.



Thus the appropriate linear system is

$$\frac{d}{dt} X(t) = -k_1 X - k_3 X + k_2 Y + I, \quad \frac{d}{dt} Y(t) = k_1 X - k_2 Y$$

where  $I$  is the rate at which inulin is injected into the bloodstream.

- a. Find the equilibrium point  $(X_e, Y_e)$  of this system.
- b. Translate the system, moving the equilibrium to the origin (as in Example 6.2.1), and find the eigenvalues of the translated system  $\mathbf{x}' = \mathbf{A}\mathbf{x}$ , where  $\mathbf{x}(t) = (x(t), y(t))^T$ , with  $x(t) = X(t) - X_e$  and  $y(t) = Y(t) - Y_e$ .
- c. Show that the equilibrium  $\mathbf{x}(\mathbf{0})$  of the translated system is always an asymptotically stable node.

[HINT: If you let  $s = k_1 + k_2 + k_3$ , the discriminant in your solution for the eigenvalues should be  $s^2 - 4k_2k_3$ . Clearly this is less than  $s^2$ , but you will have to prove that it is always positive.]

- d. Verify your result in c. for the values  $k_1 = 0.01$ ,  $k_2 = 0.02$ ,  $k_3 = 0.005$ , all measured in  $\text{hour}^{-1}$ . If  $I = 1$  gram per hour, find the equilibrium levels of inulin in each compartment for these values.
- e. Sketch the phase portrait of the original system for  $(X(t), Y(t))$  for the values of the constants given in d.

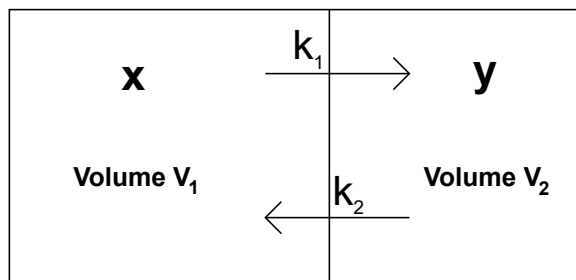
*Suggestion:* Do this by using Maple to sketch the phase portrait of the translated system on  $-1 \leq x \leq 1$  and  $-1 \leq y \leq 1$ . Then just make a hand-drawn sketch for the portrait of  $(X, Y)$  with the equilibrium shifted to  $(X_e, Y_e)$ .

[Reference for Problem 8: *Differential Equations: A Modeling Perspective*, ed 2, by R.L.Borrelli and C.S. Coleman, John Wiley & Sons, Inc, 2004, pages 351-352]

## Diffusion Through a Membrane

9. Eukaryotic cells are composed of several membrane-bound compartments. Certain molecules (e.g., oxygen) diffuse across these membranes according to Fick's law, which states that the rate of flow is proportional to the difference in concentration, with the flow being from the compartment with the higher concentration to the one with lower concentration of that molecule.

Consider the simple case of two adjacent compartments of constant volumes  $V_1$  and  $V_2$ , where molecules of a certain type diffuse freely across the separating membrane. Let  $x$  and  $y$  be the concentrations (i.e., mass per unit volume) of this species in the two compartments; thus the total masses are  $xV_1$  and  $yV_2$  respectively.



Then Fick's law implies the system

$$\frac{d}{dt}(V_1 x(t)) = D(y(t) - x(t)), \quad \frac{d}{dt}(V_2 y(t)) = D(x(t) - y(t)),$$

or  $\frac{dx}{dt} = \frac{D}{V_1}(y - x) = k_1(y - x), \quad \frac{dy}{dt} = \frac{D}{V_2}(x - y) = k_2(x - y),$

where  $D$  is a positive constant which quantifies how easily the molecules diffuse across the membrane,  $k_1 = \frac{D}{V_1}$ , and  $k_2 = \frac{D}{V_2}$ .

- Explain how the terms in the model (initial version) are implied by Fick's law.
- Recalling that  $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$ , find an implicit equation for the orbits of this system, and show that this solution is simply a statement of conservation of total mass in the two compartments.
- Write the system in the form  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ , where  $\mathbf{x}(t) = (x(t), y(t))^T$ . Find the eigenvalues and eigenvectors of  $\mathbf{A}$  and hence the general solution.
- Describe the longterm behaviour of this system (i.e., what happens as  $t \rightarrow \infty$ ) for any initial condition  $\mathbf{x}(0) = (x_0, y_0)^T$ , and give a physical interpretation.
- Sketch the phase portrait in the positive quadrant for each of the three cases (i)  $V_1 < V_2$ , (ii)  $V_1 = V_2$ , (iii)  $V_1 > V_2$ . [HINT: Find the equilibria first.]

[Reference for Problem 9: *Mathematical Modeling in Systems Biology*, by Brian P. Ingalls, MIT Press, 2013, pages 74-75]

### Orbits of a Stable Center

- 10.** Before beginning this problem, review Exercise 6.2.2 in your lectures, which guides you through the derivation of the implicit and explicit solutions of the model for the simple pendulum with no damping. If you have not yet completed the exercise, it would be very helpful if you did so now.

Consider the linear system  $\mathbf{x}'(t) = \mathbf{A}\mathbf{x} = \begin{pmatrix} -2 & \frac{5}{2} \\ -2 & 2 \end{pmatrix} \mathbf{x}$ , where  $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ .

- Show that the eigenvalues of this system imply a stable center.
- Show that the system is equivalent to an implicit equation for the orbits,

$$\frac{dy}{dx} = \frac{-4x + 4y}{-4x + 5y}.$$

- Write the DE in part b. in the form  $M(x, y) + N(x, y)y' = 0$ , and prove that it is exact. Then solve for the family of integral curves. (These curves must contain the orbits of the given linear system.)

- d. The equation  $\frac{(ax - by)^2}{A^2} + \frac{(cx - fy)^2}{B^2} = 1$  represents a ‘tilted’ ellipse. By completing the square, show that the integral curves in part c. form a family of such ellipses. (This calculation is easier if you start with your integral curves in the form  $x^2 + \dots = C$ .) Plot three typical integral curves,  $C = 1, 4, 9$ , using Maple’s *implicitplot* command (or otherwise). (A window  $x = -7..7, y = -7..7$  works well for these values.) Then use the original system to determine the orbit nullclines, and the direction of the orbits, and add this information to your plot.
- e. Find the equation of the particular integral curve through  $(2, 0)$ . Show that, while the explicit functions  $x(t) = 4 \cos \omega t + 2 \sin \omega t$ , and  $y(t) = 4 \cos \omega t$  satisfy this equation for any constant  $\omega$ , they do not satisfy the given system  $\mathbf{x}'(t) = \mathbf{A}\mathbf{x} = \begin{pmatrix} -2 & \frac{5}{2} \\ -2 & 2 \end{pmatrix} \mathbf{x}$  unless  $\omega = 1$ .
- (This tells you that the DEs in the system impose a constraint on the period of the orbit which is not present in the implicit equation.)
- It is true in general that the orbits of a stable center are ellipses.

### Critical Transitions

11. Consider the linear system  $\mathbf{x}'(t) = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & \alpha \\ 1 & 1 \end{pmatrix} \mathbf{x}$ , where  $\alpha$  is a constant.
- a. Show that the eigenvalues of this system are  $\lambda = 1 \pm \sqrt{\alpha}$ .
- b. Explain how the orbits of the system  $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}$  change as  $\alpha$  passes through zero from negative to positive, but remains less than 1.
- c. Illustrate your answer in part b. by plotting phase portraits of the given system for  $\alpha = -0.49$ ,  $\alpha = 0$ , and  $\alpha = 0.49$ .
- d. Repeat part b. for the transition through  $\alpha = 1$ . Illustrate your answer by plotting phase portraits of the given system for  $\alpha = 0.81$ ,  $\alpha = 1$ , and  $\alpha = 1.44$ . [HINT: Use a symmetric window, say  $-1 \leq x \leq 1$  and  $-1 \leq y \leq 1$  for the portraits in parts c. and d.; you will likely need to vary the initial conditions for different values of  $\alpha$  in order to get ‘nice’ phase portraits for each one.]

## Part III: Homogeneous Second Order DEs and Applications

Do problems 12, 13, 14, and one of 15 or 17

### Solution Methods

12. Consider the constant coefficient second order linear DE

$$a y'' + b y' + c y = 0.$$

- a. Assuming  $a \neq 0$ , write down the equivalent dynamical system for the given DE. Show that the eigenvalues of this system are equivalent to the roots of the characteristic equation  $Z(\lambda) = 0$  of the given DE.
- b. For each of the DEs **i.**, **ii.**, and **iii.**:
- find the characteristic roots and state the general solution  $y(t)$ ;
  - use the fact that the characteristic roots are the eigenvalues to classify the equivalent dynamical system as to type and stability, and state the vector solution  $\mathbf{x}(t) = (x_1(t), x_2(t))^T = (y(t), y'(t))^T$ ;
  - sketch the phase portrait (use Maple if desired).
- i.**  $2y'' - 3y' + y = 0$     **ii.**  $y'' - 2y' + 2y = 0$     **iii.**  $y'' + 5y' = 0$

### Mass-Spring-Damper Systems

13. A mass  $m = 0.1$  kg stretches a spring 0.05 m when hanging at rest.

- a. Find the value of the spring constant  $k$ .
- b. If the mass is set in motion from equilibrium  $y_0 = 0$  with a downward velocity  $v_0 = 0.1$  m s<sup>-1</sup>, find the position  $y(t)$  of the mass at any time  $t \geq 0$ . State the amplitude, frequency, and period of the motion.
- c. What is the time  $t^*$  when the mass first returns to equilibrium?
- d. Suppose the above mass moves in a resistive medium which exerts a drag force of  $\gamma v$ , where  $\gamma = 8$  kg s<sup>-1</sup>. Will the mass still oscillate, i.e., be underdamped? Explain.

14. Consider the standard mass-spring-damper model

$$m y'' + \gamma y' + k y = 0.$$

- a Show that, with the choice of dimensionless variable  $\tau = \omega_0 t = \sqrt{\frac{k}{m}} t$ , this DE becomes

$$\frac{d^2 y}{d\tau^2} + 2\zeta \frac{dy}{d\tau} + y = 0, \quad (\star)$$

where  $\zeta = \frac{\gamma}{2\sqrt{k m}}$ .

- b. Prove that the characteristic roots of the DE  $(\star)$  are either negative reals, or have a negative real part. (Thus such motion always dies off as  $t \rightarrow \infty$ , and the corresponding dynamical system is always asymptotically stable.)

### Critical and Near-Critical Damping

15. Recall that critical damping occurs when the characteristic roots are real and equal. It marks the transition from overdamping to underdamping.

- a. Suppose the system governed by the IVP

$$m y'' + \gamma y' + k y = 0, \quad y(0) = y_0, \quad v(0) = v_0,$$

is critically damped. Write down the general solution  $y(t)$  for this case, and show that there is a positive time  $t^*$  such that  $y(t^*) = 0$  (i.e., the mass passes back through equilibrium), if and only if the initial conditions satisfy

- $y_0 v_0 < 0$  (i.e.,  $y_0$  and  $v_0$  are of opposite sign), AND
- $y_0(v_0 + \frac{\gamma}{2m} y_0) < 0$  (i.e., either  $y_0 < 0$  and  $v_0 + \frac{\gamma}{2m} y_0 > 0$ , OR  $y_0 > 0$  and  $v_0 + \frac{\gamma}{2m} y_0 < 0$ ).

Sketch two such solutions on the same diagram, one for  $y_0 > 0, v_0 < 0$ , and one for  $y_0 < 0, v_0 > 0$ .

- b. Assume the vertical motion of your aging jalopy is governed by the DE

$$m y'' + \gamma y' + k y = 0.$$

The knowledgeable people at Snappy Muffler Queen normally install shock absorbers which provide very slight overdamping of the suspension (i.e., close to critical damping, so when you hit a bump, the car just returns as quickly as possible to equilibrium). This time, an enthusiastic employee encourages you to select a new set of shocks with built-in coil springs to stiffen the suspension (i.e., increase the constant  $k$ ). What effect will this have on the way your car rides? [HINT: Look at what happens to the characteristic roots.]

## Cauchy-Euler DEs (*Optional*)

**16.** Consider a DE of the form  $a x^2 y'' + b x y' + c y = 0$ , with unknown  $y(x)$ .

**a.** Show that, if the DE has solutions of the form  $y = x^p$ , then the constant  $p$  must satisfy the equation  $a p^2 + (b - a)p + c = 0$ .

**b.** Use the idea in part a. to find the general solution on  $x > 0$  for each of the following DEs.

**i.**  $x^2 y'' + 4 x y' + 2 y = 0$       **ii.**  $x^2 y'' + 3 x y' + 5 y = 0$

[HINT: For complex values of  $p$ , you will need to write  $x^{\alpha+\beta i} = x^\alpha x^{\beta i}$ , and expand the second exponential as  $x^{\beta i} = e^{(\ln x)\beta i} = \cos(\beta \ln x) + i \sin(\beta \ln x)$ . Then recall that if  $y(x) = u(x) + iw(x)$  is a solution of a linear DE, then each of  $u(x)$  and  $w(x)$  is a solution.]

## Measuring Viscosity

**17.** Consider the underdamped solution of the standard mass-spring-damper model  $m y'' + \gamma y' + k y = 0$ , namely

$$y(t) = R e^{-\gamma t/2m} \cos(\nu t - \delta).$$

**a.** Explain why the time between any two successive maxima is

$$T_d = \frac{2\pi}{\nu} \quad (\text{the } \textit{pseudo-period}).$$

**b.** Show that, if  $y_1$  and  $y_2$  are two successive maxima, then

$$\ln \left( \frac{y_1}{y_2} \right) = \frac{\gamma\pi}{m\nu} \quad (\text{the } \textit{logarithmic decrement}).$$

**c.** While it is not easy to directly measure the viscosity  $\alpha$  of a fluid, it is known (through Stoke's Law) that a sphere of radius  $a$  and mass  $m$ , hanging on a spring of constant  $k$ , will oscillate with pseudo-frequency

$$\nu = \sqrt{\frac{k}{m} - 9 \frac{\pi^2 \alpha^2 a^2}{m^2}}$$

in such a fluid. Since  $y_1$ ,  $y_2$ , and  $\nu$  are easy to measure, explain how you could use such an experiment to measure the viscosity  $\alpha$  of a specific fluid.