MATH 650 : Mathematical Modeling

Spring, 2019 - Final Quiz

Due by 11:59 p.m. EST on Wednesday, August 7th, 2019

Instructions:

- This final quiz contains no new concepts or solution methods. It is intended to provide an overview of the course, including both techniques and some (hopefully) interesting applications. The emphasis is on using known results in new situations, and deriving and interpreting graphical and analytical information about a given application.
- You will benefit maximally by completing these problems on your own. That said, if you get stuck and consult other resources (human or otherwise), please cite them on the first page of your assignment.
- Presentation is important. Please write your solutions in clear sentences which convey your reasoning. (See the Sample Solutions for a handy guide.) Remember that I can't know what you're thinking...I can only know what you tell me. Conciseness is also very much appreciated. Hand-written solutions are just fine, but they must be well-organized and legible, with the problems in numerical order. (I will send the LaTeX file if requested.)
- Several of the problems use specific Maple files which will be made available. (I'm assuming you can do simple Maple commands such as *plot* and *implicitplot* on your own; email me if you need help.)
- **I suggest you spend no more than 10 hours in total on this assignment, including writing up your solutions.** While you might not complete all the problems you'd like to, give it your best try...but don't worry about it beyond that time span. Partially completed problems are acceptable; I've included the marks for the individual parts of each problem.
- There are 7 problems on this assignment, two on first order DEs, one on dimensional analysis, two on second order DEs and/or linear systems of two first order DEs, and two on nonlinear systems of two first order DEs and their linearizations.
  Submit only ONE of 1a. or 1b., 2a. or 2b., and 3a. or 3b; do not submit 'extra' problems.
- Submit your solutions in the usual manner in LEARN.

Available Marks for Each Problem

Problem	Marks	
1a. or 1b.	10	(a. 10, b. 10)
2a. or 2b.	12	(a. 12, b. 12)
3a. or 3b.	11	(a. 11, b. 11 + 4)
4	13	(a. 5, b. 6, c. 2)
5	13	(a. 10, b. 1, c. 2)
6	18	(a. 1, b. 3, c. 3, d. 3, e. 7)
7	18	(a. 2, b. 7, c. 7, d. 1)
Presentation	5	

	100	

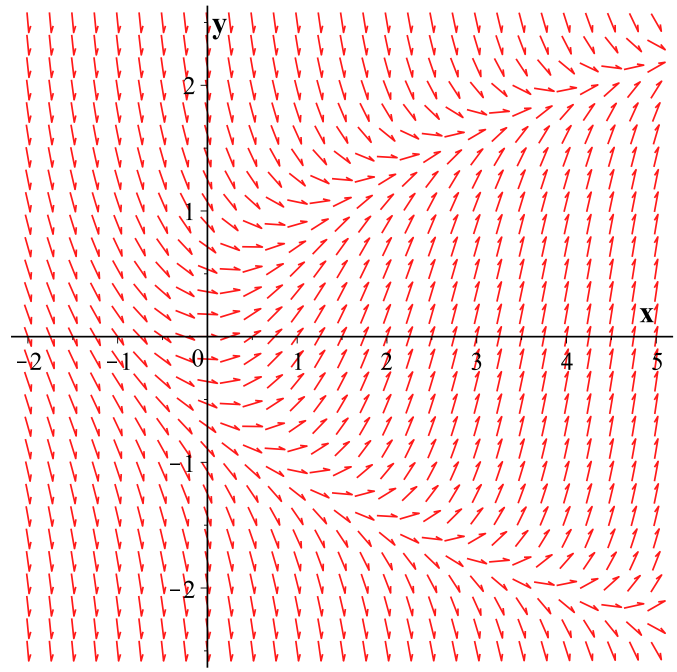
NOTE: Problem 3b. is a more difficult problem than 3a., involving a partial differential equation. Thus it has 4 ‘bonus’ marks attached to it, should it be your choice.

Using Maple to Explore First Order DEs

1.a. Consider the nonlinear first order DE $\dfrac{dy}{dx} = x - y^2$, with slope field as shown below. (You may do the required sketches by hand, or use Maple, as you wish.)

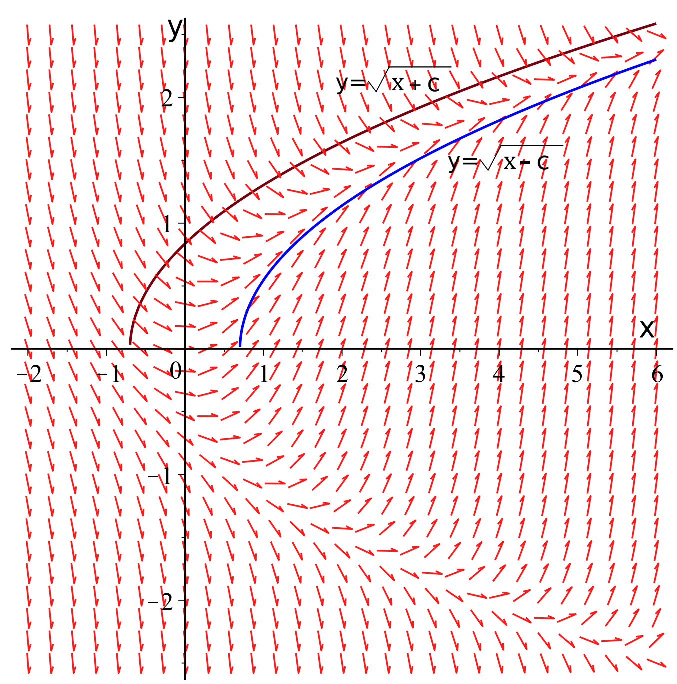
i. Find the nullcline ($\frac{dy}{dx} = 0$) and sketch it carefully on the direction field plot.

ii. Sketch the solutions with initial conditions (-1,2), (-1,0), (0,2), (2,2), (0,0), (0,-1), (1,0), (2,-1.5), (2,-2), following the direction field carefully, and extending each solution to the boundaries of the window.



iii. By examining the behaviour of $\frac{dy}{dx}$ as given by the DE, explain why, once a solution enters the region $x > y^2$, it can never leave that region as x increases, i.e., it is *trapped.*

iv. At the right is a plot of the same direction field, along with two typical isoclines for $y > 0$, namely $\frac{dy}{dx} = \pm c$, giving $y = \sqrt{x \mp c}$, for some positive constant c. Observe that solutions enter the region between these two curves from above and below, creating a *funnel* which appears to be narrowing as x increases.



Show that

$$\lim_{x\to\infty} |\sqrt{x+c} - \sqrt{x-c}| = 0,$$

and hence deduce the unique curve $y = \phi(x)$ to which all other *trapped* solutions are asymptotic as $x \to \infty$.

v Describe the two types of behaviour of the solutions of this DE as x increases. What do you think determines which type of behaviour occurs for a given solution? [Hint: What prevents a solution from entering the region $x > y^2$?]

Do only ONE of problems 1a. OR 1b.

b. We have explored the logistic model for the growth of a population of size $N(t)$, namely, $\frac{dN}{dt} = rN(1-\frac{N}{K})$, $N(0)=N_0$. Suppose that the underlying logistic growth has $r=0.1$ per year and $K=1000$, and the population $N(t)$ is subject to 'seasonal' harvesting, with period 1 year. A suitable model is the IVP

$$\frac{dN}{dt} = 0.1N(1-\frac{N}{1000}) - A(1+\sin(2\pi t)\,, \quad N(0)=N_0\,,$$

where A is a constant which determines the maximum amount ($2A$) of harvesting which occurs at any time.

i. Show that the harvesting occurs, on average, at rate $H_{avg} = A$. Use the plot command given in the Maple file Math650FQStudentPlots to explore the behaviour of the solutions for various values of A between 5 and 30. Observe carefully how the long term behaviour of the solutions depends on both A and on N_0; perhaps make a table of your observations for different values of A.

ii. Do you get the same longterm behaviour if the intrinsic growth rate is $r=0.2$ per year? Does the dependence on N_0 change? Explain any differences you observe.

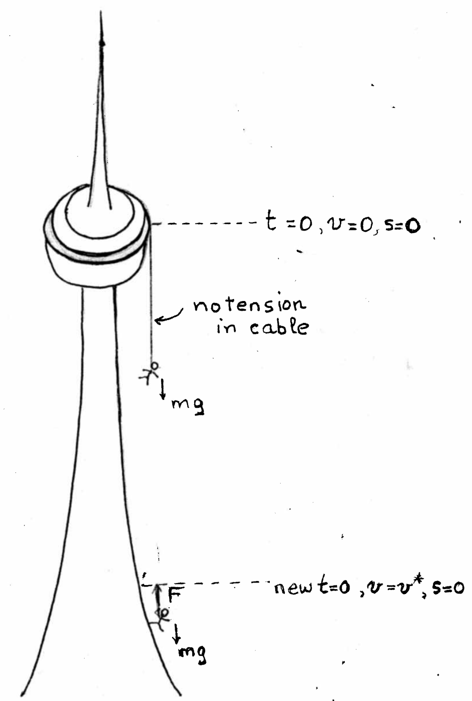
iii. Write a paragraph about your results, considering for what values of A the population is viable for at least some initial populations N_0, how the longterm behaviour of the population changes as A increases, and how this depends on N_0, and on r.
Suggestion: Suppose that harvesting occurs at a rate equal to the average value of $A(1+\sin(2\pi t))$; to what model might you compare this one? This may prove helpful in finding constraints on A and N_0 for survival.

iv. If you were a conservation officer, what recommendation(s) would you make to preserve this population?

(Submit NO plots for this question...just your well-organized thoughts and conclusions.)

Dar Robinson's CN Tower Jump

2a. In 1980, stuntman Dar Robinson jumped from the observation deck of the CN Tower, a height of about 366 metres, his only safety mechanism a single cable attached to a harness. (To see a video, do a web search for 'Dar Robinson CN Tower jump'.)



Stage 1: Suppose he is in free fall for the first 300 m, with negligible air resistance. Assume he falls from rest at $t = 0$.

- At what time $t^\star$ will he reach the 300 m point?
- What will be his velocity $v^\star$ at that point?

Stage 2: Now it is time to brake his fall. Reset the clock to $t = 0$ and the displacement to $s = 0$ at the 300 m point, and set $v(0) = v^\star$.
Assume an increasing braking force $F(t) = c\,m\,t$ N is applied from this point onward, where m is his mass.

i. Newton's Law gives the IVP $m\,\dfrac{dv}{dt} = mg - F,\ v(0) = v^\star$ for Stage 2. Solve for $v(t)$, and hence find the time t_f when his velocity $v(t_f) = 0$. (Your answer will be in terms of $g, v^\star$, and c.)

ii. Ideally, we want his velocity $v = 0$ just as he touches down at $s = 66\,\text{m}$. Show that this will occur if an equation of the following form is satisfied:

$$66 = \text{a function of } \ t_f, g, c, v^\star$$

Then use the commands for Problem 2 in the Maple file Math650FQStudentPlots to solve this equation for the value of the constant c, and hence to find the time t_f when he lands.

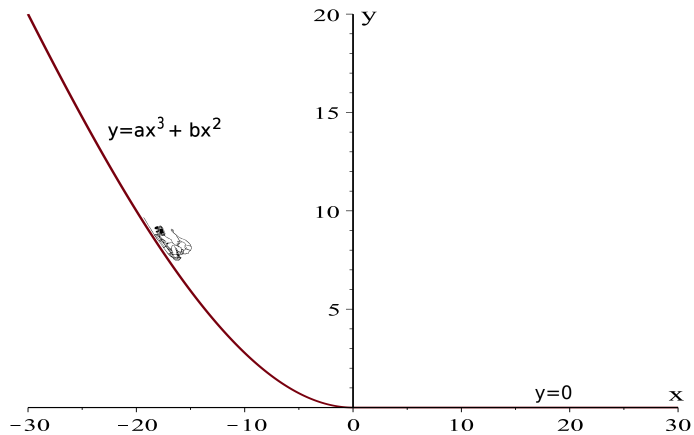
- What is the total time $t^\star + t_f$ for his fall?
- If his mass is 80 kg, what is the maximum force F on the cable during Stage 2? How many 'Gs' would he thus survive? (One G equals his weight.) How does this compare to the maximum experienced by an astronaut on re-entry? To the maximum known to have been survived?

iii. Comment on any aspects of the physical situation that have been ignored in the above model for this situation. Do you think a braking force $F = kms$, where k is a positive constant, would be better or worse for Dar?

Do only ONE of problems 2a. OR 2b.

A Sled Ride

2b. Suppose a child on a sled travels down a hill 20 m high, with shape $y = ax^3 + bx^2$, for $-30 \le x \le 0$, then slowly glides to a stop for $0 \le x \le 30$ along $y = 0$.



i. If she starts at $(-30, 20)$, passes through $(-20, 10)$, and her path is smooth at $(0, 0)$, show that $a = \frac{1}{3600}\,\mathrm{m}^{-2}$, and $b = \frac{11}{360}\,\mathrm{m}^{-1}$.

ii. Suppose the sleigh starts from rest at $(-30, 20)$ and slides down the hill to $(0, 0)$. If air resistance and frictional forces reduce her speed at $(0, 0)$ to $\frac{1}{4}$ of what it would have been with NO resistive forces, use conservation of energy to show that her speed at $(0, 0)$ is $v_0 = \frac{7}{\sqrt{2}}\,\mathrm{ms}^{-1}$. (Use $g = 9.8\,\mathrm{ms}^{-2}$.)

iii. Suppose that over $0 \le x \le 30$, air resistance is negligible, and the frictional resistive force of the snow has magnitude μmg N, where μ is the coefficient of kinetic friction. Use Newton's Law of Motion and a suitable transformation to derive the DE

$$v\frac{dv}{dx} = -\mu g, \quad \text{with} \quad v = v_0 \text{ at } x = 0,$$

and then find $v(x)$. (Your answer will contain μ.)

iv. If the sled stops at $x = 30$ m, what is the value of μ? What are its dimensions?

v. How long does the sled take to slow to a stop from $x = 0$?

Dimensional Analysis (Do only ONE of problems 3a. OR 3b.)

3a. The driver of a car of mass m travelling at constant velocity v_0 spots a squirrel crossing the road. She brakes with a constant force F for a distance s (in a straight line), reducing her speed to v_1 to avoid hitting the squirrel.

i. Write down the dimensional matrix, find its rank, and apply the Pi Theorem to show that $s = \dfrac{mv_0^2}{F} f(\dfrac{v_0}{v_1})$. Explain why this answer makes sense, physically.

ii. Two simple integrations of Newton's Law for this problem reveal that, during the braking period, $v(t) = -\dfrac{F}{m} t + v_0$, and $s(t) = t\,(-\dfrac{F}{m}\dfrac{t}{2} + v_0)$. Using these results (no need to prove them), show that

$$s\,F = \frac{1}{2}\,(mv_0^2 - mv_1^2)\,.$$

Explain this result in terms of the work done by the force F and the kinetic energy of the car. Then rearrange the equation to the form in part a., and hence find the function $f(\dfrac{v_0}{v_1})$.

3b. Dimensional analysis can sometimes help to solve a differential equation, as for the partial diffierential equation (PDE) known as the *heat equation.*

Consider a uniform rod of very long length, with a heat source which maintains one end at temperature u_0. We idealize the rod as extending along the positive x-axis, as shown, and assume that the rod is initially at temperature 0 along its full length. The problem of finding the rod's temperature $u(x,t)$ is then posed as follows (where κ is the *thermal conductivity* of the rod, with $[\kappa] = \mathcal{L}^2\mathcal{T}^{-1}$).

The PDE and *initial condition* (IC) are

$$u_t = \kappa\, u_{xx},$$
$$u(x,0) = 0 \text{ for } x > 0,$$

with *boundary conditions* (BCs)

$$u(0,t) = u_0 \text{ for } t > 0, \text{ and}$$
$$u(x,t) \to 0 \text{ as } x \to \infty.$$

i. Use dimensional analysis of the five quantities u, x, t, u_0, κ to justify the statement that the solution has the form $u(x,t) = u_0 f(\eta)$, where $\eta = \frac{x}{2\sqrt{\kappa t}}$.

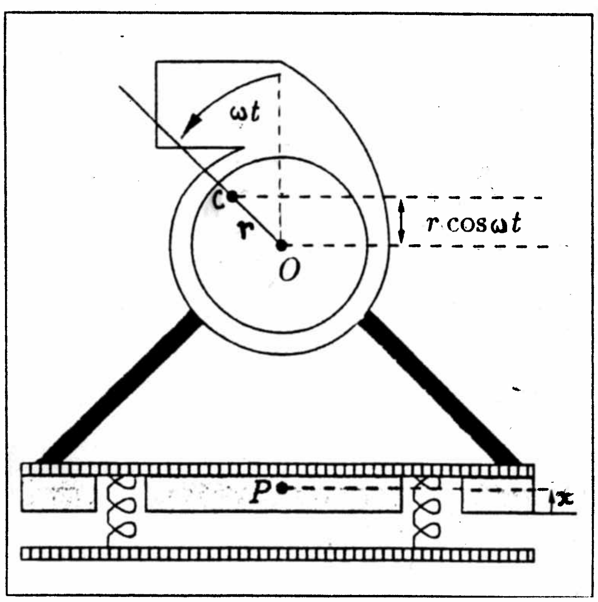
ii. Observe that the Chain Rule gives $u_t = u_0 f'(\eta)\eta_t$, and $u_{xx} = u_0 f''(\eta)(\eta_x)^2$. Use these results to show that the PDE for $u(x,t)$ becomes an ODE for $f(\eta)$,

$$f''(\eta) + 2\eta f'(\eta) = 0.$$

iii. Solve the DE in part ii. for $f(\eta)$ (by reduction of order). Then apply the IC and BCs (converted to η) to obtain the solution $u(x,t) = u_0(1 - erf(\frac{x}{2\sqrt{\kappa t}}))$. (Recall that $erf(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-z^2}\, dz$, and $\lim_{\eta\to\infty} erf(\eta) = 1$.)

Inertial Damping and Inflation

4. In order to prevent excess vibration from the large fans essential to air distribution in commercial structures, mechanical engineers frequently make use of a system such as the one shown in the simplified diagram below. (For example, this technique was used in the construction of the Quantum-Nano Centre at the University of Waterloo, where laboratory research is exceptionally sensitive to external vibrations.)



The cement block to which the frame is bolted acts as an 'inertial damper'; it simply increases the overall mass m of the system so much that $\omega_0^2 = \frac{k}{m}$ is quite small. The springs can be modelled as a single spring with spring constant k.
It can be shown that the motion of the centre of mass P of the block frame structure is governed by the DE

$$\frac{d^2x}{dt^2} + \omega_0^2\, x = A\, \cos(\omega t)\,.$$

In this DE, x is the displacement of P from equilibrium, and $A = r\,\omega^2\,\frac{m_0}{m}$, m_0 being the mass of the fan, and m the total mass. Thus we see that the rotating fan creates a periodic 'lift' with amplitude proportional to the square of its angular frequency ω, i.e., the faster the fan rotates, the larger the force exerted.

a. By substitution into the given DE, show that, with ICs $x(0) = 0$, $x'(0) = 0$, the solution of this DE is

$$x(t) = \frac{A}{\omega_0^2 - \omega^2}\,(\cos \omega t - cos\omega_0 t)\,.$$

b. The behaviour of the solution in part a. depends critically on the value of the input frequency ω compared to the natural frequency ω_0. Two phenomena of interest are

- **i.** *beats*, which occur when $|\omega_0 - \omega|$ is small, and
- **ii.** *resonance*, which occurs when $\omega = \omega_0$.

Describe each of these phenomena in your own words, with hand-drawn sketches to illustrate each, and then decide whether they would be good or bad in this situation, giving reasons. Are they likely to occur here?

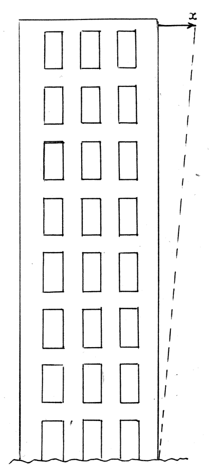
c. In Module 7, Example 7.4.3, we discussed a model for the interaction between supply and demand which led to a second order DE for the 'oversupply' $x(t)$ (i.e., the supply over and above some equilibrium supply level), namely

$$\frac{d^2x}{dt^2} + k_1\, k_2\, x = k_2\, f(t)\,,$$

where k_1 and k_2 are positive constants, and $f(t)$ is the inflation rate. Consider a periodic inflation rate $f(t) = f_0 \cos \omega t$. Show that, with suitable definitions of ω_0^2 and A, this DE is equivalent to the above model, and describe what happens economically, according to this model, if $\omega = \omega_0$. Is this prediction realistic? Explain.

Shaking Things Sideways

5. Consider the lateral vibrations of a tall structure due to external forces (e.g., wind, or earthquakes). (See *https : //www.youtube.com/watch?v = f1U4SAgy60c.*)



It can be shown that such vibrations are modelled by the DE

$$x'' + 2\,\delta x' + \omega_0^2 x = f(t)\,,$$

where δ and ω_0 are positive constants determined by the design parameters of the building, and $f(t)$ is the external force per unit mass.

a. Rewrite the associated homogeneous DE $x'' + 2\,\delta x' + \omega_0^2 x = 0$ as a system $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$ of two first order DEs, where $x_1 = x$, $x_2 = x'$. Find the eigenvalues of the matrix $\mathbf{A}$, then show that the equilibrium $\mathbf{0}$ of the homogeneous system is asymptotically stable for all positive δ and ω_0. Identify the type of equilibrium for each of the three possible cases of the eigenvalues.

b. Suppose a brief gust of wind hits the structure, giving an initial state $x(0) = \ell$, $x'(0) = 0$ to the system. What do your results in part a. predict will be the long-term behaviour of $x(t)$? (No need to solve the IVP; just use your stability result from part a., and the fact that $x(t) = x_1(t)$.) Describe the motion of the building for the overdamped case.

c. Now consider the effects of an earthquake, for which the input function is $f(t) = A \cos \omega t$. The positive constants A and ω measure the strength and frequency of the lateral vibrations induced by the earthquake. Research has shown that the best earthquake resistance occurs if the following two conditions are met:

1. the unforced vibrations are *underdamped*, and
2. the forced vibrations are such that the steady-state amplitude A_{ss} decreases as ω increases (i.e., $\dfrac{d}{d\omega}(A_{ss}) < 0$ for all $\omega > 0$), so there is no amplitude resonance.

Prove that both these conditions hold if the constants δ and ω_0 satisfy $\dfrac{\omega_0}{\sqrt{2}} < \delta < \omega_0$ by following these two simple steps.

i. What will guarantee that condition 1. is met? (Examine your results from part a.)

ii. We have already seen (in the course lectures) that the steady-state amplitude of the particular solution of a DE of the form $x'' + 2\,\delta x' + \omega_0^2 x = A\,\cos\omega t$ is given by

$$A_{ss} = \frac{A}{\omega_0^2\sqrt{(1-r^2)^2 + 4\zeta^2 r^2}}\,,$$

where $r = \dfrac{\omega}{\omega_0}$, and $\zeta = \dfrac{\delta}{\omega_0}$. You also showed on a previous assignment that amplitude resonance cannot occur for any $\omega > 0$ if $\zeta > \dfrac{1}{\sqrt{2}}$. Rewrite this condition in terms of δ and ω_0 to determine what will guarantee condition 2., completing the proof.

Two-Species Interactions

6. Consider the following model for the interaction between two species.

$$\frac{dN_1}{dt} = r_1 N_1 (1 - \frac{N_1}{K_1}) - aN_1N_2\,, \quad \frac{dN_2}{dt} = r_2 N_2 (1 - \frac{N_2}{K_2}) + bN_1N_2\,.$$

a. Clearly the underlying growth of each species is logistic, i.e., in the absence of one another, they would each thrive. What do the two interaction terms tell you about the nature of their interaction, as compared to the competition model we have previously discussed? Name this interaction, giving reasons.

b. Show that, with the choice of dimensionless variables $x = \frac{N_1}{K_1}$, $y = \frac{N_2}{K_2}$, $\tau = r_1 t$, and dimensionless constants $A = \frac{aK_2}{r_1}$, $B = \frac{bK_1}{r_2}$, $R = \frac{r_2}{r_1}$, the model becomes

$$\frac{dx}{d\tau} = x(1 - x - Ay)\,, \quad \frac{dy}{d\tau} = Ry(1 - y + Bx)\,.$$

Give biological interpretations of x and y, and of the constants A, B, and R.

c. Show that the system in part b. has four equilibrium points, with the one in the positive quadrant being $P : \left(\dfrac{1-A}{1+AB}, \dfrac{1+B}{1+AB}\right)$, provided $A < 1$.

d. Suppose that the equilibrium P is asymptotically stable for all $x(0) > 0$ and $y(0) > 0$. Given the equilibrium you found above, show that, if $0 < A < 1$ and $B > 0$, the long term population of species N_1 will always be below K_1, while that of species N_2 will always be above K_2. Explain why this makes sense biologically.

e. Use the plot command for Problem 6 in the Maple file Math650FQStudentPlots to plot phase portraits for each of the following cases:

i. $A = \dfrac{1}{2}$, $B = 1$, $R = \dfrac{1}{8}$; **ii.** $A = \dfrac{6}{5}$, $B = 1$, $R = \dfrac{1}{8}$.

In each case, linearize about the asymptotically stable equilibrium to confirm the type of equilibrium you observe in the phase portrait. State the the biological outcome, and justify it, based on the parameter values for A and B and their biological meaning.

7. In a biological interaction between two molecular species X and Y, the species Y undergoes *autocatalysis*, that is, it activates its own production as well as benefitting from the presence of X. Production is governed by

$$\frac{dx}{dt} = 8 - (1 + y^n)x\,, \quad \frac{dy}{dt} = (1 + y^n)x - 5y\,,$$

where $x(t)$, $y(t)$ are the concentrations of X and Y, and n is a positive constant called the Hill coefficient.

a. Would species Y survive without species X? Would species X survive without species Y? How does the presence of Y affect the production of X? Of Y?

b. Use suitable hand calculations plus Maple for Problem 7 in Math650FQStudentPlots to explore the behaviour of this system for $n = 2$, as follows.

i. State the equation of the horizontal isocline, and of the vertical isocline. Then find the single equilibrium (x_e, y_e) of the system.

ii. Use Maple to create a plot of the direction field of this system on the domain $0 \le x \le 4.5$, and $0 \le y \le 5$. Print a copy of your plot in a suitable size. Then carefully sketch the three orbits with ICs $(x_0, y_0) = (0, 1)$, $(2, 0)$, and $(0, 3)$, following the direction field as closely as possible. Make a conjecture about the nature of the equilibrium point.

iii. Explore the linearization about (x_e, y_e) and hence describe its type and stability. Describe the behaviour of the species concentrations $x(t)$ and $y(t)$ over time.

c. Now suppose that $n = 2.5$, and explore the given system as follows.

i. Find the single equilibrium (x_e, y_e) of the system (you'll need a calculator for this).

ii. Use Maple to plot the phase portrait. What does the phase portrait imply about the long term behaviour of the species concentrations $x(t)$ and $y(t)$?

iii. Explore the linearization about (x_e, y_e) and state its type and stability. Use Maple to plot an orbit with IC near (x_e, y_e), and describe its behaviour. Does this behaviour coincide with that of the nonlinear system?

iv Sketch by hand a typical graph of $x(t)$ and of $y(t)$ as functions of t if their ICs are near (x_e, y_e). (This is a qualitative question; your sketches do not have to be numerically accurate. Look at the phase portrait and interpret the behaviour you see there to sketch the two component graphs.)

d. It seems clear from your results in b. and c. that the qualitative behaviour of this system changes dramatically for some value of n between 2 and 2.5. Describe the change that occurs in the stability of the equilibrium (x_e, y_e), and the resulting change in the phase portrait. Which example in the lectures on the Phase Portrait Gallery in Module 8, Section 1.1 does our system most resemble? Now go to Wikipedia and read about the Hopf bifurcation (the animation of the Brusselator is especially intriguing).

[Reference for Problem 7: *Mathematical Modeling in Systems Biology*, by Brian P. Ingalls, MIT Press, 2013, pages 108-110.]