MATH 650 : Mathematical Modeling

Spring, 2019

Electronic Assignment #9

Due by 11:59 p.m. EST on Tuesday, July 25, 2019

Instructions:

- Ensure you have reviewed Module 8, Sections 1 and 2, and any other activities therein. You will require this knowledge to answer the questions on this assignment.
- Read and think about the following assignment problems.
- Print and complete the following assignment. Record your answers on the printed copy so you have a record of your solutions.
- Once you are satisfied with your answers, submit your solutions online as follows:
  - Go to UW's course management website at `learn.uwaterloo.ca`
  - Enter your **QUEST Username** and **Password** in the space provided and click **Login**.
  - Once inside the LEARN course environment, click on the link for **MATH 650 : Mathematical Modeling**.
  - Click on the **Submit** → **Quizzes** tab at the top of the page.
  - Click **Electronic Assignment 9**, and follow the instructions provided. An answer key for this assignment will appear where you can fill-in your solutions. Please email your instructor immediately if you encounter any problems.
  - Click on the **SUBMIT QUIZ** button when you are done. You have only 1 attempt to submit your solutions. Any assignment submitted after midnight (Waterloo, Ontario time) will be considered **late** and will not be counted toward your final grade (no exceptions).

Note that this assignment has 17 questions, 15 True/False and 2 Multiple Choice.

The following questions are based on Module 8, Sections 1 and 2

Part 1: True or False (1 mark each)

Indicate whether the following statements are true (a) or false (b).

1. The system $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} x\cos t + y^2 \\ \sqrt{x} - ty \end{pmatrix}$ is an autonomous nonlinear system in the x-y plane.

a. True

b. False

2. The non-linear system $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -x + y^2 \\ x^2 - y \end{pmatrix}$ has exactly two equilibrium points.

a. True

b. False

3. The graphical process of 'zooming' in on an equilibrium point $\mathbf{x}_0$ of a nonlinear system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ often reveals 'local' phase portraits similar to the phase portraits of linear systems.

a. True

b. False

4. In the two-species population model of Portrait 2 of Section 8.1 of the course lectures, if you changed the term $0.5yx$ in y' to $-0.5yx$, the model would represent competition rather than a predator-prey relationship.

a. True

b. False

5. The *basin of attraction* of the equilibrium point $(\frac{2}{3}, \frac{5}{3})$ in Portrait 2 is the set of points $\{(x, y) | \, x \geq 0, y \geq 0\}$.
[HINT: Look carefully at the phase portrait: for what ICs do the orbits approach $(\frac{2}{3}, \frac{5}{3})$?]

a. True

b. False

6. In Portrait 3 of Section 8.1 of the course lectures (the 'Fish'), the black curve is made up of three separate orbits linked together by the unstable equilibrium at $\mathbf{0}$.

a. True

b. False

7. There are no stable equilibrium points in the 'Fish' (the nonlinear system of Portrait 3 of Section 8.1).

a. True

b. False

8. The equilibrium at $\mathbf{0}$ of the 'Fish' (the nonlinear system of Portrait 3 of Section 8.1) locally resembles a saddle point.

a. True

b. False

9. A periodic orbit $\mathcal{C}$ which is the limit as $t \to \infty$ of all other non-constant orbits 'near' $\mathcal{C}$ is called an *asymptotically stable limit cycle*.

a. True

b. False

10. In Portrait 4 of Section 8.1 of the course lectures, several phase portraits of the the Van der Pol dynamical system are displayed. It is evident from those plots that both the width and the height of the limit cycle decrease as the parameter μ decreases.

a. True

b. False

11. If $\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \tan x + y \\ y \ln(x^2) \end{pmatrix}$, then we would expect the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ to have a unique solution through any point of the domain $\{(x, y) |\, 0 < x < \pi, y > 0\}$.

a. True

b. False

12. Suppose the constant b in the predator-prey model $N_1'(t) = r_1 N_1 - a\, N_1 N_2$, $N_2'(t) = -r_2 N_2 + b\, N_1 N_2$ has the value $b = 0.02$. This would mean that there is a 2% increase, per unit time, in the predator population N_2 for each prey N_1.

a. True

b. False

13. If a system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ represents an interaction between two species, and the orbits of this system are closed curves in the positive quadrant, then the model predicts periodic cycles for each species over time.

a. True

b. False

14. For the undamped nonlinear pendulum model $\dfrac{d^2\theta}{dt^2} + \omega_0^2 \sin\theta = 0$, the equilibrium at $\theta = 0$ is asymptotically stable.

a. True

b. False

15. The undamped pendulum of Example 8.2.2 simply oscillates about the equilibrium $\theta = 0$ no matter how great its kinetic energy.

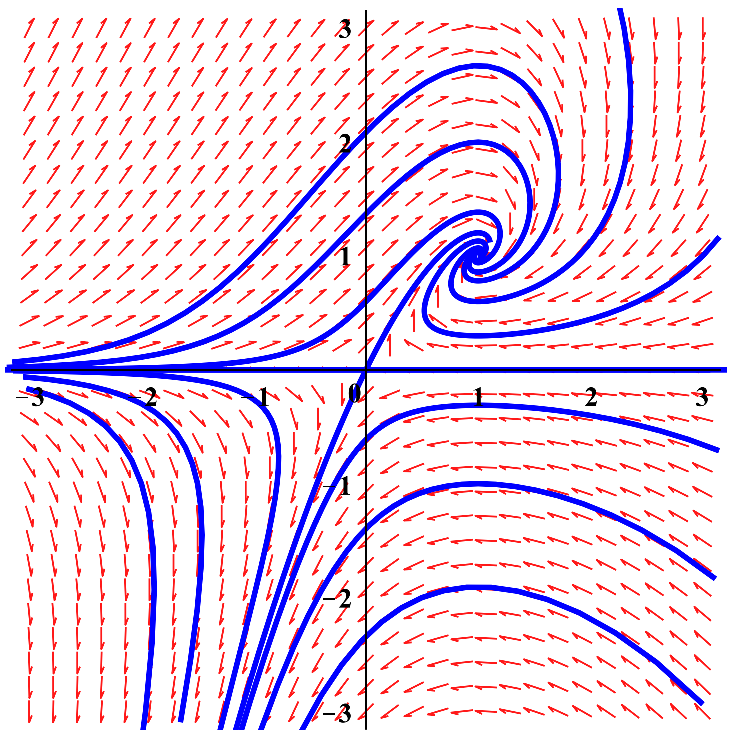
a. True

b. False

Part 2: Multiple Choice (1 mark each)

Choose the **best** answer for each question.

16. Consider the nonlinear system $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -x+y \\ -xy+y \end{pmatrix}$, for which a phase portrait is shown below.



a. The equilibrium points are $(0,0)$ and $(1,1)$.

b. Any orbit with initial data (x_0, y_0) where $y_0 > 0$ appears to approach $(1,1)$ as $t \to \infty$.

c. There are exactly two orbits which approach the equilibrium $\mathbf{0}$ as $t \to \infty$.

d. All of the above statements are true.

e. Two of the above statements are true.

17. Consider the nonlinear dynamical system $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 3x - x^2 \\ 2xy - 3y + 2 \end{pmatrix}$.

a. This system has two equilibrium points, $(0, \frac{2}{3})$ and $(0,0)$.

b. The orbits have horizontal tangents as they cross the curve $y = \dfrac{2}{2x-3}$.

c. There are two vertical orbits along the $y-$axis, each of which approaches the equilibrium $(0, \frac{2}{3})$. [HINT: If you set $x = 0$ in the DE, what can you say about all the tangents? What about the sign of y' along $x = 0$?]

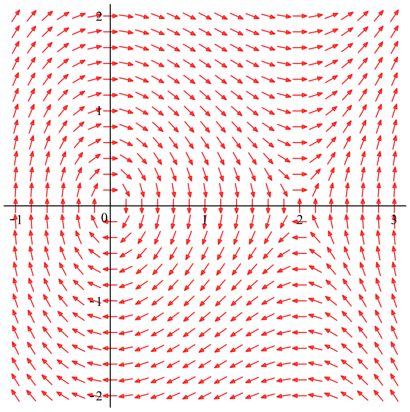
d. The orbits are contained within the family of curves $3xy - x^2y - 2x = C$.

[HINT: The system implies that $\dfrac{dy}{dx} = \dfrac{2xy - 3y + 2}{3x - x^2}$, an exact DE.]

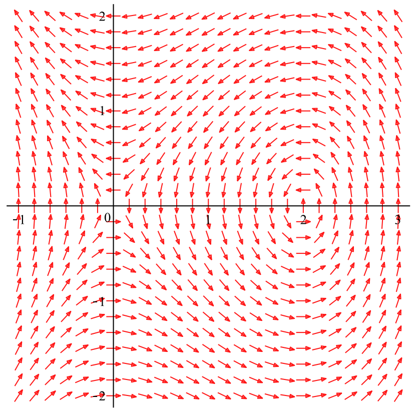
e. Two of the above statements are true.

18. Each of the direction fields shown below is that of one of the following nonlinear dynamical systems:

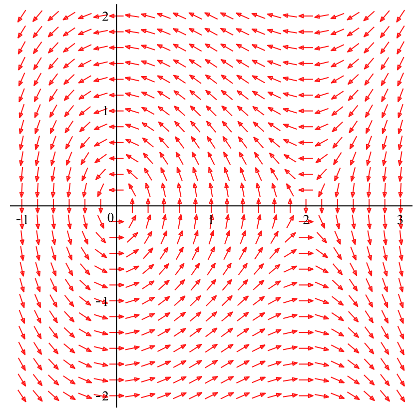
1. $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -y \\ 2x - x^2 \end{pmatrix}$ **2.** $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} y \\ x^2 - 2x \end{pmatrix}$ **3.** $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -y \\ x^2 - 2x \end{pmatrix}$



Direction Field A



Direction Field B



Direction Field C

A careful examination of the signs of $x'(t)$ and $y'(t)$ in the various regions of the plane for these direction fields reveals that

a. The x-axis contains all the equilibrium points of each of Systems 1., 2., and 3., and is the horizontal isocline of each system.

b. Systems 1., 2., 3., correspond to Direction Fields B, A, C, respectively.

c. Systems 1., 2., 3., correspond to Direction Fields C, A, B, respectively.

d. Systems 1., 2., 3., correspond to Direction Fields A, B, C, respectively.

e. Exactly two of the above statements are true.