MATH 650 : Mathematical Modeling

Spring, 2019

Electronic Assignment #7

Due by 11:59 p.m. EST on Tuesday, July 2, 2019

Instructions:

- Ensure you have reviewed Module 6, Section 3, and Module 7, Section 1 and any other activities therein. You will require this knowledge to answer the questions on this assignment.
- Read and think about the following assignment problems.
- Print and complete the following assignment. Record your answers on the printed copy so you have a record of your solutions.
- Once you are satisfied with your answers, submit your solutions online as follows:
  - Go to UW's course management website at `learn.uwaterloo.ca`
  - Enter your **QUEST Username** and **Password** in the space provided and click **Login**.
  - Once inside the LEARN course environment, click on the link for **MATH 650 : Mathematical Modeling**.
  - Click on the **Submit** → **Quizzes** tab at the top of the page.
  - Click **Electronic Assignment 7**, and follow the instructions provided. An answer key for this assignment will appear where you can fill-in your solutions. Please email your instructor immediately if you encounter any problems.
  - Click on the **SUBMIT QUIZ** button when you are done. You have only 1 attempt to submit your solutions. Any assignment submitted after midnight (Waterloo, Ontario time) will be considered **late** and will not be counted toward your final grade (no exceptions).

Note that this assignment has 20 questions, 13 True/False and 7 Multiple Choice.

The following questions are based on Module 6, Section 3 and Module 7, Section 1

Part 1: True or False (1 mark each)

Indicate whether the following statements are true (a) or false (b).

1. If two solutions $\mathbf{x_1}(t)$ and $\mathbf{x_2}(t)$ of the linear system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ have a non-zero Wronskian on an interval $\mathcal{I}$, then there exists a nonzero constant k such that $\mathbf{x_1}(t) = k\mathbf{x_2}(t)$ for all $t \in \mathcal{I}$.

a. True

b. False

2. The two functions $\mathbf{x_1}(t) = \begin{pmatrix} \dfrac{1}{t} \\ 0 \end{pmatrix}$ and $\mathbf{x_2}(t) = \begin{pmatrix} e^{-t}(1 + \dfrac{1}{t}) \\ -e^{-t} \end{pmatrix}$ have a zero Wronskian on the interval $(0, \infty)$.

a. True

b. False

3. If you are given that the functions $\mathbf{x_1}(t)$ and $\mathbf{x_2}(t)$ in question 2 are solutions of a linear system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, then the general solution of this system is $\mathbf{x}(t) = c_1 \begin{pmatrix} \dfrac{1}{t} \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 + \dfrac{1}{t} \\ -1 \end{pmatrix}$.

a. True

b. False

4. The equation $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}$ is a homogeneous linear system for the eigenvector components. Thus there are non-trivial solutions for any constant λ.

a. True

b. False

5. If a matrix $\mathbf{A}$ has a negative eigenvalue, then the equilibrium $\mathbf{0}$ of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is asymptotically stable.

a. True

b. False

6. A singular 2×2 matrix $\mathbf{A}$ always has eigenvalues $\lambda_1 = 0$, and $\lambda_2 = tr(\mathbf{A})$.

[HINT: Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find the eigenvalues, and apply the condition for singularity of $\mathbf{A}$.]

a. True

b. False

7. If the 2×2 matrix $\mathbf{A}$ has complex conjugate eigenvalues, then the complex solution $\mathbf{x}(t) = \mathbf{u}(t) + i\mathbf{v}(t)$ of the corresponding system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is such that the real-valued functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ form a linearly independent (*fundamental*) set of solutions.

a. True

b. False

8. If $\mathbf{A} = \alpha \mathbf{I}$ for some real constant $\alpha \neq 0$, then any vector $\mathbf{v}$ is an eigenvector, and the origin $\mathbf{0}$ is a *star point* or *proper node* of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

a. True

b. False

9. In the dimensional model $I\,\dfrac{d^2\theta}{dt^2} + c\,\dfrac{d\theta}{dt} + k\,\theta = 0$ of Example 7.1.1, the damping constant c has dimensions $\mathcal{M}\mathcal{L}^2\mathcal{T}^{-1}$.
[HINT: Recall that $[I] = \mathcal{M}\mathcal{L}^2$.]

a. True

b. False

10. The **dimensionless** swinging door model of Example 7.1.1, and the model $\theta'' + \omega_0^2\theta = 0$ for the simple pendulum, are each equivalent to a dynamical system of the form $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -a & -b \end{pmatrix}\mathbf{x}$ for positive constants a and b.

a. True

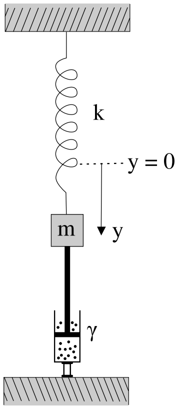
b. False

11. A 100 g mass on a spring with constant $k = 0.02\,\mathrm{kg\ s^{-2}}$ is set in motion from equilibrium with a downward velocity of $v_0 = 10\,\mathrm{cm\ s^{-1}}$.

It is also attached to a viscous damper which exerts a force of 3 N when the mass has velocity $3\ \mathrm{m\,s^{-1}}$. An appropriate IVP for the displacement y of the mass, downward from equilibrium, is

$$0.1\,\frac{d^2y}{dt^2} + 0.6\,\frac{dy}{dt} + 0.02\,y = 0\ \ , y(0) = 0,\ \ v(0) = 0.1\,.$$

[HINT: What does the given information tell you about the damping constant γ?]



a. True

b. False

12. In linearizing the models for the motion of a pendulum, both with and without damping, the approximation $\sin\theta \approx \theta$ is used. As long as the oscillations are 'small', i.e, $|\theta| \leq 0.25$ radians, the maximum error induced by this approximation is less than 0.0026.

a. True

b. False

13. The Existence Uniqueness Theorem for second order linear DEs guarantees that the IVP

$$t\,y'' + 2\,y' + e^t\,y = t\,\sin t\,,\ \ y(1) = y_0\,,\ \ y'(1) = v_0$$

will have a solution valid for all real t, for any real y_0 and v_0.

a. True

b. False

Part 2: Multiple Choice (1 mark each)

Choose the **best** answer for each question.

14. Suppose that $y_1(t) = e^{-t}\cos t$ and $y_2(t) = e^{-t}\sin t$ are two known solutions of a certain linear homogeneous second order DE.

- **a.** The Wronskian $W[y_1, y_2]$ of the two functions is equal to zero.
- **b.** The two solutions are linearly dependent.
- **c.** Every solution of this DE is unbounded as $t \to \infty$.
- **d.** The solution with initial conditions $y(0) = 1$ and $y'(0) = 0$ is $y(t) = e^{-t}(\cos t + \sin t)$.
- **e.** None of the above

In problems 15 to 18, use the eigenvector method to solve the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the given matrix $\mathbf{A}$. Then decide which of the given statements is true.

15. $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- **a.** The eigenvector associated with the eigenvalue $\lambda = 3$ is $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
- **b.** The origin is a saddle point of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- **c.** The particular solution with ICs $\mathbf{x}(0) = (0, 1)^T$ is $\mathbf{x}(t) = \frac{1}{2}\begin{pmatrix} e^{3t} - e^t \\ e^{3t} + e^t \end{pmatrix}$.
- **d.** The tangent to the solution through $(1, 0)$, at that point, is $(\frac{dx}{dt}, \frac{dy}{dt}) = (2, 2)$.
- **e.** None of the above

16. $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$.

- **a.** The eigenvectors of $\mathbf{A}$ are mutually orthogonal (perpendicular).
- **b.** The origin is a saddle point of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- **c.** The general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is $\mathbf{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- **d.** As $t \to \infty$, the slopes of nearly all solutions approach 2.

  [Hint: Recall that the slope of $\mathbf{x}(t) = (x_1(t), x_2(t))^T$ is just the ratio $\dfrac{x_2'(t)}{x_1'(t)}$.]
- **e.** Only two of the above statements are true.

17. $\mathbf{A} = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$.

a. The tangent field of the orbits is vertical along the line $x_2 = x_1$.

b. The eigenvalues of $\mathbf{A}$ are $\lambda = 1 \pm 2\,i$.

c. The orbits of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ wind clockwise about the origin.

d. The orbits of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are spirals winding counterclockwise into the asymptotically stable equilibrium at the origin.

e. None of the above

18. $\mathbf{A} = \begin{pmatrix} -1 & -\frac{1}{2} \\ 2 & -3 \end{pmatrix}$.

a. The matrix $\mathbf{A}$ has repeated eigenvalues.

b. The origin is an asymptotically stable improper node of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

c. The general solution is $\mathbf{x}(t) = e^{2t}\begin{pmatrix} c_1 + c_2 t \\ 2c_1 - 2c_2 + 2c_2 t \end{pmatrix}$.

d. All nonzero solutions approach the equilibrium at $\mathbf{0}$ tangent to the line $x_2 = 2x_1$.

e. Exactly one of the above statements is incorrect.

19. Examine the four diagrams below and compare them to the results you found for the systems in questions 15 - 18. Then decide which of the following statements are true.

[Hint: Be careful! Check the stability, the directions of the eigenvectors, the nullclines of the direction field, and the direction of increasing t along the orbits.]

a. Diagram 1 is a phase portrait for the system in question 16.

b. Diagram 2 is a phase portrait for the system in question 17.

c. Diagram 3 is a phase portrait for the system in question 18.

d. Diagram 4 is a phase portrait for the system in question 15.

e. All of the above

Diagram 1

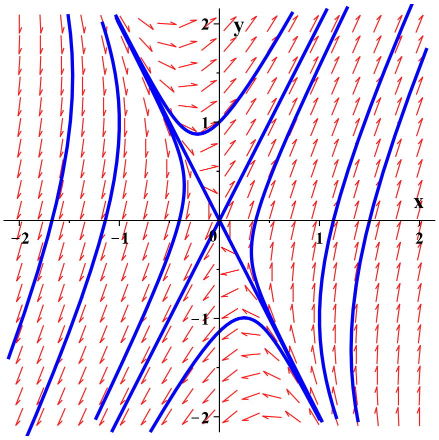


Diagram 2

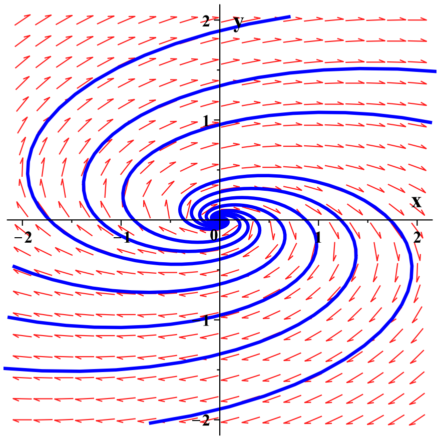


Diagram 3

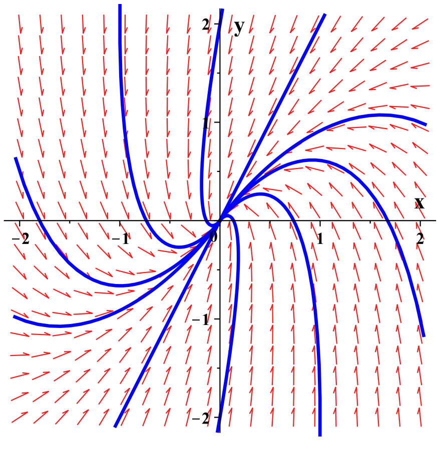
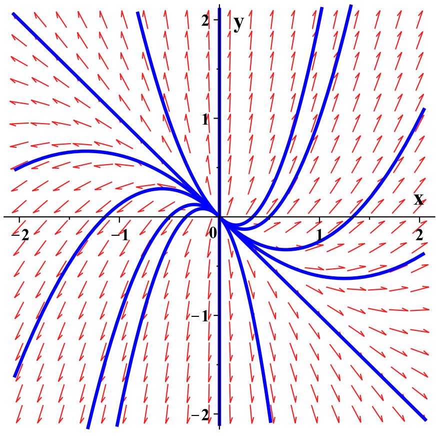


Diagram 4



20. Consider the following statements regarding the equilibrium points of the system $\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \mathbf{x}$.

1. If $\alpha < 0$, then $\mathbf{0}$ is a spiral sink.
2. The equilibrium $\mathbf{0}$ is stable for all values of α.
3. If $\alpha = 0$, then $\mathbf{0}$ is a stable centre.
4. If $\alpha > 1$, then $\mathbf{0}$ is an unstable saddle.
5. If $0 < \alpha < 1$, then $\mathbf{0}$ is an asymptotically stable node.
6. If $\alpha = 1$, then every point on the line $x_2 = -x_1$ is an equilibrium point, and all orbits are half-lines with slope 1.

a. Statements 1, 2, 4, and 6 are true.

b. Statements 2, 3, 5, and 6 are true.

c. Statements 1, 3, 4, and 5 are true.

d. Statements 1, 4, 5, and 6 are true.

e. None of the above