MATH 650 : Mathematical Modeling

Spring, 2019

Electronic Assignment #6

Due by 11:59 p.m. EST on Tuesday, June 25, 2019

Instructions:

- Ensure you have reviewed Module 6, Sections 1 and 2, and any other activities therein. You will require this knowledge to answer the questions on this assignment.
- Read and think about the following assignment problems.
- Print and complete the following assignment. Record your answers on the printed copy so you have a record of your solutions.
- Once you are satisfied with your answers, submit your solutions online as follows:
  - Go to UW's course management website at `learn.uwaterloo.ca`
  - Enter your **QUEST Username** and **Password** in the space provided and click **Login**.
  - Once inside the LEARN course environment, click on the link for **MATH 650 : Mathematical Modeling**.
  - Click on the **Submit** → **Quizzes** tab at the top of the page.
  - Click **Electronic Assignment 6**, and follow the instructions provided. An answer key for this assignment will appear where you can fill-in your solutions. Please email your instructor immediately if you encounter any problems.
  - Click on the **SUBMIT QUIZ** button when you are done. You have only 1 attempt to submit your solutions. Any assignment submitted after midnight (Waterloo, Ontario time) will be considered **late** and will not be counted toward your final grade (no exceptions).

Note that this assignment has 20 questions, 15 True/False and 5 Multiple Choice.

The following questions are based on material in Module 6, Sections 1 and 2

Part 1: True or False (1 mark each)

Indicate whether the following statements are true (a) or false (b).

1. All solutions of the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$, with $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, lie on the line $x = 2y$.

a. True

b. False

2. Cramer's Rule for solving the nonhomogeneous system $\mathbf{Ax} = \mathbf{b}$ yields the solution for any matrix $\mathbf{A} \neq \mathbf{0}$.

a. True

b. False

3. The system $\mathbf{Ax} = \begin{pmatrix} \frac{1}{3} & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$ has no solutions because the matrix $\mathbf{A}$ is singular.

a. True

b. False

4. The matrix $\mathbf{A} = \begin{pmatrix} 1 & -\alpha \\ 2\alpha & 3 \end{pmatrix}$ has complex eigenvalues if $|\alpha| > \dfrac{1}{\sqrt{2}}$.)

a. True

b. False

5. The eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 4 & -6 \\ 1 & -3 \end{pmatrix}$ are $\mathbf{v_1} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$, and $\mathbf{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

a. True

b. False

6. For the Beauti-Bowl problem of Example 6.2.1, suitable dimensionless variables for $\mathbf{Q} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ might be $x = \dfrac{Q_1}{25\,000}$ and $y = \dfrac{Q_2}{10\,000}$.

a. True

b. False

7. The system of DEs $\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ has a critical point (equilibrium solution) $\mathbf{x_e} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

a. True

b. False

8. The equilibrium point of $\mathbf{x}' = \mathbf{Ax} + \mathbf{b}$ is $\mathbf{x_e} = -\mathbf{A^{-1}b}$ for any matrix $\mathbf{A} \neq \mathbf{0}$.

a. True

b. False

9. In Example 6.2.1 (Beauti-Bowl Mixing), the system of DEs for the quantities $\mathbf{Q} = \begin{pmatrix} Q_1(t) \\ Q_2(t) \end{pmatrix}$ of Beauti-Bowl in the tank and bowl, respectively, predicts an equilibrium state $\mathbf{Q_e} = \begin{pmatrix} 2.5 \\ 1 \end{pmatrix}$ ml. However, the equilibrium level of the concentration of Beauti-Bowl is the same in both tank and bowl, namely 0.0001 (or 0.01 %).

a. True

b. False

10. The horizontal nullcline of the system $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ (i.e., the set of points where the tangents to the orbits are horizontal) is the line $y = 5x$.

a. True

b. False

11. For a constant matrix $\mathbf{A}$, a solution $\mathbf{x}(t)$ of the IVP $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x_0}$, can be envisioned as a parametric curve through the point $\mathbf{x_0}$ with tangent vector $(\frac{dx}{dt}, \frac{dy}{dt})^T = \mathbf{A}(x(t), y(t))^T$ at each point along the curve.

a. True

b. False

12. If the system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ has an equilibrium $\mathbf{x_e}$ which is a stable centre, the orbits of the system do not approach $\mathbf{x_e}$, but rather are unbounded as $t \to \infty$.

a. True

b. False

13. In the Beauti-Bowl model of Example 6.2.1 of the course lectures, there is *Conservation of Mass*, while in the simple pendulum model of Example 6.2.2, the *total mechanical energy* (kinetic plus potential) is conserved. [HINT: Friction is assumed negligible in the latter model.]

a. True

b. False

14. The Existence-Uniqueness Theorem predicts that the IVP $\mathbf{x}' = \begin{pmatrix} \frac{1}{\sqrt{t}} & 1 \\ -1 & \frac{1}{\sqrt{t-2}} \end{pmatrix}\mathbf{x}$, $\mathbf{x}(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ will have a unique solution on the interval $t > 2$. [HINT: What is the value of t_0 in this IVP?]

a. True

b. False

15. Direction field analysis reveals that the orbits of the linear autonomous system $\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 5 & -3 \end{pmatrix}\mathbf{x}$ could be closed, bounded ellipses or clockwise spirals.

a. True

b. False

Part 2: Multiple Choice (1 mark each)

Choose the **best** answer for each question.

16. If the flow (leak) rate in the Beauti-Bowl model of Example 6.2.1 is 2 litres per hour instead of 1 litre per hour, then the system of DEs for Q_1 and Q_2 will change. Suppose the tank already contains 0.05 ml of Beauti-Bowl at $t = 0$.

a. The revised IVP for Q_1 is $Q_1' = 0.1 - 0.08Q_1$, $Q_1(0) = 0.05\,\text{ml}$.

b. The revised DE for Q_2 is $Q_2' = 0.08Q_1 - 0.2Q_2$.

c. The new equilibrium levels are exactly twice those of the original equilibrium state.

d. All of the above

e. Statements a. and b. are true.

17. For the linear autonomous system $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}\mathbf{x}$,

a. The vertical nullcline is $y = 2x$.

b. In the region $2x < y < 0$, the tangent field is sloped up and to the left.

c. For $y > 0$, no orbits have horizontal tangents.

d. Direction field analysis predicts that the orbits could be counter-clockwise spirals.

e. None of the above

18. In Diagram 1 below are four orbits of a linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. In Diagram 2 are partial component plots $x(t)$ and $y(t)$ which may correspond to one of those orbits. If they correspond to one of the orbits, decide which one.

[Hint: Wherever the two component plots intersect (e.g., both have value -1 at around $t = 21$), you know that $x(t) = y(t)$ must be on the matching orbit (e.g., (-1,-1)). Where else does this happen?]

a. Orbit A.

b. Orbit B.

c. Orbit C.

d. Orbit D.

e. None of the above

Diagram 1

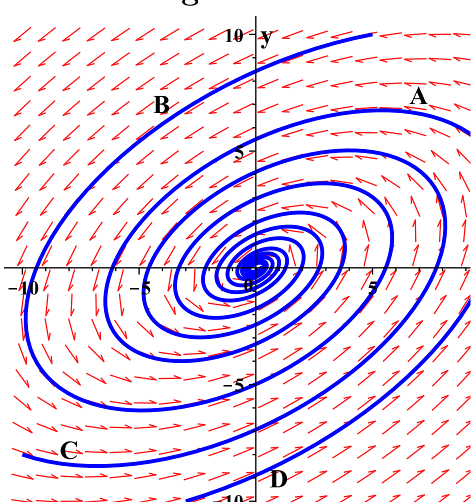
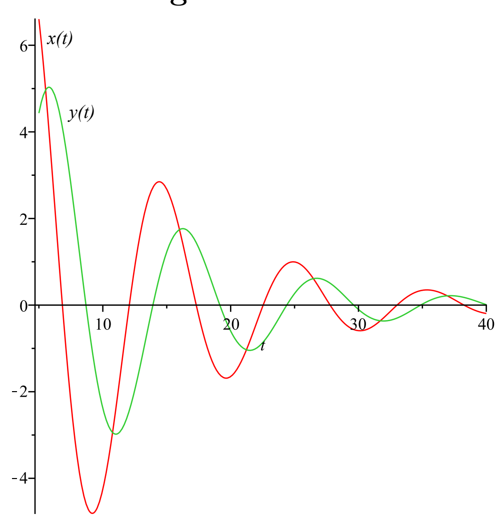


Diagram 2



19. Suppose competition occurs between two otherwise thriving species X and Y, with populations $x(t)$ and $y(t)$ respectively. A very simple model for this interaction would be

$$\begin{aligned}\frac{dx}{dt} &= ax - by \\ \frac{dy}{dt} &= -cx + dy\end{aligned},$$

where a, b, c, and d are positive constants.

[HINT: For parts c. and d., a sketch and direction field analysis will be very helpful.]

a. The constant b is the loss inflicted on population X per unit time by each member of population Y.

b. If $a = 0.1$, $b = 0.4$, $c = 0.3$, and $d = 0.2$, then the tangent vectors along the vertical nullcline point downward for $x > 0$.

c. With the constants a, b, c, and d as in part b., if the populations have positive initial values on the vertical nullcline, the direction field indicates that species Y will survive.

d. Two of statements a., b., c., are true.

e. All of the statements a., b., c., are true.

20. The MIT Mathlet demonstration reveals the following about possible phase portraits of the system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix} \mathbf{x} \, :$$

a. They include both asymptotically stable and unstable nodes.

b. With $d = -4$, as c changes from negative to positive, the equilibrium changes from an asymptotically stable node to an unstable saddle.

c. The phase portrait never consists entirely of straight lines.

d. With $c = -1.75$, as d passes through 0 from negative to positive, a stable centre occurs as the orbits transition from inward- to outward-winding spirals.

e. Exactly three of the above statements are correct.