MATH 650 : Mathematical Modeling

Spring, 2019

Electronic Assignment #5

Due by 11:59 p.m. EST on Thursday, June 18, 2019

Instructions:

- Ensure you have reviewed Module 5, Sections 1,2, and 3, and any other activities therein. You will require this knowledge to answer the questions on this assignment.
- Read and think about the following assignment problems.
- Print and complete the following assignment. Record your answers on the printed copy so you have a record of your solutions.
- Once you are satisfied with your answers, submit your solutions online as follows:
  - Go to UW's course management website at `learn.uwaterloo.ca`
  - Enter your **QUEST Username** and **Password** in the space provided and click **Login**.
  - Once inside the LEARN course environment, click on the link for **MATH 650 : Mathematical Modeling**.
  - Click on the **Submit** → **Quizzes** tab at the top of the page.
  - Click **Electronic Assignment 5**, and follow the instructions provided. An answer key for this assignment will appear where you can fill-in your solutions. Please email your instructor immediately if you encounter any problems.
  - Click on the **SUBMIT QUIZ** button when you are done. You have only 1 attempt to submit your solutions. Any assignment submitted after midnight (Waterloo, Ontario time) will be considered **late** and will not be counted toward your final grade (no exceptions).

Note that this assignment has only 17 questions.
(Some relief after all your hard work last week :-))

The following questions are based on material in Module 5, Sections 1, 2, and 3

Part 1: True or False (1 mark each)

Indicate whether the following statements are true (a) or false (b).

1. The dimensions of *derived* quantities are products of powers of the dimensions of *fundamental* quantities.

a. True

b. False

2. The *Principle of Dimensional Homogeneity* states that quantities with the same dimensions can be multiplied or compared.

a. True

b. False

3. Characteristic values used to scale dimensional variables in order to obtain dimensionless variables must be positive constants.

a. True

b. False

4. A body falling with velocity v is subject to a drag force of magnitude βv^2. Then $[\beta] = \mathcal{MT}^{-1}$.

a. True

b. False

5. The dimensionless model $y = -\lambda(\tau - 1)^2 + 1 + \lambda$ for vertical motion (as obtained in Example 1.3.2 and revisited in Module 5) clearly always has a maximum at $\tau = 1$. This corresponds in the dimensional model to the maximum height $s = \dfrac{v_0^2}{2g} + s_0$ achieved by the object at time $t = \dfrac{v_0}{g}$.

a. True

b. False

6. The dimensions of the kinetic energy of a body moving vertically near the surface of the earth are the same as the dimensions of its gravitational potential energy.

a. True

b. False

7. Assume we have a complete list of all parameters relevant to a given problem. Then creating the dimensional matrix $\mathcal{D}$ and applying the Pi Theorem allows us to predict the number of *essential* dimensionless parameters.

a. True

b. False

8. The dimensionless variables for a given problem are not generally unique (i.e., there may be more than one way to choose suitable dimensionless variables).

a. True

b. False

9. In a problem with 6 dimensional quantities having dimensions involving only length and time, the dimensional matrix will be a 6×2 matrix.

a. True

b. False

10. If we transform the dimensional model $\frac{d^2s}{dt^2} = -g,\ v(0) = v_0,\ s(0) = s_0$ to dimensionless variables $y = \frac{s}{s_0},\ \tau = \frac{t}{v_0/g}$, the initial conditions become $y(0) = 1\,,\ \frac{dy}{d\tau}|_{\tau=0} = 2\,\lambda\,$, where $\lambda = \frac{v_0^2}{2gs_0}\,.$

a. True

b. False

11. A certain problem involves 5 functionally related dimensional quantities $Q_1,\ Q_2, ...Q_5\,.$ If you discover that the rank of the dimensional matrix is 3, then that functional relationship can be expressed in terms of two dimensionless quantities.

a. True

b. False

12. For any problem involving n dimensional quantities with dimensions involving mass, length, and time, the Pi Theorem predicts the existence of $n - 3$ dimensionless quantities.

a. True

b. False

Part 2: Multiple Choice (1 mark each)

Choose the **best** answer for each question.

13. In the model for population growth $\frac{dp}{dt} = rp - K\,,\ p(0) = p_0 > 0\,,$

a. $y = \frac{p}{p_0}\,,\ \tau = rt$ are suitable dimensionless variables.

b. any power of $\lambda = \frac{K}{rp_0}$ is dimensionless.

c. the dimensionless form of the solution is $y = \lambda + (1 - \lambda)e^{\tau}\,.$

d. if $K > rp_0\,,$ the population becomes extinct.

e. All of the above

14. You decide to see whether you can deduce Toricelli's law using dimensional analysis, with dimensional quantities $h,\ v,\ g,$ and the density ρ of the fluid in the tank.

a. The dimensional matrix is a 3×4 matrix of rank 2.

b. The dimensional matrix reveals that the density ρ is not relevant in this context.

c. The Pi Theorem predicts 2 dimensionless quantities.

d. The Pi Theorem predicts that $h,\ v,\ g,$ are related by an equation of the form $v^2 = C\,g\,h\,,$ where C is a dimensionless constant.

e. Only two of the above statements are correct.

15. According to the result in Example 5.3.1, the period of a simple pendulum depends on its length ℓ, the initial displacement θ_0, and the local gravitational acceleration g, according to $T = \sqrt{\frac{\ell}{g}}\, f(\theta_0)$ for some function f. Here are some values of g in various locations: Earth - 9.8 m s^{-2}; Moon - 1.625 m s^{-2}; Mars - 3.728 m s^{-2}; Jupiter - 25.93 m s^{-2}; Uranus - 9.01 m s^{-2}.

In each of the following statements, assume a fixed θ_0 for all pendulums (i.e., $f(\theta_0)$ is constant).

[HINT: First show that for two locations 1 and 2, the periods are related by $\frac{T_1}{T_2} = \sqrt{\frac{g_2}{g_1}\,\frac{\ell_1}{\ell_2}}$.]

a. If two clocks on Earth have pendulums of length $\ell_1 = 72\,\text{cm}$ and $\ell_2 = 18\,\text{cm}$ respectively, then the period one is 4 times the period of the other.

b. A pendulum on the Moon has a period of about 2.5 times as long as the same pendulum on Earth.

c. For a given pendulum, the longest period of any of the five locations would occur on Jupiter.

d. For a pendulum on Uranus to have a period twice as long as a pendulum on Mars, the pendulum on Uranus would have to be about twice as long as the pendulum on Mars.

e. Only two of the above statements are correct.

16. An infestation of borers is spreading in a forest with a total biomass of T ash trees according to the DE $\frac{dm}{dt} = \beta m(T - m)$, where m is the biomass of infected trees, β is a positive constant, and t is time.

a. The variable defined by $y = \frac{m}{T}$ would be a suitable dimensionless measure of the infected trees.

b. The constant β has dimensions $\mathcal{T}^{-1}$.

c. The Pi Theorem predicts that there exists an equivalent DE involving two dimensionless variables.

d. If even one tree becomes infected, then the whole forest eventually becomes infected.

e. One of the above statements is not true.

17. You are cooking a roast of volume V at oven temperature T_a. The time t for it to reach a final temperature T_f depends on its initial temperature T_i and a physical constant κ of the roast, called the *thermal diffusivity*, which has dimensions $\mathcal{T}^{-1}\mathcal{L}^2$. Thus the dimensions involved in this problem are temperature $\mathcal{U}$, length $\mathcal{L}$, and time $\mathcal{T}$.

a. The 3×6 dimensional matrix of the quantities V, κ, t, T_a, T_i, T_f has rank 2.

b. The Pi Theorem predicts that there are only four essential dimensionless ratios among the six dimensional quantities.

c. Two of the dimensionless ratios are $\pi_1 = \frac{T_i}{T_f}$ and $\pi_2 = \frac{t\,\kappa}{V^{1/3}}$.

d. The cooking time t is proportional to $V^{2/3}$.

e. All of the above