MATH 650 : Mathematical Modeling

Spring, 2019

Electronic Assignment #4

Due by 11:59 p.m. EST on Tuesday, June 4, 2019

Instructions:

- Ensure you have reviewed Module 3, Sections 2 and 3, and any Text Readings and other activities therein. You will require this knowledge to answer the questions on this assignment.
- Read and think about the following assignment problems.
- Print and complete the following assignment. Record your answers on the printed copy so you have a record of your solutions.
- Once you are satisfied with your answers, submit your solutions online as follows:
  - Go to UW's course management website at `learn.uwaterloo.ca`
  - Enter your **QUEST Username** and **Password** in the space provided and click **Login**.
  - Once inside the LEARN course environment, click on the link for **MATH 650 : Mathematical Modeling**.
  - Click on the **Submit** → **Quizzes** tab at the top of the page.
  - Click on **Electronic Assignment 4**, and follow the instructions provided. An answer key for this assignment will appear where you can fill-in your solutions. Please email your instructor immediately if you encounter any problems.
  - Click on the **SUBMIT QUIZ** button when you are done. You have only 1 attempt to submit your solutions. Any assignment submitted after midnight (Waterloo, Ontario time) will be considered **late** and will not be counted toward your final grade (no exceptions).

Note that this assignment has fewer questions than previous assignments.

BUT there's more thinking required!

The following questions are based on material in Module 3, Sections 2 and 3

Part 1: True or False (1 mark each)

Indicate whether the following statements are true (a) or false (b).

1. A smooth, dense object of cross-sectional area A falling at velocity v through a medium of density ρ is subject to a resistive force of magnitude $\rho A v^2$.

a. True

b. False

2. The antiderivative $\int \frac{1}{a^2+x^2}\,dx = \arctan\left(\frac{x}{a}\right)$.

a. True

b. False

3. Assuming positive displacement upward, the appropriate DE for an object of mass m falling downward under the influence of gravity and Newtonian drag is $m\frac{dv}{dt} = -mg + \beta v^2$, where β is a positive constant.

a. True

b. False

4. In general, the actual escape velocity from the surface of the Earth is less than $\sqrt{2gR}$, where R is the radius of the Earth and g is the gravitational acceleration.

a. True

b. False

5. The fact that $f(y) = y^{1/3}$ is continuous on $\Re^2$ guarantees that the DE $y'(t) = y^{1/3}$ has a unique solution through any point (t_0, y_0) in $\Re^2$.

a. True

b. False

6. A horizontal translation of any solution of an autonomous DE $y'(t) = f(y)$ gives another solution of the same DE.

a. True

b. False

7. All equilibrium solutions of the DE $y' = c - y^2$ are either asymptotically stable, or unstable, regardless of the value of the parameter c.

a. True

b. False

8. According to the E-U Theorem, the domain of validity of the solution of the IVP $(t-2)y' - (\ln t)\, y = t\,,\ y(1) = 1$ is $0 < t < 2$.

a. True

b. False

9. A model for the growth of a population $p(t)$ is given by $\frac{dp}{dt} = kp(M-p)(p-N)$, where $k > 0$ and $0 < N < M$. This model results in the same long-term behaviour as logistic growth with carrying capacity M, as long as the initial population is above the *threshold* N.

a. True

b. False

Part 2: Multiple Choice (1 mark each)

Choose the **best** answer for each question.

10. For a body moving upward, acted on by gravity and Newtonian drag, the appropriate DE for velocity is $\dfrac{dv}{dt} = -g - \dfrac{\beta}{m}v^2$; let the time to reach maximum height be t^*. On the other hand, for negligible drag, the DE for velocity is $\dfrac{dv}{dt} = -g$; let the time to reach maximum height be t^{**}.
Decide which of the following is true.
[HINT: Compare the two expressions $\dfrac{dv}{dt} = -g$, and $\dfrac{dv}{dt} = -g - \dfrac{\beta}{m}v^2$ and think about what this tells you about the velocity graphs. Based on your thoughts, make a rough sketch of each $v(t)$ on the same diagram, with $v(0) = v_0$ for both; recall that at maximum height, the velocity is $v = 0$.]

a. $t^{**} > t^*$

b. $t^{**} = t^*$

c. $t^{**} < t^*$

d. The two times can't be compared

e. None of the above

11. If a spaceship is launched from an altitude of 200 km, then

a. the drag force due to air resistance is greatly reduced.

b. the gravitational force is less than g, per unit mass.

c. the escape velocity is less than at the Earth's surface.

d. All of the above

e. None of the above

12. Consider a population $y(t)$ which obeys the IVP $y'(t) = f(y) = -0.1(1 - \dfrac{y}{10})\, y$, $y(0) = y_0$. Assume that you have found that $y'' = f(y)f'(y) = 0.01y\,(\dfrac{y}{5} - 1)(\dfrac{y}{10} - 1)$.
Use this information and qualitative analysis to decide which statements, if any, are true.

a. The population thrives for all $y_0 > 10$.

b. The solution graphs are decreasing and concave up in the region $5 < y < 10$.

c. For all $y_0 < 10$, extinction occurs in finite time.

d. The long-term behaviour of the solutions depends continuously on the initial population y_0. [HINT: What happens for values of y_0 near $y = 10$?]

e. None of the above

13. For the IVP $\dfrac{dy}{dt} + \dfrac{1}{t}y = \dfrac{e^t}{t-2}$, $y(t_0) = y_0$, WITHOUT solving the DE (simply apply the E-U Theorem), decide which of the following is true.

- **a.** A solution exists for any t_0 in $\Re$.
- **b.** With IC $y(1) = 2$, the solution $y = \phi_1(t)$ will have domain of validity $0 < t < 2$.
- **c.** With IC $y(-1) = 2$, the solution $y = \phi_2(t)$ will have domain of validity $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$.
- **d.** All of the above
- **e.** None of the above

14. For the IVP $y'(t) = -3t^2y^2$, $y(0) = y_0$ (Module 3, Example 3.3.3):

- **a.** if $y_0 = 0$, there is a vertical asymptote at t = $y_0^{-1/3}$.
- **b.** as $y_0 \to \infty$, solutions of this IVP approach $y = \dfrac{1}{t^3}$, which is also a solution of the DE.
- **c.** no solution of the given IVP, for any y_0, is defined for all t in $\Re$.
- **d.** All of the above
- **e.** None of the above

15. For the IVP $y'(t) = f(t, y) = \dfrac{2 + \sin t}{y - 5}$, $y(t_0) = y_0$:

- **a.** solutions exist for all t_0 in $\Re$.
- **b.** the numerator of $y'(t)$ satisfies $2 + \sin t \geq 1$ for all real t.
- **c.** $f(t, 5)$ is undefined for all t in $\Re$.
- **d.** No solution exists for any t_0 if $y_0 = 5$.
- **e.** All of the above

16. Solve the IVP $y'(t) = f(t, y) = -ty^3$, $y(0) = y_0$ to decide which of the following statements are true.

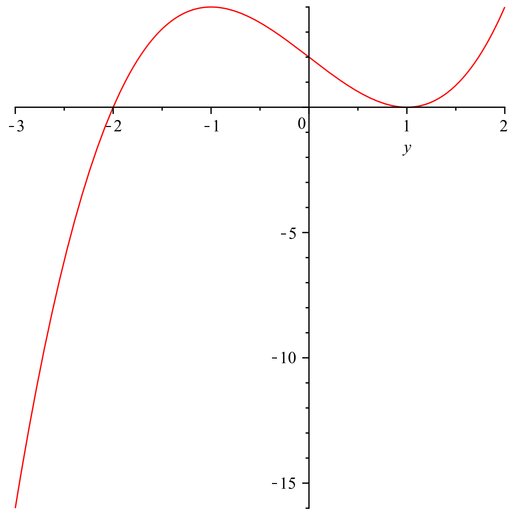
- **a.** The E-U Theorem predicts that a unique solution exists through any point for which $y_0 \neq 0$.
- **b.** The DE has exactly one equilibrium solution.
- **c.** For any $y_0 \neq 0$, the solution approaches the equilibrium solution as $t \to \infty$ and as $t \to -\infty$.
- **d.** Each solution with $y_0 \neq 0$ has domain of validity $-\frac{1}{y_0} < t < \frac{1}{y_0}$.
- **e.** Exactly three of the above statements are true.

17. Solve the IVP $y'(t) = f(t, y) = -\dfrac{t}{4y}$, $y(0) = y_0$ to decide which of the following statements are true.

- **a.** The integral curves are ellipses with semi-major axis $2|y_0|$ and semi-minor axis $|y_0|$.
- **b.** Each value of $y_0 > 0$ determines a unique solution with domain of validity $-2y_0 < t < 2y_0$.
- **c.** Each value of $y_0 < 0$ determines a unique solution with domain of validity $-2y_0 < t < 2y_0$.
- **d.** The solutions depend continuously on the IC y_0 for $y_0 > 0$.
- **e.** All of the above statements are true.

18. At right is a sketch of a function $f(y)$. Below are four plots of solutions, one of which shows solutions of the DE $\dfrac{dy}{dt} = f(y)$ for this function. Decide which is the correct diagram.

Plot of $f(y)$ versus y



a. Diagram 1

b. Diagram 2

c. Diagram 3

d. Diagram 4

e. None of the above

Diagram 1

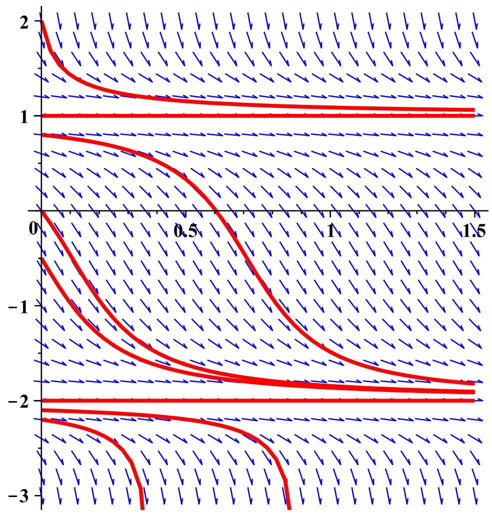


Diagram 2

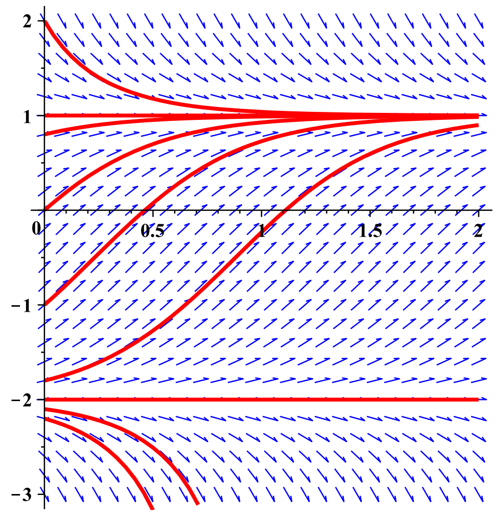


Diagram 3

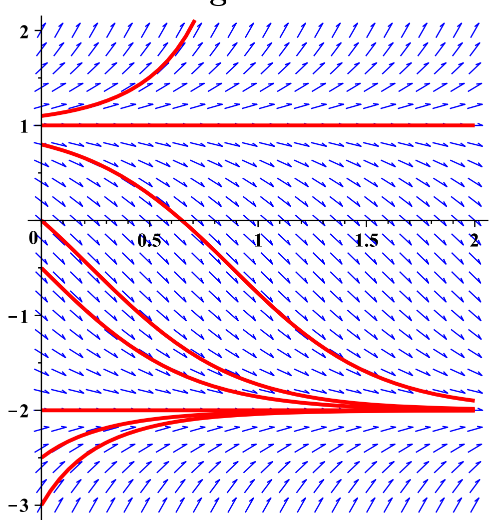


Diagram 4