MATH 650 : Mathematical Modeling

Spring, 2019

Electronic Assignment #3

Due by 11:59 p.m. EST on Tuesday, May 28, 2019

Instructions:

- Ensure you have reviewed Module 2, Section 2, Module 3, Section 1, and any Text Readings and other activities therein. You will require this knowledge to answer the questions on this assignment.
- Read and think about the following assignment problems.
- Print and complete the following assignment. Record your answers on the printed copy so you have a record of your solutions.
- Once you are satisfied with your answers, submit your solutions online as follows:
  - Go to UW's course management website at `learn.uwaterloo.ca`
  - Enter your **QUEST Username** and **Password** in the space provided and click **Login**.
  - Once inside the LEARN course environment, click on the link for **MATH 650 : Mathematical Modeling**.
  - Click on the **Submit** → **Quizzes** tab at the top of the page.
  - Click on **Electronic Assignment 3**, and follow the instructions provided. An answer key for this assignment will appear where you can fill-in your solutions. Please email your instructor immediately if you encounter any problems.
  - Click on the **SUBMIT QUIZ** button when you are done. You have only 1 attempt to submit your solutions. Any assignment submitted after midnight (Waterloo, Ontario time) will be considered **late** and will not be counted toward your final grade (no exceptions).

The following questions are based on material in Module 2, Section 2, and Module 3, Section 1

Part 1: True or False (1 mark each)

Indicate whether the following statements are true (a) or false (b).

1. To develop a valid mathematical model for a specified phenomenon requires a thorough understanding of what physical variables and parameters are essential, and what existing laws/principles govern their relation to one another.

a. True

b. False

2. If a model fails to predict what we observe, one possibility is to conduct experiments to obtain more realistic parameter values, and hence revise the model.

a. True

b. False

3. A model can be approximated by simplifying the assumptions and/or finding a solution numerically.

a. True

b. False

4. It is more economical to collect real data and perform laboratory experiments than to conduct numerical simulations which convey the effects of varying the parameters in a problem.

a. True

b. False

5. It is physically accurate to assume homogeneous mixing of pollutants in a lake.

a. True

b. False

6. In a tank problem with equal inflow and outflow rate $r_i = r_o = r$, input concentration c_i of a toxic substance, and total volume V_0 of mixture in the tank, the appropriate DE for the quantity Q of toxin in the tank is $\dfrac{dQ}{dt} = r(c_i - \dfrac{Q}{V_0})$.

a. True

b. False

7. The model for Sunken Treasure (Example 2.2.3 of Module 2) assumes that the water exerts a Newtonian drag force on the chest and barrel.

a. True

b. False

8. The curve $x^2 + y^2 = 4$ is the solution of the IVP $\dfrac{dy}{dx} = -\dfrac{x}{y}$, $y(0) = 2$.

a. True

b. False

9. The domain (interval of validity) of the solution of the IVP $\dfrac{dy}{dx} = -\dfrac{x}{y}$, $y(0) = 3$ is $-3 \leq x \leq 3$.

a. True

b. False

10. A significant difference between the falling raindrop and the rising rocket is that the raindrop gains both mass and momentum, while the rocket loses mass but gains momentum.

- **a.** True
- **b.** False

11. Every point on an integral curve of the DE $M(x) + N(y)\,\dfrac{dy}{dx} = 0$ is also a point on an explicit solution of $\dfrac{dy}{dx} = -\dfrac{M(x)}{N(y)}$.

- **a.** True
- **b.** False

12. Every integral curve $x^3 - 3y + y^3 = C$ of the separable DE $x^2 + (y^2 - 1)\,y' = 0$ contains exactly two explicit solutions of the DE $\dfrac{dy}{dx} = \dfrac{x^2}{1 - y^2}$.

- **a.** True
- **b.** False

Part 2: Multiple Choice (1 mark each)

Choose the **best** answer for each question.

13. For an object of density ρ and volume V in a fluid of density ρ_f, Archimedes Law of Buoyancy implies that, if the object is placed gently in the fluid, then

- **a.** if $\rho > \rho_f$, the object will float.
- **b.** if $\rho_f > \rho$, the object will sink.
- **c.** if $\rho < \rho_f$, the object will float, displacing less than volume V of fluid.
- **d.** if $\rho = \rho_f$, the object will float, displacing less than volume V of fluid.
- **e.** None of the above

14. In the model for Sunken Treasure (Example 2.2.3 of Module 2), suppose that the drag force is $0.25\,v$, rather than $0.2\,v$. Then

- **a.** the chest and barrel will fall at a lower velocity than before.
- **b.** the terminal velocity is about 18 m s$^-$1.
- **c.** the terminal velocity is about 57.9 km per hour.
- **d.** the drag force is greater than the buoyant force for all $t > 0$.
- **e.** Exactly two of the above statements are true.

15. In the general model for mixing problems, $\dfrac{dQ}{dt} = r_i c_i - r_o \dfrac{Q(t)}{V(t)}$, $Q(0) = Q_0$, assume that r_i and r_o are constant flow rates. Then

a. if $r_i = r_o$, $V(t)$ is constant.

b. if $r_i > r_o$, the tank will overflow eventually.

c. if $r_i = r_o = r$, and c_i is constant, $Q(t)$ approaches an equilibrium level as $t \to \infty$.

d. All of the above

e. None of the above

16. Which of the following DEs is NOT separable?

a. $(\tan y)\,\dfrac{dy}{dx} = x + 2xy^2$

b. $\dfrac{dy}{dt} = y\,(5 - y)$

c. $\dfrac{dy}{dx} = e^{x+y}$

d. $\dfrac{dy}{dx} = 3\sqrt{x^2y^2}$

e. $(2x + y)\,\dfrac{dy}{dx} = 1$

Note: For problems 17 - 20, it is essential that you understand the terms *integral curve, explicit solution, interval of validity* and how they arise in solving a first order DE.

17. Consider the DE $\dfrac{dy}{dx} + 12 = \dfrac{5}{x}$:

a. Solving this DE involves finding the general antiderivative of the function $\dfrac{5}{x} - 12$.

b. The explicit solution through $(-1, 0)$ is $y = 5\ln(-x) - 12x - 12$.

c. No solution has domain (interval of validity) $-2 < x < 2$.

d. All of the above

e. None of the above

18. Consider the DE $y\,\dfrac{dy}{dx} = (1 - y^2)\,\sin x$. Decide which of the following statements is true.

a. The DE has only one equilibrium solution.

b. The direction field is anti-symmetric about the y-axis (i.e., $y'(-x) = -y'(x)$), so the integral curves will be symmetric about the y-axis. [Hint: Think about this geometrically.]

c. All solutions have positive slopes if $|y| < 1$.

d. The explicit solution through $(0, 2)$ is $y = \sqrt{1 + 2\,e^{2\cos x - 2}}$.

e. None of the above

19. Referring to the details of Example 2 on pages 40 - 41 of the course Text, $-2\,(y-1)\,\dfrac{dy}{dx}+3x^2+4x+2=0$, answers to these questions can be found by inspection:

a. The explicit solution through $(0, 2)$ is $y^2 - 2y = x^3 + 2x^2 + 2x$.

b. The explicit solution of the IVP $\dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)}$, $y(0) = 0$, is the same as the explicit solution for IC $y(0) = 2$.

c. The domain of the explicit solution of the IVP $\dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)}$, $y(0) = 2$ is $x > -1$, and its range is $y > 1$.

d. All of the above

e. None of the above

20. Consider the DE $\dfrac{dy}{dx} = -2xy^2$:

a. The integral curves of the DE are $x^2 + \dfrac{1}{y} = C$ for arbitrary real C .

b. The general solution of the DE is $y = \dfrac{1}{C} - \dfrac{1}{x^2}$.

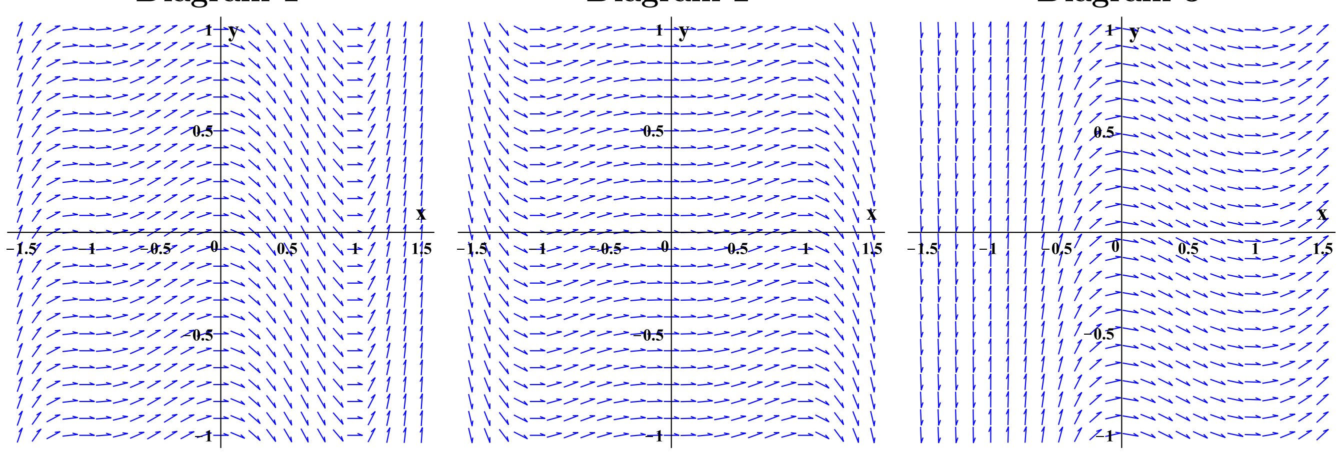
c. The solution of the IVP $\dfrac{dy}{dx} = -2xy^2$, $y(0) = -1$ has domain $|x| < 1$.

d. The solution of the IVP $\dfrac{dy}{dx} = -2xy^2$, $y(0) = -1$ has domain $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

e. None of the above

21. A certain separable DE has the form $\dfrac{dy}{dx} = f(x)$, where $f(-1) = 0$, $f(0) = 0$, and $f(1) = 0$. Below are plots of three direction fields. Decide which one(s) could correspond to this DE.

Diagram 1 **Diagram 2** **Diagram 3**



a. Diagram 1

b. Diagram 2

c. Diagram 3

d. None of the plots corresponds to the DE

e. Either of Diagrams 1 or 2 could correspond to the DE

22. Suppose that, in Example 2.2.1 of Module 2, iodine is accidentally leaking into the flushing water at 1 gram per minute (i.e., 1 gram in every 5 litres of fluid, so $c_i = 0.0002\,\text{kg}$ per litre). Then

a. The revised model is $\dfrac{dQ}{dt} = 0.001 - 2\,\dfrac{Q}{200-t}$, $Q(0) = 5$ kg.

b. Solving the DE requires finding $\int \dfrac{0.001}{(200-t)^2}\,dt$.

c. By the time the tank empties, the concentration of iodine in the tank has reached 0.2 gram per litre.

d. All of the above

e. None of the above

23. Consider the DE $x\,\dfrac{dy}{dx} - 2y = 0$; note that the DE is NOT given in standard form:

a. The integral curves of the DE are $y = Cx^2$ for arbitrary real C.

b. The function $y = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$ is a solution of the given DE for all real x.

c. The function $y = \begin{cases} -C_1\,x^2 & \text{if } x < 0 \\ C_2\,x^2 & \text{if } x \geq 0 \end{cases}$ is a solution of the given DE for all real x and any constants C_1 and C_2.

d. None of the above

e. All of the above

24. In solving the separable DE $M(x) + N(y)\,\dfrac{dy}{dx} = 0$, sometimes we can't find antiderivatives for $M(x)$ and $N(y)$ in terms of elelmentary functions. One such example is the IVP $\dfrac{dy}{dx} = y^2\cos(x^2)$, $y(0) = 1$. Without using series, the **very best** we can do for the solution on $y > 0$ is:

a. $1 - \dfrac{1}{y} = \int_0^x \cos(s^2)\,ds$

b. $\ln(y^2) = \int_0^x \cos(s^2)\,ds$

c. $\int_1^y \dfrac{1}{s^2}\,ds = \int_0^x \cos(s^2)\,ds$

d. All of the above

e. None of the above

25. Suppose we make the substitution $z = t + y$ in the DE $\dfrac{dy}{dt} = (t+y)^2$. Then

a. The resulting DE for $z(t)$ is separable.

b. The general solution for $z(t)$, transformed back to $y(t)$, gives $y(t) = \tan(t + C) - t$ for arbitrary constant C.

c. The soution of the IVP $\dfrac{dy}{dt} = (t+y)^2$, $y(0) = 1$, has domain $-\dfrac{3\pi}{4} < t < \dfrac{\pi}{4}$.

d. Every solution $y(t)$ has a finite domain.

e. All of the above