MATH 650 : Mathematical Modeling

Spring, 2019

Electronic Assignment #2

Due by 11:59 p.m. EST on Tuesday, May 21, 2019

Instructions:

- Ensure you have reviewed Module 1, Sections 3 and 4, Module 2, Section 1, and any Text Readings and other activities therein. You will require this knowledge to answer the questions on this assignment.
- Read and think about the following assignment problems.
- Print and complete the following assignment. Record your answers on the printed copy so you have a record of your solutions.
- Once you are satisfied with your answers, submit your solutions online as follows:
  - Go to UW's course management website at `learn.uwaterloo.ca`
  - Enter your **QUEST Username** and **Password** in the space provided and click **Login**.
  - Once inside the LEARN course environment, click on the link for **MATH 650 : Mathematical Modeling**.
  - Click on the **Submit** $\rightarrow$ **Quizzes** tab at the top of the page.
  - Click **Electronic Assignment 2**, and follow the instructions provided. An answer key for this assignment will appear where you can fill-in your solutions. Please email your instructor immediately if you encounter any problems.
  - Click on the **SUBMIT QUIZ** button when you are done. You have only 1 attempt to submit your solutions. Any assignment submitted after midnight (Waterloo, Ontario time) will be considered **late** and will not be counted toward your final grade (no exceptions).

The following questions are based on material in Module 1, Sections 3 and 4, and Module 2, Section 1

Part 1: True or False (1 mark each)

Indicate whether the following statements are true (a) or false (b).

1. An appropriate choice of units for the velocity of a nuthatch climbing a tree trunk is km per hour.

a. True

b. False

2. We could expect to find an expression of the form e^v in a model for the motion of an object with velocity $v\,\mathrm{m\ s^{-1}}$.

a. True

b. False

3. The solution $s(t) = -\dfrac{1}{2}gt^2 + v_0 t + s_0$ for the displacement of an object in vertical motion with no drag and constant gravity can be rewritten in the form $s(t) = -\dfrac{1}{2}g(t - \dfrac{v_0}{g})^2 + s_0 - \dfrac{v_0^2}{2g}$.

a. True

b. False

4. The Principle of Dimensional Homogeneity confirms that we can add, subtract, equate, or compare expressions with like dimensions.

a. True

b. False

5. If τ is a dimensionless variable, and $\tau = \dfrac{t}{t_c}$ where t is time, then t_c can be any non-zero constant with dimension $\mathcal{T}$.

a. True

b. False

6. The reason for using dimensionless variables is that they normalize the model to a simplified form, revealing the essential physical parameters as dimensionless ratios.

a. True

b. False

7. The DE $t\,y' - 2y = \sin t$ is a linear DE in standard form.

a. True

b. False

8. The DE $\cos t\, y' + y \sin t = \cos^2 t$ has a singular point at $t = \pi$.

a. True

b. False

9. Euler's Method uses a succession of tangent line segments to approximate the solution of the IVP $\dfrac{dy}{dt} = f(t, y),\ y(t_0) = y_0$.

a. True

b. False

10. The second step of Euler's Method, $y_2 = y_1 + hf(t_1, y_1)$, is always tangent to the solution $\phi(t)$ of the IVP $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$.

a. True

b. False

11. Referring to Exercise 1.3.2 on slide 56 of Module 1, in converting the population model $\frac{dp}{dt} = rp - K$ to dimensionless variables $z = \frac{r}{K}p$, $\tau = rt$, the derivative $\frac{dp}{dt}$ becomes $K\frac{dz}{d\tau}$.

a. True

b. False

12. In the DE $\frac{dv}{dt} = -g - \frac{\gamma}{m}v$ for the velocity v of an object of mass m moving vertically with constant gravity and viscous drag, the drag coefficient γ has dimensions $\mathcal{M}\mathcal{T}^{-1}$.

a. True

b. False

Part 2: Multiple Choice (1 mark each)

Choose the **best** answer for each question.

13. The IVP $\frac{du}{dt} = -k(u - T_a)$, $u(0) = u_0$ for Newton's Law of Temperature Change has solution $u(t) = T_a + (u_0 - T_a)e^{-kt}$.
[HINT: What are the dimensions of k, u_0, and T_a in the DE and solution?]

a. A suitable choice of dimensionless variables is $y = \frac{u}{ku_0}$, $\tau = kt$.

b. With the choice in part a., the solution becomes $y = \beta + (1-\beta)e^{-\tau}$ for a dimensionless parameter β.

c. With dimensionless variables $z = \frac{u}{T_a}$, $\tau = kt$, the solution becomes $z = 1 + (\lambda - 1)e^{-\tau}$ for a dimensionless parameter λ.

d. All of the above

e. None of the above

14. Referring to the solution $s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$ (as above in True/False problem 3), assume $s_0 = 0$ in considering the following statements.

a. Changing the time units is equivalent to a vertical scaling of the graph of $s(t)$ versus t.

b. Changing the length units is equivalent to a horizontal scaling of the graph of $s(t)$.

c. If you double v_0, the object goes 4 times as high, but takes twice as long to get there.

d. If you double g, the object goes half as high in twice the time.

e. All of the above

15. Suppose that Euler's method is applied to an IVP $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$ which has exact solution $y = \phi(t)$. [Hint: Text Reading 1.4.1, and Maple Exploration 1.4.1 may be helpful.]

a. Euler's Method always approaches the exact solution as t increases.

b. Euler's Method copes well with vertical asymptotes in the solution $y = \phi(t)$.

c. Reducing the size of the stepsize h always reduces the error in the approximation.

d. If the family of solutions of the DE $\frac{dy}{dt} = f(t, y)$ is converging as t increases, then Euler's Method gives better approximations as t increases.

e. None of the above

16. For the linear DE $(\tan t)y' + (\sin t)y = t\cos t$:

a. When written in standard form, the coefficient functions are $p(t) = \cos t$, and $q(t) = t\cos^2 t$.

b. There is a singular point at $t = \frac{\pi}{4}$.

c. The integrating factor is $e^{\sin t}$.

d. All of the above

e. None of the above

17. A linear DE $\frac{dy}{dt} + p(t)y = g(t)$ is defined for t in an interval $\mathcal{I}$ of real numbers.

a. The DE is homogeneous if $g(t) = 0$ on $\mathcal{I}$.

b. The DE is homogeneous if $p(t) = 0$ on $\mathcal{I}$.

c. The DE is non-homogeneous if $p(t) \neq 0$ on $\mathcal{I}$.

d. A particular solution $y_p(t)$ of the DE must satisfy $\frac{dy_p}{dt} + p(t)y_p = g(t)$.

e. Exactly two of the above statements are true.

In problems 18 - 22, solve the linear DE and then choose the best answer.

18. For the DE $y' = -5y + 2$:

a. A particular solution is $y_p(t) = \frac{2}{5}$.

b. The homogeneous solution is $y_h(t) = Ce^{-5t}$.

c. The general solution is $y(t) = Ce^{-5t} + \frac{2}{5}$.

d. As $t \to \infty$, all solutions approach the equilibrium solution.

e. All of the above

19. For the DE $t\,\dfrac{dy}{dt} + y = e^t$:

a. The integrating factor is e^t.

b. With IC $y(1) = -1$, the solution is negative for all $t > 0$.

c. With IC $y(1) = e$, the domain (interval of validity) of the solution is $t > 0$, and the range of the solution is $e \le y < \infty$. [HINT: The range is the interval from least to greatest value of $y(t)$.]

d. All of the above

e. None of the above

20. For the DE $t\,\dfrac{dy}{dt} = -3y + \dfrac{\sin t}{t^2}$:

a. The homogeneous solution is $y_h(t) = \dfrac{C}{t^3}$.

b. A particular solution is $y_p(t) = \dfrac{-\cos t}{t^3}$.

c. Every solution $y(t)$ satisfies $\lim_{t\to\infty} y(t) = 0$.

d. All of the above

e. None of the above

21. For the DE $\dfrac{dy}{dt} = -y + 2t + 1$:

a. The integrating factor is e^{-t}.

b. The general solution is $y(t) = 2t - 1 + Ce^t$.

c. All solutions $y(t)$ for $C \neq 0$ approach the particular solution $y = 2t - 1$ as $t \to \infty$.

d. All solutions $y(t)$ cross the line $y = 2t + 1$, and do so horizontally.

e. Exactly two of the above statements are true.

22. For the DE $\dfrac{dy}{dt} + 2y = \cos t + 3\,t$:

a. The homogeneous solution is $y_h(t) = Ce^{-2t}$.

b. For undetermined coefficients, a suitable form to assume for the particular solution is $y_p = A\sin t + B\cos t + Ct + D$.

c. The particular solution is $y_p(t) = \dfrac{1}{5}\sin t + \dfrac{2}{5}\cos t + \dfrac{3}{2}\,t - \dfrac{3}{4}$.

d. As $t \to \infty$, every solution oscillates about the straight line $y = \dfrac{3}{2}\,t - \dfrac{3}{4}$.

e. All of the above

23. For the DE $y' - y = t^2 - 2t + \sin \pi t$:

a. The homogeneous solution is Ce^t.

b. For undetermined coefficients, a suitable particular solution is $y_p = At^2 + Bt + C + D \sin \pi t$.

c. For undetermined coefficients, a suitable particular solution is $y_p = At^2 + Bt + C \sin \pi t + D \cos \pi t$.

d. By examining the DE, we would predict that most solutions are bounded for all t.

e. None of the above

24. Solve the IVP $\dfrac{dy}{dt} = -\dfrac{2}{t}\, y + \dfrac{\sin t}{t^2}$, $y(\dfrac{\pi}{2}) = a$:

a. The integrating factor is t^2.

b. A particular solution is $y_p(t) = -\dfrac{\cos t}{t^2}$.

c. For any real number a, the solution approaches 0 as $t \to \infty$.

d. If $a > 0$, then the solution $y(t)$ satisfies $y(t) > y_p(t)$ on its domain.

e. All of the above

25. We know that the general solution of the linear DE $\dfrac{dy}{dt} + p(t)\, y = g(t)$ is

$$y(t) = C\, e^{-P(t)} + e^{-P(t)} \int e^{P(t)} g(t)\, dt, \qquad \text{where } P(t) = \int p(t)\, dt.$$

Sometimes one or both of these antiderivatives cannot be expressed in terms of elementary functions. Consider the innocent-looking DE $\dfrac{dy}{dt} + y = \dfrac{1}{t+1}$:

a. The integrating factor is e^{-t}.

b. The general solution is $y(t) = C\, e^{-t} + e^{-t} \int \dfrac{e^t}{t+1}\, dt$.

c. The horizontal isocline (where $\dfrac{dy}{dt} = 0$) is the curve $y = \dfrac{-1}{t+1}$.

d. The solutions have positive slopes if $y > \dfrac{1}{t+1}$.

e. Exactly two of the above statements are false.