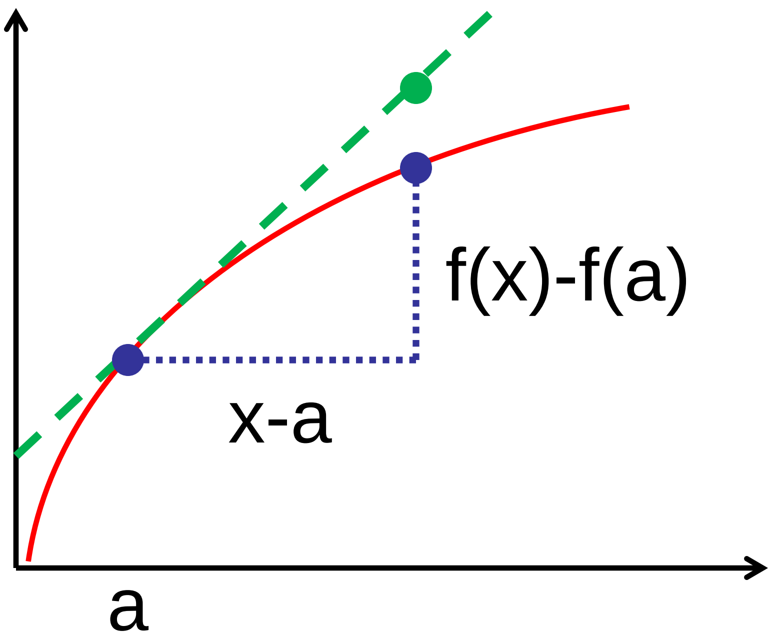
11.10 – Taylor (and Maclaurin) Series

In MATH 127, we used the linearization of a function to approximate it near a point $x = a$ as a constant term plus a linear term.

$$L(x) = f(a) + f'(a)(x - a)$$



Notice, this looks like the first two terms in a power series centered at $x = a$. Perhaps we can build a better approximation by including higher order contributions (e.g., quadratic, cubic, etc.).

Let's try this with $f(x) = e^x$ about $x = 0$. From the linearization we know $f(x) \approx 1 + x$. Let's now add an arbitrary quadratic term $c_2 x^2$,

$$f(x) \approx 1 + x + c_2 x^2$$

What should the coefficient c_2 be to make this approximation effective?

Notice what we get if we take two derivatives:

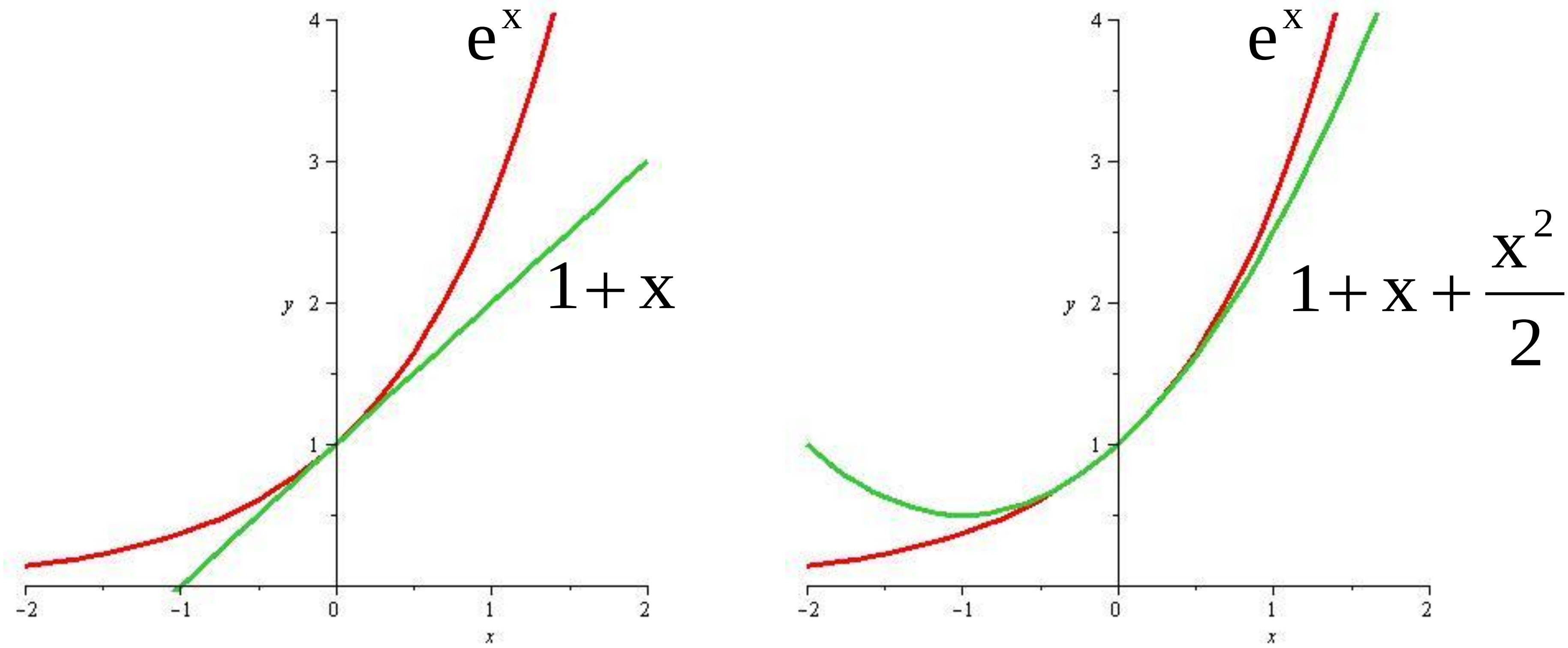
$$f''(x) \approx 0 + 0 + 2c_2 \quad \Rightarrow \quad f''(0) \approx 2c_2$$

But if $f(x) = e^x$, then $f''(x) = e^x \Rightarrow f''(0) = 1$

So for our approximation to have a second derivative with the correct behaviour we need

$$2c_2 = 1 \quad \Rightarrow \quad c_2 = \frac{1}{2} \quad \Rightarrow \quad f(x) \approx 1 + x + \frac{x^2}{2}$$

Plotting the linearization versus this new quadratic approximation reveals it is a better fit.



We can extend this procedure to higher order. To fix the *n*-th order coefficient of a trial power series we take *n* derivatives and match to the derivative of our actual function where our series is centered.

Let's see how this works for a generic function $f(x)$ that we wish to represent with a power series centered at $x = 0$. That is, let

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

Now compute derivatives and evaluate them at $x = 0$

$$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots \quad \Rightarrow \quad f(0) = c_0$$

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \ldots \quad \Rightarrow \quad f'(0) = c_1$$

$$f''(x) = 2c_2 + 6c_3 x + 12c_4 x^2 + \ldots \quad \Rightarrow \quad f''(0) = 2c_2$$

If we continue, we find using the *n*-th derivative

$$f^{(n)}(0) = n! c_n$$

Solving for the coefficients gives

$$c_n = \frac{f^{(n)}(0)}{n!}$$

With these coefficients, we have the following power series representation for our function

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Maclaurin Series

We call this power series representation of a function centered at $x = 0$ a **Maclaurin series**.

When the series converges, it serves as an "easier-to-calculate-with" version of the function!

As we know, power series are most effective near their centers so we repeat the above arguments with a trial power series centered at $x = a$. Let

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

Now we compute derivatives at $x = a$ to solve for the coefficients:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

These coefficients give the **Taylor series** for the function

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor Series

Notice, a Maclaurin series is actually just a Taylor series centered at $x = 0$.

Now that we know how to compute the series representation for a function, let's go beyond finding an approximation for $f(x) = e^x$ and instead find a complete power series representation.

For now, let's compute the Maclaurin series. This means we need the coefficients

$$c_n = \frac{f^{(n)}(0)}{n!}$$

But $f^{(n)}(x) = e^x$ for all *n* ➡ $f^{(n)}(0) = 1$ for all *n*.

Therefore, for $f(x) = e^x$, the Maclaurin series coefficients are

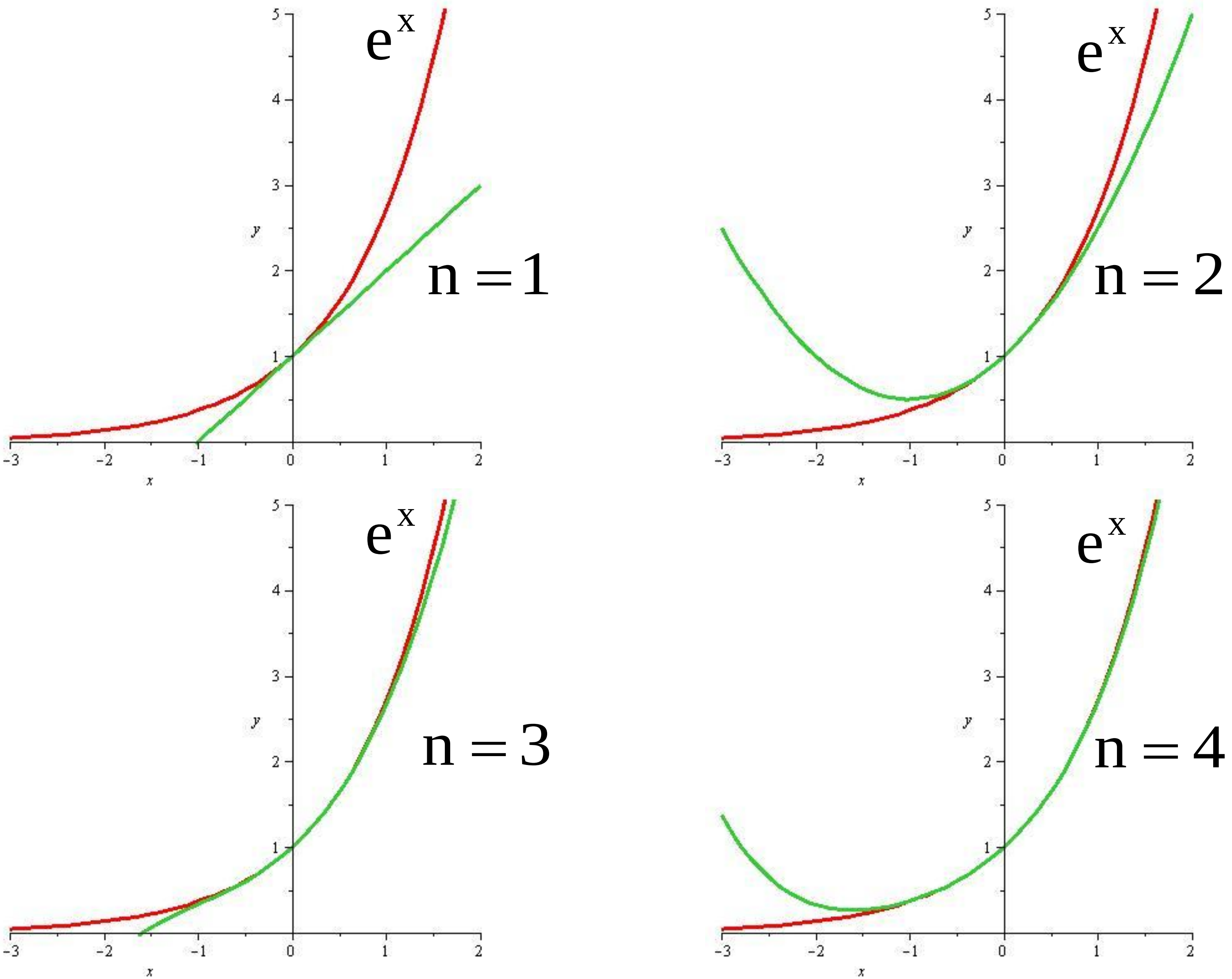
$$c_n = \frac{1}{n!}$$

This gives us the Maclaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Notice, our linearization and quadratic approximation appear as the first few terms. In fact, we can show this series converges (using the Ratio Test) for all x so this power series serves as an exact substitution for e^x.

Plot of e^x against first n terms in Maclaurin series



Example:

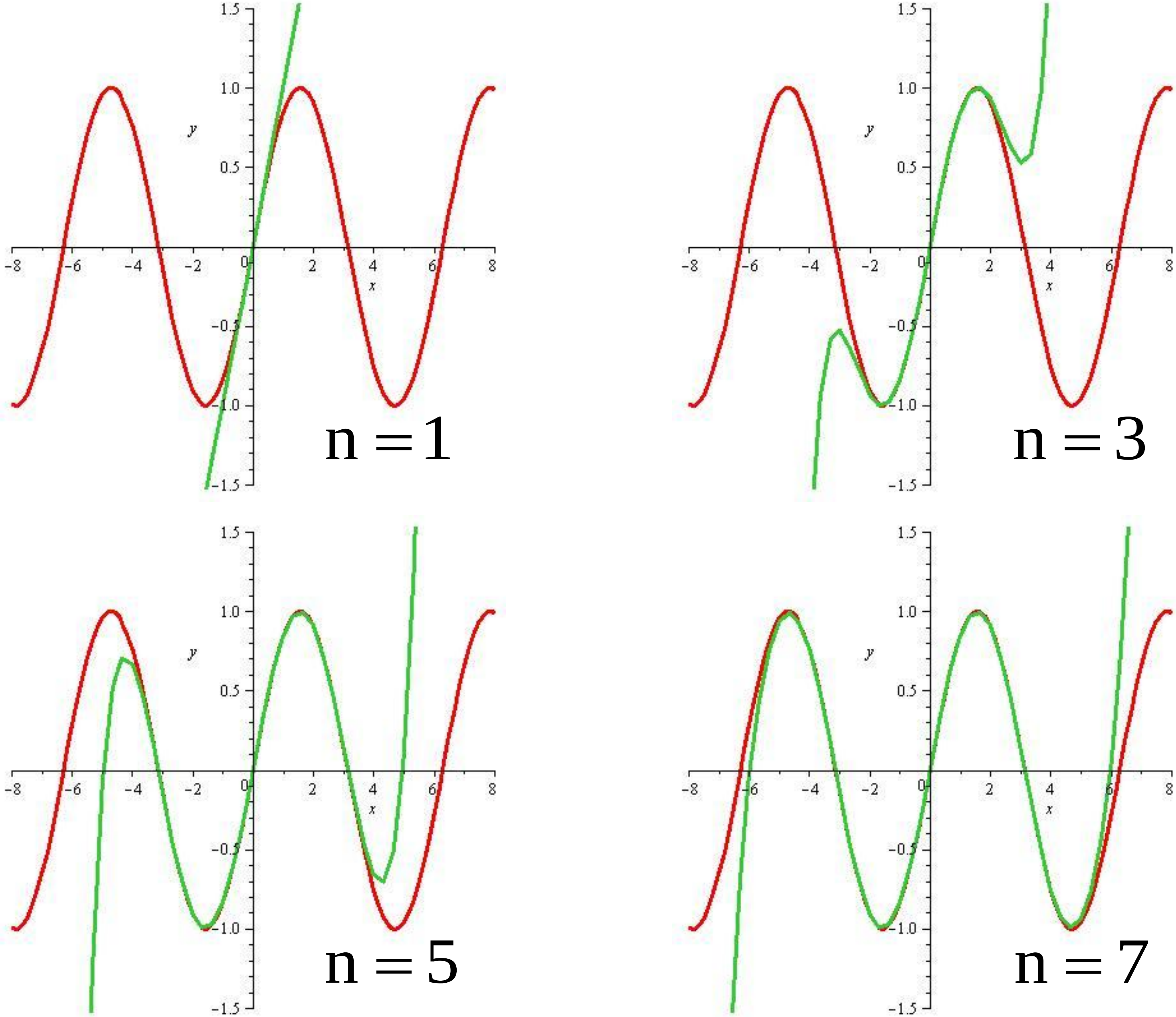
Find the Maclaurin series for $\sin(x)$ and $\cos(x)$.

$$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

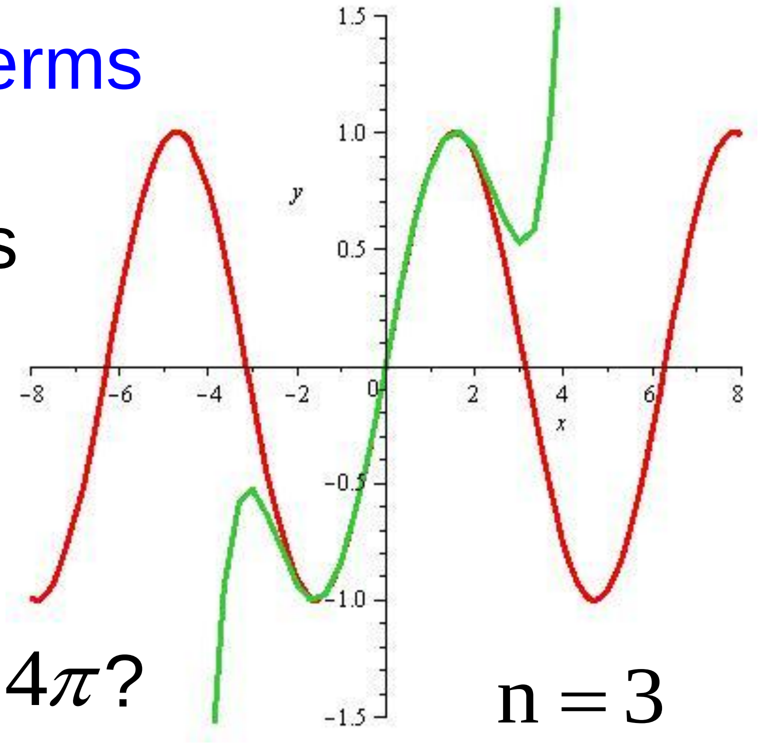
$$\cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$$

The Maclaurin series for e^x, $\sin(x)$, and $\cos(x)$ come up so frequently that it is worth knowing how to derive them and/or memorizing them.

Plot of $\sin(x)$ against first n terms in Maclaurin series



Notice, using only the first three terms of the Maclaurin series for $\sin(x)$ gives an approximation that works great for $-2 < x < 2$ but is terrible beyond this interval.



$n = 3$

What should we do if we want to approximate $\sin(x)$ near say $x = 4\pi$?

We *could* use more terms from the Maclaurin series (after all, we know the series converges so infinitely many terms will exactly reproduce $\sin(x)$) but this would get very messy.

A smarter strategy is to move the series centre and compute the Taylor series about $x = 4\pi$.

The Taylor series for $f(x) = \sin(x)$ about $x = 4\pi$ is

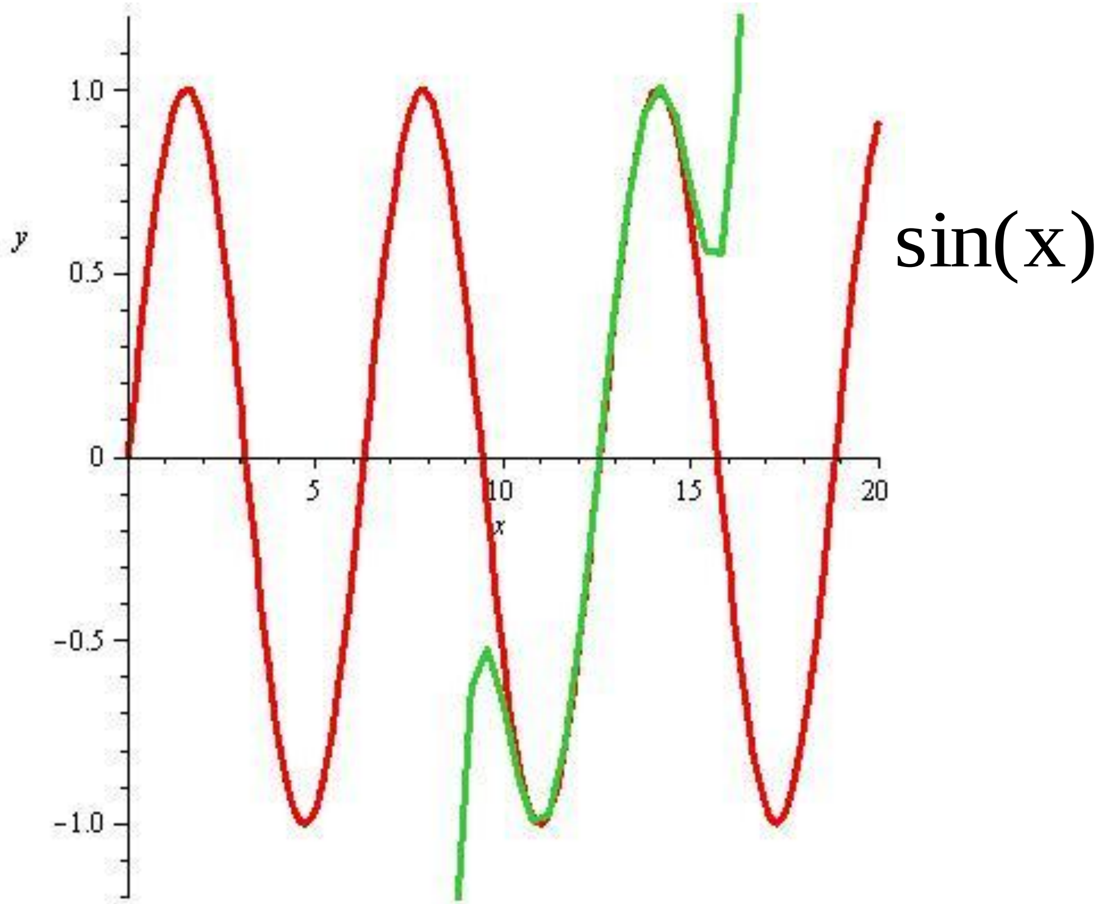
$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(4\pi)}{n!}(x-4\pi)^n$$

Since $f(x)$ is 2π-periodic, so are its derivatives. As such, we end up with the same coefficients as the Maclaurin series – all that changes is we get a series in powers of $(x-4\pi)$.

$$\Rightarrow \quad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-4\pi)^{2n+1}}{(2n+1)!}$$

$$= (x-4\pi) - \frac{(x-4\pi)^3}{3!} + \frac{(x-4\pi)^5}{5!} + \ldots$$

Sketching the first three terms, we see re-centering the series at $x = 4\pi$ (i.e., using a Taylor series) allows us to approximate $\sin(x)$ much more effectively in that domain.



$$(x-4\pi)-\frac{(x-4\pi)^3}{3!}+\frac{(x-4\pi)^5}{5!}$$

Example:

Compute the first three non-zero terms of the Taylor series of $f(x) = e^{-x^2}$ about $x = 1$. How well do they approximate $f(1.1) \approx 0.2982$?

From the previous examples, we see that we can often make do for practical purposes with just the first few terms of a Taylor series.

As such, it is useful to define the **Taylor polynomial**

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Thinking of the Taylor series like any other series – the Taylor polynomial just gives the partial sum.

And if the Taylor series converges, then

$$\lim_{n\to\infty} T_n(x) = f(x)$$

The difference between the *n*-th Taylor polynomial and the function itself defines the remainder

$$R_n(x) = f(x) - T_n(x)$$

The remainder tells us how well the first *n* terms in the Taylor series approximate the function (similar to the Alternating Series Remainder).

Note, as long as the Taylor series converges to the function (i.e., x is in the interval of convergence) , the remainder should go to zero as $n \to \infty$.

$$\lim_{n\to\infty} R_n(x) = 0 \quad \text{for} \quad |x-a| < R$$

It is also useful to place an upper bound on $R_n(x)$ because this quantifies how well a Taylor polynomial approximates a function. It takes a fair bit of work to prove (see textbook), but one can show

Taylor's Inequality

If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for} \quad |x-a| \le d$$

Taylor's Inequality also provides an avenue to prove convergence of a Taylor series since

$$\lim_{n\to\infty} \frac{|x-a|^n}{n!} = 0 \quad \text{for all} \quad x \in \Re$$

Example:

Compute the second degree Taylor polynomial, $T_2(x)$, for $f(x) = \ln(x)$ about $x = 1$. Use it to approximate $\ln(1.5)$ and then use Taylor's Inequality to find a bound on the error, $R_2(1.5)$, for this approximation.

$$T_2(x) = (x-1) - \frac{1}{2}(x-1)^2$$

$$R_2(x) = \frac{1}{3}(x-1)^3$$

for $1 \le x \le 1.5$

$\Rightarrow$ $$\ln(1.5) = 0.375 \pm 0.042$$

Taylor's Inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

if $|f^{(n+1)}(x)| \le M$

for $|x-a| \le d$

Warm-Up Example:

Using an appropriate Taylor series representation of $\cos(x)$, compute the following limit:

$$\lim_{x \to 0} \frac{1-\cos(x)}{x^2}$$

Binomial Approximation:

In special relativity, the energy of a mass m moving at velocity v is given by

$$E = \frac{mc^2}{\sqrt{1-(v/c)^2}}$$

where c is the speed of light.

However, in Newtonian ("everyday") physics, the energy of a moving mass is just

$$E = \frac{1}{2}mv^2$$

How do we reconcile these two results?

First off, let's rewrite the relativistic formula as

$$E = mc^2\left[1-(v/c)^2\right]^{-1/2}$$

(Notice, for an object with zero velocity, the energy is

$$E = mc^2$$

We call this the "rest" mass energy – it is the energy stored in the mass itself.)

But what if the velocity isn't zero?

Let's compute a Taylor polynomial about $v = 0$…

For simplicity, we replace $\left(\frac{v}{c}\right)^2 \to x$ and just consider the square root term

$$\Rightarrow \quad f(x) = (1-x)^{-1/2}$$

With a bit of work we find:

$$f(x) = (1-x)^{-1/2} \quad \Rightarrow \quad f(0) = 1$$

$$f'(x) = \frac{1}{2}(1-x)^{-3/2} \quad \Rightarrow \quad f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{3}{4}(1-x)^{-5/2} \quad \Rightarrow \quad f''(0) = \frac{3}{4}$$

$$\Rightarrow \quad T_2(x) = 1 + \frac{1}{2}x + \frac{3}{8}x^2$$

Swapping back to $x \to \left(\frac{v}{c}\right)^2$

$$E = mc^2\left[1-\left(\frac{v}{c}\right)^2\right]^{-1/2} \approx mc^2\left[1+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4\right]$$

$$= mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2}$$

Rest mass energy

Newtonian kinetic energy

First relativistic correction

A Taylor series of Einstein's formula reproduces Newton's result but yields new information about the nature of energy!

The function we had to Taylor expand in this problem was a special case of

$$f(x) = (1+x)^k$$

which occurs frequently in practical applications.

Fortunately, we can compute the Maclaurin series for arbitrary values of k.

$$f(x) = (1+x)^k \Rightarrow f(0) = 1$$

$$f'(x) = k(1+x)^{k-1} \Rightarrow f'(0) = k$$

$$f''(x) = k(k-1)(1+x)^{k-2} \Rightarrow f''(0) = k(k-1)$$

$$\vdots \qquad\qquad \vdots$$

$$\Rightarrow f^{(n)}(0) = k(k-1)\ldots(k-n+1)$$

This gives us the Maclaurin series

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

Binomial Series

where $\binom{k}{n} = \frac{k(k-1)\ldots(k-n+1)}{n!}$ are called binomial coefficients.

Note, if k is a positive integer

$$\binom{k}{n} = \frac{k(k-1)\ldots(k-n+1)}{n!} = \frac{k!}{n!(k-n)!}$$

and, moreover, when $n > k$ the coefficient contains a term $(k-k) = 0$ so the series will terminate at $n = k$ (as you would expect if you expanded the function).

We can readily show that the binomial series converges for $|x| < 1$ by the Ratio Test:

$$\Rightarrow \lim_{n\to\infty}\left|\frac{\binom{k}{n+1}x^{n+1}}{\binom{k}{n}x^{n}}\right| = |x|\lim_{n\to\infty}\left|\frac{n!(k-n)!}{(n+1)!(k-(n+1))!}\right|$$

$$= |x|\lim_{n\to\infty}\left|\frac{(k-n)}{(n+1)}\right|$$

$$= |x|$$

However, convergence at the endpoints depends on the exponent of k so needs to be checked on a case-by-case basis.

Note, the binomial series is particular useful for quickly producing the linear approximation

$$(1+x)^n \approx 1 + nx$$

Binomial Approximation

which is accurate as long as $|x| << 1$.

Examples:

Apply the binomial approximation to the following functions:

a) $(1+x)^{1/2}$

b) $(2-x)^{10}$

c) $(x-x^2)^{-3/2}$

Summary of common Maclaurin Series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \qquad R = \infty$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad R = \infty$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad R = \infty$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2} x^2 + \ldots \qquad R = 1$$

11.11 – Taylor Polynomial Applications

As we have seen, Taylor polynomials can be used to find approximate solutions to various problems.

In this section, we will look at some more examples where Taylor polynomials are useful with extra attention to the accuracy of the approximation.

In particular, we will quantify the error one of two ways:

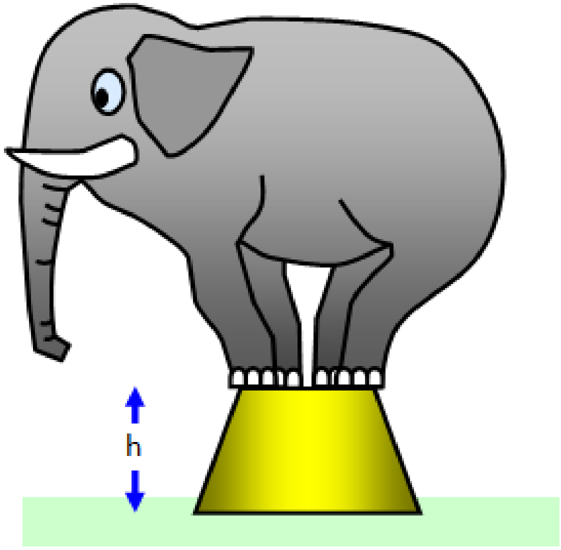
(i) Using the Alternating Series Estimation Theorem when our Taylor series is alternating.

(ii) Using Taylor's Inequality when the series is *not* alternating.

Example:

For what values of x does the second degree Taylor polynomial for $\cos(x)$ with centre $a = 0$ give an error of less than 0.01 (i.e., $|R_2(x)| \leq 0.01$)?

Lifting an object with mass m gives it some gravitational potential energy. If we lift it a height h, then the potential energy is given by



$$E_g = mgh$$

where $g = \frac{GM_E}{R_E^2}$ is the acceleration due to gravity.

But where does this formula come from?

Well, another way of calculating the energy is to ask “how much work (energy) do we have to do to lift the mass”?

Work is usually just "force times distance" but here the force is gravity which varies with height. This means, to be exact, we have to do an integral.

In particular, to raise an object from the Earth's surface R_E to a height h requires work equal to

$$W = \int_{R_E}^{R_E+h} \frac{GMm}{r^2} dr = mgh\left(\frac{1}{1+\left(\frac{h}{R_E}\right)}\right) = E_g$$

This is the *actual* expression for the gravitational potential energy. We see the familiar mgh factor out front but there is more to the story.

Why, in practice, do we ignore the $\left(\frac{1}{1+\left(\frac{h}{R_E}\right)}\right)$ piece?

Let’s Taylor expand to find out. Notice this is just an alternating geometric series

$$\frac{1}{1-\left(-\frac{h}{R_E}\right)} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{h}{R_E}\right)^n = 1 - \frac{h}{R_E} + \left(\frac{h}{R_E}\right)^2 - \ldots$$

$$\Rightarrow \quad E_g = mgh - \frac{mgh^2}{R_E} + \ldots$$

So the formula $E_g = mgh$ is really just the zeroth degree Taylor polynomial for the gravitational potential energy.

This raises concern - how accurate could the zero degree Taylor polynomial possibly be?

Well, since this is an alternating series, the error on any partial sum is no greater than the size of the next term in the series.

When we move objects around our own environment, h is on the order of meters. However, the radius of Earth is $\sim 6{,}400$ kilometers.

So if we raise an object by say 6.4 meters, then

$$\frac{h}{R_E} = \frac{6.4}{6.4 \times 10^6} = 10^{-6}$$

Therefore, using the zeroth degree Taylor polynomial for everyday physics is accurate to parts per million. However, if we want to talk about sending a satellite into space, we cannot use $E_g = mgh$.

Example:

Find the 1st degree Taylor polynomial for $f(x) = x^{-1/2}$ about $a = 9$. Next, use Taylor's Inequality to find an upper bound on the remainder on the interval $8 \le x \le 9$. Finally, approximate $(8)^{-1/2}$ and give a bound on the error of this approximation.

Example:
Show that $\sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(-\frac{\pi^2}{4}\right)^n = 1 - \frac{\pi^2}{8} + \frac{\pi^4}{384} - \ldots = 0.$

Example:
Approximate $\int_{x=0}^{1} x^2 \sin(x)\,dx$ with $|\text{error}| \le 0.00002$.

Example:
Starting with $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, express $\tan^{-1}(\sqrt{x})$ as an infinite series.

Example – Electric Dipole:

Using the binomial approximation (assume $d << R$), compute, at lowest non-zero order, the electric field at the point P in the diagram.

$$\Rightarrow \quad E = \frac{q}{R^2} - \frac{q}{(R+d)^2} \approx \frac{2qd}{R^3}$$