Recognizing Totally Dual Integral Systems is Hard

Notes on Ding, Feng and Zang's proof and some remaining problems

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These are notes about Ding, Feng and Zang's proof [5]. The proof of their result is not new, the only difference with them is the starting point: we work directly on their gadget graph encoding a SAT problem and not on more general graphs. This allows to shortcut some parts of the original proof that become superfluous. After proving their theorem I clarify some points about total dual integrality in order to put forward some remaining interesting open problems.

1 The Reduction

For any input I of SAT on n variables and m clauses we define the graph $G(I)$ (see the Figure) where

– the variables x_i ($i = 1...n$) are represented by disjoint 4-cycles $a_i v_i b_i \bar{v}_i a_i$ ($i = 1, ..., n$), where $a_i, v_i, b_i, \bar{v}_i, a_i$ are vertices.

– the clauses $j = 1, \dots, m$ are represented by two vertices s_j, t_j and one path of length 2 between s_j and t_j for each variable i in the clause, with a middle vertex denoted u_{ij}, independently of whether the variable or its negation are present in the clause.

– we identify b_i with a_{i+1} ($i = 1, ..., n-1$), b_n with s_1, and t_j with s_{j+1} ($j = 1, ...m-1$).

– If variable x_i is present in clause j, we introduce two "twin" vertices y_{ij}, z_{ij}, join both to vertices u_{ij} and v_i or $\bar{v}_i$, depending on which of the two is in clause j. The vertices y_{ij}, z_{ij} and the 4 incident edges will be called *connectors.*

  Denote C_j the set of indices $i \in \{1, \dots, n\}$ for which x_i is present in clause j (with or without negation sign) ($j = 1, \dots, m$).

  So far the defined graph is bipartite.

– Add now the edge $e_0 = a_1 t_m$.

Let us summarize more formally the definition of $G(I)$. Its vertex set is

$$V := \{a_1, b_1 = a_2, ..., b_{n-1} = a_n, b_n\} \cup \{v_1, \bar{v}_1, ..., v_n, \bar{v}_n\} \cup \{s_2 = t_1, ..., t_n\}$$

$$\bigcup \cup_{j=1}^{m} \cup_{i \in C_j} \{u_{ij}, y_{ij}.z_{ij}\}.$$

Every odd circuit of G contains the edge e_0.

For the graph $G = G(I)$ define matrix $A = A(I)$ to have the incidence vectors of the edges of $G(I)$ as columns. The *edge cover constraints* are

$$(EC) \qquad\qquad Ax \geq 1, x \geq 0.$$

The *edge cover polytope* is

$$P = P(I) := \{x : Ax \geq 1, x \geq 0\}.$$

Theorem 1 *The following statements are equivalent:*

(i) The system (EC) is box-TDI for $A = A(I)$.

(ii) The system (EC) is TDI for $A = A(I)$.

(iii) P(I) is an integer polytope.

(iv) I is not satisfiable.

Proof. The statements $(i) \implies (ii)$, $(ii) \implies (iii)$ are trivial (from Edmonds-Giles).

To prove $(iii) \implies (iv)$ we have to prove that satisfiability of I implies non-integrality of $P(I)$. Suppose I is satisfiable, and fix a truth assignment which makes it true.

Let $v_i' := v_i \in V$ if variable x_i is false , and $v_i' := \bar{v}_i \in V$ if x_i is true. Pick for all $j = 1, \ldots, m$ an index $i \in \{1, \ldots, n\}$ such that variable x_i is present in clause j with the sign that makes it true. Since the fixed truth assignment satisfies I, for all $j = 1, \ldots, m$ there exists such an i. Let $u_j' := u_{ij}$.

$$U := \cup_{i=1}^{n} \{a_i, v_i'\} \bigcup \cup_{j=1}^{m} \{s_j, u_j'\} \cup \{t_m\}; \;\; U' := \{v_i' : i = 1, ..., n\} \cup \{u_j' : j = 1, ..., m\},$$

where clearly, $U' \subseteq U$.

Claim 1: U is the vertex-set of the circuit C_U defined as the edge-set

$$C_U := \{a_i v_i' : i = 1, \ldots, n\} \cup \{s_j u_j' : j = 1, \ldots, m\} \cup \{a_1 t_m\},$$

where we have $|U| = |C_U| = 2m + 2n + 1$, and every connector vertex has at most one neighbor in U.

Indeed, the only statement that requires an effort is that any connector vertex y_{ij} (or z_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, m$) has at most one neighbor in U. This is true because if y_{ij} has two neighbors in U, then these are v_i' and u_j', meaning that x_i or $\bar{x}_i$ is present in clause j depending on whether $v_i' = v_i$ or $v_i' = \bar{v}_i$. Which ever is present its truth value is 'true' because of the definition of u_j', and it is false because of the definition of v_i'. This contradiction proves the claim.

Let now $c(e) := 1$ if $e \in C$, or e is (a connector) incident to a vertex of U (then in fact to U'), and 0 otherwise.

Claim 2: The minimum of $c^T x$ on (EC) is $m + n + 1/2$.

Indeed, define a primal solution X by $X(e) := 1/2$ if $e \in C_U$; 0 if e is a connector incident to U'; 1 otherwise – that is, if e is not incident to U. Now if $v \in V$ is neither contained in an edge e of value $X(e) = 1$, nor two edges of X-value $1/2$, then $v \notin U$, but its set of neighbors $N(v) \subseteq U$. Then v is a connector vertex and according to Claim 1 it has only one neighbor in U, a contradiction.

So X is a solution, and its value is $m+n+1/2$. The dual solution $Y(v) = 1/2$ if $v \in U$ has the same value. So the optimum is $m + n + 1/2$, as claimed.

Since $m+n+1/2$ is noninteger, $P(I)$ is not an integer polyhedron, finishing the proof of the implication $(iii) \implies (iv)$.

$(iv) \implies (v)$: Suppose I is not satisfiable, and let c be an arbitrary integer objective function, and Y a dual solution for a problem completed with box inequalities. We will call the set of dual variables that belong to the rows of A, that is, the vertices of G, the *vertex-part* of Y.

Again, we show that there exists an optimal dual solution where the support of the vertex-part induces a bipartite graph. Let Y be an optimal dual solution with a maximum number of dual variables set to 0 value among the non-connector vertices. For a contradiction suppose that the vertex-set U of the odd circuit C is contained in the support of the vertex-part of Y.

Claim 3: There exists a connector vertex whose both neighbors are in U.

Indeed, since e_0 is contained in every odd circuit,

- either U contains a connector vertex (with its two incident connectors)
- or U contains v_i or $\bar{v}_i$ for each variable $(i = 1, \ldots, n)$ and s_j, t_j , moreover one x_{ij} denote it u_j for all $j = 1, \ldots, m$.

If the first case holds then we are done, since the degree of connector vertices is 2, and therefore their two neighbors are also on C. So we can suppose that the second case holds.

We show that the following truth assignment provides a feasible solution of the given instance of SAT: for all $i = 1, \ldots, n$ if $\bar{v}_i \in U$ then x_i :=true, $v_i \in U$, then x_i :=false. Since I is not satisfiable, this is not a feasible truth assignment, that is, there exists a clause j for which the variable $\iota \in U \cap C_j$ is false. This means that both neighbors v'_ι and u'_j of $y_{\iota j}$ and $z_{\iota j}$ are in U, and the Claim is proved.

Keeping the notations at the end of the Claim's proof, we got: $Y(v'_i) > 0,$, $Y(u'_j) > 0$. Let $\Delta := \min\{Y(v'_i), Y(u'_j)\}$. Decreasing both $Y(v'_i)$ and $Y(u'_j)$ by Δ and increasing $Y(y_{ij})$ and $Y(z_{ij})$ by Δ we get a solution of the same value, and the number of 0 values on non-connector vertices of the dual solution increases (namely at least one of $Y(v'_i)$ or $Y(u'_j)$ becomes 0), contradicting the choice of Y. $\square$

Surprisingly the TDI property of *minimal systems* describing some polyhedra is proved to be NP-hard, which means that we did not even exploit the possible additional difficulties of redundant inequalities in TDI systems. As far as the box TDI property is also present, this is not so surprising, since this latter depends only on the polyhedron [1].

2 Remarks, Problems

Since the matrix $A = A(I)$ is a $0-1$ matrix, (EC) defines a set-covering polyhedron. Such a polyhedron, or the matrix A, are called *ideal*, if this polyhedron has integer vertices;

A is called *Mengerian* if (EC) is TDI or *box-Mengerian* if (EC) is box-TDI. With these terms, the theorem can be restated as follows:

Corollary 1 *For all input I of the satisfiablility problem, $A(I)$ is ideal if and only if it is Mengerian, or box-Mengerian, and in turn, any (equivalently all) of these properties holds if and only if I is not satisfiable.*

Let us also restate the main complexity results:

Corollary 2 *It is NP-hard to test if a given matrix A is ideal or Mengerian.*

Two important questions come into one's mind related to this corollary.

2.1 High dimension, small number of vertices

It is well-known [10] that any full dimensional polyhedron has a description with a TDI system of linear inequalities also called *Schrijver system.* TDI-ness is then characterized by the emptiness of the set of linear inequalities that have to be added to the system.

The Schrijver system, and already the TDI property, are closely related to *Hilbert-bases*, that is, sets of integer vectors which can represent any integer vector in the cone they generate, with a non-negative integer combination. Indeed, complementary slackness easily implies (see [10]) that a system is TDI if and only if for every vertex (or minimal face) v of the given polyhedron, the set of normal vectors F_v of facets containing v form a Hilbert basis.

It follows that the Schrijver system arises by adding an inequality for each element of the minimal Hilbert basis of F_v (v a minimal face). Since the minimal Hilbert basis of a pointed cone is uniquely determined, so is the minimal TDI defining system of a system of linear inequalities.

It is somewhat surprising that (EC) for $A(I)$ for the inputs I of the satisfiability problem provided NP-hardness result, since *linear programs on (EC) or on its integer points can be solved in polynomial time* (with a well-known and easy reduction to matchings, see [11], volume 3, Chapter 34) furthermore the minimal system describing the 'edge-cover polytope' is TDI !

The only difficulty of checking for integer vertices or the TDI property is that for each input I of the satisfyibility problem, $P(I)$ can have many vertices – one for each potential "blossom", that is for factor-critical subgraph, in particular

$$(*) \qquad x(C) + x(\delta(U) \geq k + 1$$

for each circuit C with vertex-set U, $|U| = 2k + 1$ of $G(I)$, $x \geq 0$. (We do not want to go into matching theory and factorcritical subgraphs; those who are not familiar with these can skip the proof of the following proposition.)

Proposition 1 *The Schrijver system of (EC) is (*).*

Indeed, using the ear-decomposition of factor-critical graphs it is easy to see that the only factor-critical subgraphs of $A(I)$ (I is an input of the satisfyability problem), are the odd circuits: every odd circuit contains e_0, and in an ear decomposition there is no possibility of a second ear, since all other ears are even.

Therefore, in situations when we can list all the vertices of our polyhedron, Theorem 1 does not tell us anything negative:

Problem 1 *What is the complexity of testing for the TDI property of a system of linear inequalities that defines a polyhedron with a fixed number of vertices.*

The most particular case of this problem is when the defined polyhedron has only one vertex:

Problem 2 *What is the complexity of testing whether a set of given vectors forms a Hilbert basis.*

These two problems are clearly equivalent.

Since still nobody knows how to decide for odd holes in polynomial time, and since odd circuits are not only very present in the above arguments, but we are in fact detecting whether all odd circuits have some kinds of chords, it could be worth retrying the following:

Problem 3 *With some modification $G'(I)$ of $G(I)$ (that is constructed in polynomial time) of $G(I)$, could the existence of an odd hole be also equivalent to the satisfiability of I ?*

2.2 Could Mengerian clutters still be characterized ?

Despite the NP-completeness result of Corollary 1 for ideal clutters, *a co-NP characterization* of ideal clutters is known: Lehman's theorem, see [3, 11]. But is Lehman's theorem necessary for such a characterization ? Does it help at all for any complexity issue ? What about the analogous question about the TDI property ?

There are slight differences in the use of the term "NP-complete" by different sources. We say that a problem is *NP-complete* or *coNP-complete*, if it is NP-hard and in NP (coNP).

There is a lot of confusion already about the characterizations of "ideal" clutters: there are some obvious coNP characterizations (but what do they use ?), there is Lehman's combinatorial characterization for which some corollaries are stated concerning computational complexity, and there are some conjectures [3] ...

Suppose $V := \{1, \dots, n\}$ and that we are studying clutters $\mathcal{A} \subseteq 2^V$. What one should keep in mind in the algorithmic considerations is *not to suppose that a clutter $\mathcal{A}$ is given explicitely*, but only that a *filter oracle* is given [9]: an oracle that tells for any given set as input, whether the set contains a member of $\mathcal{A}$; a filter oracle for the clutter immediately provides one for its blocker, since $X \subseteq V$ does not contain an element of the clutter if and only if $V \setminus X$ contains an element of the blocker.

Indeed, in most applications the input data is more condensed than the explicit formulation of the linear program: most usually the clutter is defined with the help of a graph and has exponentially many elements in terms of the input of a graph: for instance odd circuits of a graph or transversals of directed cuts of a digraph, etc. In all of these applications the filter oracle can be very efficiently computed.

It is easy to fall in the catch of saying: "a coNP characterization of integrality is obvious, since one has to certify "only" a fractional vertex x, that can be done by *showing n linearly independent valid inequalities satisfied by x with equality.*" The error in this naive claim is that *x may violate some inequalities of the system.* To check whether x satisfied all inequalities needs the solution of the membership problem which - at our level of the discussion – is the same as the so called *separation oracle* !

Let us call a class of polyhedra *solvable* if the optimization or (actually and) the separation problem can be solved for them in polynomial time, see [7]. If the system of linear inequalities is given explicitly, or with a separation (equivalently optimization) oracle, then one can easily get a coNP characterization in this naive way instead.

The advantage of Lehman's theorem in these terms is that *it provides a polynomial certificate of the nonintegrality of a polytope* even if it is not given explicitly, and does not have to rely on solvability in terms of [7], only on the filter oracle [9].

Briefly summarizing the results about ideal clutters (for analogy) :

Proposition 2 *Recognizing ideal matrices is obviously in coNP for explicitly given or more generally for solvable polyhedra. It is in coNP for set-covering polyhedra given with a filter oracle by Lehman's theorem. It is coNP-complete even for explicitly given set-covering polyhedra.*

For a comparison, if we reverse the direction of the inequality in (EC), integral polyhedra defined with inequalities of this form are called *perfect*, and graphs whose clique matrix is perfect are called *perfect graphs*; (EC) is then called a set packing problem, and the polytope of feasible solutions a set packing polytope. How does Proposition restate for perfect matrices ?

An additional twist modifies (simplifies) the picture in this case: *there is a relation between integrality and solvability of perfect polyhedra.* The filter oracle corresponds to the following in this case (excluding only some uninteresting cases) : the constraint matrix of a set packing problem is not given explicitly, but as the clique matrix of a graph. Only the graph is given ! If its clique matrix defines a perfect polyhedron it is called *perfect.*

A set packing matrix given with a perfect graph *is solvable* [7] !

A second coincidence further simplifies the picture: there is no difference between integer vertices and the TDI property in this case [8], [**?**] ! It is easy to modify the coloration algorithm [7] so as it either finds an integral dual solution, or provides a certificate of imperfectness. The imperfectness certificate that comes out can be combinatorial (neither use the θ number nor odd holes): *showing $n - \omega$ linearly independent cliques (in terms of their incidence vectors).* The latter is a certificate of imperfectness since the differences of characteristic vectors of color classes are orthogonal to all ω-cliques; this orthogonal space is of rank at least $\omega - 1$ if the graph is ω-colorable. (Requiring this property for all induced subgraphs of a graph is equivalent to perfectness [6].)

For these reasons, and using the very recent perfecntess test the picture is slightly different for perfect matrices:

Proposition 3 *Recognizing perfect matrices is in coNP for set-packing polyhedra either by the ellipsoid method or any oracle finding a maximum clique in a graph – and* **without any oracle***, from Lovász's characterization of imperfectness with partitionable graphs [8]. It is polynomially solvable by [4].*

Now we arrive at our point: can a coNP characterization of the TDI property exist – despite the NP-completeness stated in Theorem 1 ? The answer is yes, but we have more open questions here than in the preceding two cases.

The only assertion we can be sure of is a polynomial coNP certificate using the ellipsoid method ! This is already less trivial than in the previous cases, and this time the combinatorial characterization goes to the problem section !

Theorem 2 *It is in coNP to test whether a set-covering polyhedron given explicitly is Mengerian, that is, whether (EC) is TDI.*

Proof. The certificate is $x \in \mathbb{R}^n$, and $w \in \mathbb{Z}^n$ chosen in the following way:

Let A_x be the set of rows of A that are satisfied by x with equality in (EC). Suppose x is a vertex (it may be integer as well) and w is in the minimal Hilbert basis of A_x but not among the rows of A. (Clearly, (EC) is not TDI iff there exists such a w. Since w is in the parallelepiped of A_x, it can be shown to have polynomial size.) Check the feasibility of x, and that the rank of A_x is n certifying that x is a (possibly integer) vertex.

Now check with linear programming one by one for all the rows a of A_x whether the optimum for $w - a$ is still in x: w is in the minimal Hilbert basis of A_x if x is no more optimal for any of these objective functions $w - a$. This can be checked in polynomial time. □

Problem 4 *Find a "combinatorial" coNP characterization that makes possible to test whether a set-covering polyhedron given with a filter oracle is Mengerian (that is, whether (EC) is TDI), or show that this is impossible unless NP=coNP.*

I hope this is all clear,

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