RECOGNIZING PINCH-GRAPHIC MATROIDS

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ABSTRACT. Even-cycle matroids are elementary lifts of graphic matroids. An even-cycle matroid is *pinch-graphic* if it has a signed-graph representation with a blocking pair. We present a polynomial algorithm to check if an internally 4-connected binary matroid is pinch-graphic. Combining this with a result in [6] this allows us to check, in polynomial time, if an arbitrary binary matroid is pinch-graphic.

1. INTRODUCTION

In this paper, we follow the terminology in [5] and [6]. A cycle C of a graph G is a subset of edges of G such that every vertex has even degree in the subgraph of G induced by C. A polygon of a graph is an inclusion-wise minimal non-empty cycle. A signed-graph is a pair (G, Σ) where G is a graph and $\Sigma \subseteq EG$. A cycle C of (G, Σ) is even (resp. odd) if $|C \cap \Sigma|$ is even (resp. odd). A matroid M is an *even-cycle* matroid if its circuits are the inclusion-wise minimal non-empty even cycles of some signed-graph (G, Σ). We then write $M = \mathrm{ecycle}(G, \Sigma)$ and say that (G, Σ) is a *signed-graph representation* of M. Even-cycle matroids are binary and are elementary lifts of graphic matroids [13] § 2.5. A *blocking pair* of a signed-graph (G, Σ) is a pair of vertices v, w of G such that every odd polygon of (G, Σ) contains v or w. A matroid M is *pinch-graphic* if $M = \mathrm{ecycle}(G, \Sigma)$ for some signed-graph (G, Σ) that has a blocking pair. We then say that (G, Σ) is a *BP-representation* of M. Pinch-graphic matroids are elementary projections of graphic matroids [13], Remark 2.9. A *graft* is a pair (H, T) where H is a graph and $T \subseteq VH$ such that each component of G contains even number of vertices of T. Vertices in T are called *terminals*. A cut of (H, T) is even (resp. odd) if it contains an even (resp. odd) number of terminals on each shore. A matroid M is an *even-cut* matroid if its circuits are the inclusion-wise minimal non-empty even cuts

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of some graft (H,T). We then write $M = \mathrm{ecut}(H,T)$ and say that (H,T) is a *graft-representation* of M. Even-cut matroids are binary and are elementary lifts of cographic matroids [13] § 2.5.

1.1. **Algorithmic result.** In [5], we present a polynomial algorithm to check if a binary matroid is an even-cycle matroid and we present a polynomial algorithm to check if a binary matroid is an even-cut matroid. These algorithms however, rely on the existence of a polynomial algorithm to check if a binary matroid is pinch-graphic which does not appear in [5]. The goal of this paper is to present such an algorithm. More precisely, we present an algorithm that solves the following problem,

(1) Given an internally 4-connected binary matroid M, check if M is a pinch-graphic matroid in polynomial time.

In [6] we present an algorithm that solves the following problem,

(2) Given a binary matroid M, check if M is a pinch-graphic matroid or return an internally 4-connected matroid N that is isomorphic to a minor of M such that M is pinch-graphic if and only if N is pinch-graphic, in polynomial time.

By combining algorithms (1) and (2) we get a polynomial algorithm to check if a binary matroid M is pinch-graph (thereby completing the description of the algorithm for recognizing even-cycle and even-cut matroids). Namely, we first apply algorithm (2) and establish whether M is pinch-graphic, or we construct the matroid N and use algorithm (1) to determine whether N is pinch-graphic.

For all the aforementioned algorithms, we assume that the matroid M is given in terms of its $0,1$ matrix representation A and by a polynomial algorithm, we mean an algorithm that runs in time polynomial in the number of entries of A.

An extended abstract of the content of this paper and [5, 6] appeared in [11].

1.2. **The key bound.** Consider a matroid M with rank function r. For $X \subseteq EM$, the *connectivity function* is defined as $\lambda_M(X) := r_M(X) + r_M(EM - X) - r(M)$. [1] Consider $X \subseteq EM$ where $X \neq \emptyset$ and $X \neq EM$ and let k be a positive integer. Then X is *k-separating* when $\lambda(X) \leq k-1$. It is *exactly* k-separating if equality holds. If X is exactly k-separating and $|X|, |EM - X| \geq k$ then X is a *k-separation*. A matroid is *connected* if it has no 1-separation, it is 3-*connected* it is connected and has no 2-separations. Let $\ell \geq 3$ be an integer, then M is $(4,\ell)$-*connected* if it is 3-connected and for every 3-separation X, $|X| \leq \ell$ or $|EM - X| \leq \ell$. Chain theorems for $(4,\ell)$-connected matroids are proved for $\ell = 5$ [9], $\ell = 4$ [4], and $\ell = 3$ [1]. A $(4,3)$-connected matroid is said to be *internally* 4-*connected*.

[1] For sets A, B we denote by $A - B$ the set $\{a \in A : a \notin B\}$.

The following is the key result underpinning algorithm (1),

Theorem 1. *Let M be a pinch-graphic matroid that is not graphic. If M is $(4,5)$-connected then the number of BP-representations of M is in $\mathcal{O}(|EM|^4)$.*

Note that we could remove the condition that M be pinch-graphic in the previous result as otherwise the number of BP-representations is 0 and the result trivially holds. We kept the condition to emphasize that this is a result about pinch-graphic matroids. In [5] we show that there exists non-graphic matroids that are 3-connected, that have an exponential number of BP-representations, and where for each BP-representation the graph is 3-connected. Thus the condition that M is $(4,5)$-connected is critical in the previous theorem. Indeed, this is the motivation for first reducing the recognition problem for pinch-graphic matroids to internally 4-connected matroids.

1.3. **Organization of the paper.** A chain theorem obtained by combining existing results is presented in Section 2. Algorithm (1) is presented in Section 3. We review connectivity in even-cycle and even-cut matroids in Section 4. In Section 5, we provide background material on even-cut matroids. Using this machinery we prove a variant of Theorem 1 in Section 6, modulo the proof of Propositions 23 and 24. These two propositions are proved in Section 7.

2. A CHAIN THEOREM

Let M and N be matroids. M *contains an N-minor* if for some $I, J \subseteq EM$ where $I \cap J = \emptyset$, we have that $M/I \setminus J$ is isomorphic to N. Note, F_7 denotes the Fano matroid, $M(G)$ is the graphic matroid of graph G, and we write M^* for the dual of M. Let M be a binary matroid, we say that a sequence $M_1, \ldots, M_k$ of matroid is a *good sequence* for M if

1. $M_1 = M$ and $M_k \in \{F_7, F_7^*, M(K_5)^*, M(K_{3,3})^*\}$,
2. for all $i \in [k-1]$, M_{i+1} is a single element deletion or contraction of M_i,
3. for all $i \in [k]$, M_i is $(4,6)$-connected.

Here is the key result of this section,

Proposition 2. *Let M be a binary non-graphic matroid that is $(4,5)$-connected. Then there exists a good sequence $M_1, \ldots, M_k$ for M. Moreover, if we are given M by its $0,1$ matrix representation A, then in time polynomial in the number of entries of A we can construct that good sequence.*

The proof will require the following Splitter theorems,

Theorem 3 (Seymour [14]). *Let M be a matroid that is 3-connected that is not a wheel or a whirl, and let N be a 3-connected proper minor of M. If $|EM| \geq 4$, then there exists $e \in EM$ such that $M \backslash e$ or M/e is 3-connected and contains an N-minor.*

Theorem 4 (Geelen and Zhou [3]). *Let M be a binary matroid that is $(4,5)$-connected and let N be an internally 4-connected proper minor of M. If $|EM| \geq 7$, then there exists either*

a. $e \in EM$ such that $M \backslash e$ or M/e is $(4,5)$-connected and contains an N-minor; or
b. $e, e' \in EM$ such that $M/e \backslash e'$ is $(4,5)$-connected and contains an N-minor.

To be able to find the good sequence in polynomial time we require the following results,

Proposition 5 (Cunningham [2]). *Let M be a binary matroid described by an $m \times n$ $0,1$ matrix A and let k, ℓ be fixed integers. In time polynomial in m and n, we can either find a k-separating set X where $|X|, |EM - X| \geq \ell$ or establish that none exists.*

Proposition 6 (Tutte [18]). *Let M be a binary matroid described by an $m \times n$ $0,1$ matrix A. In time polynomial in m and n we can check if M is graphic.*

Note for both results we have actual algorithms, not just proof of existence.

We are ready for the main proof of this section,

Proof of Proposition 2. Since M is not graphic there exists a minor N of M which is minimally non-graphic. It follows from Tutte's characterizations of regular matroids [17] and graphic regular matroids [16, 15] that N is isomorphic to one of F_7, F_7^*, $M(K_5)^*$, or $M(K_{3,3})^*$. Let $M_1 := M$ and let $k = |EM_1| - |EN| + 1$. Let us show that there exists a good sequence by induction on k. If $k = 1$ then $M_1 = N$ and trivially M_1 is the good sequence. Thus, we may assume that $k \geq 2$ and in particular that N is a proper minor of M_1. As $N \in \{F_7, F_7^*, M(K_5)^*, M(K_{3,3})^*\}$ it is internally 4-connected and $|EM_1| \geq |EN| \geq 7$. It follows then from Theorem 4 that there exists a $(4,5)$-connected matroid $\tilde{M}$ with an N-minor where

1. $\tilde{M} = M_1 \backslash e$ or $\tilde{M} = M_1/e$ for some $e \in EM_1$; or
2. $\tilde{M} = M_1 \backslash e/e'$ for some distinct $e, e' \in EM_1$.

If (1) occurs, then we let $M_2 := \tilde{M}$. By induction there exists a good sequence $M_2, \ldots, M_k$. But then $M_1, \ldots, M_k$ is a good sequence for M as required. Hence, we may assume that (2) occurs.

Since $\tilde{M}$ is binary and non-graphic, $\tilde{M}$ is neither a wheel or a whirl. By Theorem 3, there exists a matroid M' where either $M' = M_1 \backslash f$ or $M' = M_1/f$ for some $f \in EM_1$ such that M' is 3-connected and has a minor M'' isomorphic to $\tilde{M}$.

Claim. *M' is $(4,6)$-connected.*

Subproof. Suppose for a contradiction that there exists a 3-separation X of M' such that $|X|, |EM'-X| \geq 7$. Observe that $M'' = M'/f'$ or $M'' = M' \setminus f'$ for some $f' \in EM'$. After possibly replacing X by $EM' - X$ we may assume that $f' \notin X$. Corollary 8.2.6 in [12] implies that $\lambda_{M''}(X) \leq \lambda_{M'}(X) = 2$. Since M'' is 3-connected (as it is isomorphic to $\tilde{M}$) $\lambda_{M''}(X) = 2$. But $|X|, |EM'' - X| \geq 6$ contradicts the fact that M'' is $(4,5)$-connected. $\Diamond$

Now, we let $M_2 := M'$ and $M_3 := M''$. By induction there exists a good sequence $M_3, \dots, M_k$. But then $M_1, M_2, M_3, \dots, M_k$ is a good sequence for M as required. We leave it as an exercise to show how to find the sequence in polynomial-time by mean of Propositions 5 and 6. $\square$

A key ingredient in the analysis of Algorithm (1) will be the following result,

Theorem 7. *Let $M_1, \dots, M_k$ denote $(4,6)$-connected binary matroids where for each $i \in [k-1]$, M_{i+1} is a single element contraction or deletion of M_i. Suppose that M_k is non-graphic and that $|EM_k| \in \mathcal{O}(1)$. Then for each $i \in [k]$, the number of BP-representations of M_i is in $\mathcal{O}(|EM_i|^4)$.*

We will postpone the proof of this result until Section 6.

Combining this theorem with our chain theorem proves Theorem 1.

Proof of Theorem 1. Note M is binary as it is a pinch-graphic matroid. It follows from Proposition 2 that M admits a good sequence $M_1, \dots, M_k$. Then $|EM_k| \leq 10$, hence $|EM_k| \in \mathcal{O}(1)$. It follows by Theorem 7 that $|EM| = |EM_1| \in \mathcal{O}(|EM|^4)$, as required. $\square$

3. The algorithm

We present a description of Algorithm (1) in this section.

3.1. Minors.

We say that $\Gamma \subseteq EG$ is a *signature* of a signed graph (G, Σ) if (G, Σ) and (G, Γ) have the same even cycles. Equivalently, Γ is a signature of (G, Σ) if $\Gamma \triangle \Sigma := \Gamma \cup \Sigma - \Gamma \cap \Sigma$ is a cut of G [10]. The operation that consists of replacing a signature by another signature is called *resigning*. Consider a signed graph (G, Σ) and $I, J \subseteq EG$ where $I \cap J = \emptyset$. The minor $(G, \Sigma)/I \setminus J$ is the signed graph defined as follows: If there exists an odd polygon of (G, Σ) contained in I then $(G, \Sigma)/I \setminus J = (G/I \setminus J, \emptyset)$; otherwise there exists a signature Γ where $\Gamma \cap I = \emptyset$ and $(G, \Sigma)/I \setminus J = (G/I \setminus J, \Gamma - J)$. Note, minors are only defined up to resigning.

Consider an even-cycle matroid M with a representation (G, Σ). Then $(H, \Gamma) = (G, \Sigma)/I \setminus J$ is a representation of the minor $N = M/I \setminus J$ [13], page 21. In particular, the class of even-cycle matroids is minor-closed. We say that the representation (H, Γ) of N *extends* to the representation (G, Σ) of M. Hence, every signed-graph representation of M extends some signed-graph representations of the minor N. Observe that if (G, Σ) has a blocking pair, then so does (H, Γ). It follows in that case that (G, Σ) is a BP-representation of M and that (H, Γ) is a BP-representation of N. Hence, the class of pinch-graphic matroid is also minor-closed. Moreover, every BP-representation of M extends some BP-representation of the minor N.

3.2. **How to extend representations.** Suppose that we have binary matroids M and N where $N = M/e$ or $N = M \setminus e$. The following propositions and remark explain how to construct all BP-representations of M from the BP-representations of N.

Proposition 8 (Proposition 36 [5]). *Let M be a binary matroid, let $e \in EM$ and let $N = M \setminus e$. Let C be a cycle of M using e and let (G, Σ) be a signed-graph representation of N. Then (G, Σ) extends to a representation (H, Γ) of M if and only if for some signature Σ' of (G, Σ) we have $\Gamma = \Sigma'$ when $|C \cap \Sigma'|$ is even and $\Gamma = \Sigma' \cup e$ otherwise, and in addition either, (i) $G|C - e$ has no odd degree vertex in which case H is obtained from G by adding a loop e; or (ii) $G|C - e$ has exactly two odd degree vertices v, w in which case H is obtained from G by adding an edge $e = (v, w)$.*

Here $G|C - e$ denotes the subgraph of G induced by the edges of C that are distinct from e.

Proposition 9 (Proposition 37 [5]). *Let M be a binary matroid, let $e \in EM$ and let $N = M/e$ where N is non-graphic. Let D be a cocycle of M using e and let (G, Σ) be a signed-graph representation of N. Then (G, Σ) extends to a representation (H, Σ) of M if and only if either,*

(i) there exists a signature Γ of $(G, D - e)$; or

(ii) there exists a signature Γ of $(G, [D - e] \,\Delta\, \Sigma)$,

where for (i) and (ii), all edges of Γ are incident to some vertex v or contained in loops and for both cases H is obtained from G by uncontracting e at v according to Γ.

The definition of uncontracting e at v according to Γ can be found in [5].

We will also require the following observation,

Remark 10. *We can check in polynomial time if a signed graph has a blocking pair.*

Proof. First observe that we can check if a signed-graph is bipartite by picking a spanning tree and checking if every fundamental polygon is even. Then we check for every pair of distinct vertices u, v if the signed-graph obtained by deleting u and v is bipartite. □

3.3. **A description of the algorithm.** The algorithm will take as input a binary matroid M that is $(4,5)$-connected, described by its $0,1$ representation A and it decides in time polynomial in the size of A if M is pinch-graphic. (As internally 4-connected matroids are $(4,5)$-connected, this yields an algorithm for checking if an internally 4-connected matroid is pinch-graphic).

First we check if M is graphic (see Proposition 6). If it is, then we can stop and M is pinch-graphic. Otherwise there exists a good sequence $M_1,\dots,M_k$ for M by Proposition 2. Iteratively, we will construct the set $\mathcal{S}_i$ of all BP-representations of M_i. Since $M_k \in \{F_7, F_7^*, M(K_5)^*, M(K_{3,3})^*\}$, $|EM_k| \leq 10$ and we can find the set of BP-representations $\mathcal{S}_k$ by brute force. Suppose now that for some $i \in [k]$ where $i \neq 1$ we have constructed the set $\mathcal{S}_i$. If $\mathcal{S}_i = \emptyset$ then we stop as M is not pinch-graphic since every BP-representation of M should extend some representation from $\mathcal{S}_i$. Otherwise either (i) $M_{i-1} = M_i \setminus e$ or (ii) $M_{i-1} = M_i / e$ for some e. For case (i) we extend the BP-representations of $\mathcal{S}_i$ to $\mathcal{S}_{i-1}$ as in Proposition 8, and for case (ii) we extend the BP-representations of $\mathcal{S}_i$ to $\mathcal{S}_{i-1}$ as in Proposition 9. In both cases we use Remark 10 to only keep the BP-representations. If $\mathcal{S}_1 = \emptyset$ then M is not pinch-graphic otherwise M is pinch-graphic.

Correctness is clear. Note that the algorithm runs in polynomial time in the size of A as we can construct the good sequence in polynomial time and since by Theorem 7 each of the set $\mathcal{S}_i$ have cardinality $\mathcal{O}(|EM_i|)^4$.

4. CONNECTIVITY IN EVEN-CYCLE AND EVEN-CUT MATROIDS

In this section we translate our condition that an even-cycle or an even-cut matroid is $(4,6)$-connected in terms of its signed-graph or graft representation. In particular, we will see that the graph must be essentially 3-connected and that all 2-separations have bounded size.

4.1. **Even-cycles matroids.**

Proposition 11 ([13], Proposition 2.6)**.** *Suppose that* $\operatorname{ecycle}(G,\Sigma)$ *is* 3*-connected. Then*

a. if G has no loop then G is 2 connected;
b. G has at most one loop, which is odd;
c. if X is a 2-separation of G then X contains an odd polygon.

Given a graph G and $X \subseteq EG$, we write $G|X$ for the subgraph of G with edges X and vertices that correspond to endpoints of edges of X. We denote $\partial(X)$ the set of vertices common to $G|X$ and $G|EG - X$. We will need a special case of Proposition 8 [5], see also Lemma 2.7 [13],

Proposition 12. *Let $M = \mathrm{ecycle}(G, \Sigma)$ where (G, Σ) has at least one odd polygon. Let X, Y be a partition of EG where $G|X$ and $G|Y$ are both connected. Then*

$$\lambda_M(X) = \begin{cases} |\partial(X)| & \text{if each of } X \text{ and } Y \text{ contains an odd polygon} \\ |\partial(X)| - 1 & \text{if exactly one of } X \text{ and } Y \text{ contains an odd polygon} \\ |\partial(X)| - 2 & \text{if none of } X \text{ and } Y \text{ contains an odd polygon.} \end{cases}$$

Proposition 13. *Let $M = \mathrm{ecycle}(G, \Sigma)$ that is $(4, 6)$-connected. Then the following hold,*

a. G is 2-connected except for a unique possible odd loop;
b. if X is a 2-separation of G then X contains an odd polygon;
c. if X is a 2-separation of G then $\min\{|X|, |EG - X|\} \leq 6$;
d. (G, Σ) has no parallel edges of the same parity.

Proof. Since M is 3-connected, Proposition 11 implies (a) and (b) hold, moreover, M has no pair of parallel elements. Then (d) holds, for any pair of parallel edges of the same parity would correspond to a parallel pair in M. Suppose X is a 2-separation of G, then by Proposition 12, X is 3-separating. It follows that $\min\{|X|, |EG - X|\} \leq 6$, hence (c) holds. □

4.2. **Even-cut matroids.** An edge of a graft (H, T) is a *pin* if it has an end $v \in T$ of degree 1.

Proposition 14 ([13], Proposition 2.5)**.** *Suppose that $\mathrm{ecut}(H, T)$ is 3-connected. Then*

a. if H has no bridge, then H is 2-connected;
b. if H has a bridge e, then e is a pin and (H, T) has at most one pin;
c. if X is a 2-separation of H then $\mathcal{I}(X) \cap T \neq \emptyset$.

Denote by $\mathcal{I}(X)$ the vertices of $G|X$ that are not in $\partial(X)$.

We will need a special case of Proposition 9 [5], see also Lemma 2.7 [13],

Proposition 15. *Let $M = \mathrm{ecut}(H, T)$ where (H, T) has at least one odd cut. Let X, Y be a partition of EH where $H|X$ and $H|Y$ are both connected. Then*

$$\lambda_M(X) = \begin{cases} |\partial(X)| & \text{if each of } \mathcal{I}(X) \text{ and } \mathcal{I}(Y) \text{ contains a vertex of } T \\ |\partial(X)| - 1 & \text{if exactly one of } \mathcal{I}(X) \text{ and } \mathcal{I}(Y) \text{ contains a vertex of } T \\ |\partial(X)| - 2 & \text{if none of } \mathcal{I}(X) \text{ and } \mathcal{I}(Y) \text{ contains a vertex of } T. \end{cases}$$

Proposition 16. *Let $M = \mathrm{ecut}(H, T)$ that is $(4, 6)$-connected. Then the following hold,*

a. H is 2-connected except for a unique possible pin;

b. *if X is a 2-separation of H then $\mathcal{I}(X) \cap T \neq \emptyset$;*

c. *if X is a 2-separation of H then $\min\{|X|, |EH - X|\} \leq 6$;*

d. *H has no parallel edges;*

e. *every even cut of (H,T) has cardinality at least 3.*

Proof. Since M is 3-connected, Proposition 14 implies (a) and (b) hold, moreover, M has no loop, no pair of parallel elements and no pair of series elements. Then (d) hold for parallel edges e, f of H would be in series in M. Similarly (e) hold, for an even-cut of (H,T) of cardinality at most 2 is either: a loop, or a pair of parallel elements of M. Suppose X is a 2-separation of H, then by Proposition 15, X is 3-separating. It follows that $\min\{|X|, |EH - X|\} \leq 6$, hence (c) holds. $\square$

5. Even-cut matroids - background material

5.1. **Pinch-cographic matroids.** We observe that if an even-cut matroid has a graft-representation with at most two terminals then it is cographic, i.e. we have,

Remark 17 ([13], page 48)**.** *If $M = \mathrm{ecut}(H,T)$ where $|T| \leq 2$ then M is cographic.*

An even-cut matroid M is *pinch-cographic* if it has a representation (H,T) where $|T| \leq 4$. We say in that case that (H,T) is a *T_4-representation*. Thus pinch-cographic matroids generalize cographic matroids. Consider a graft (H,T) with terminals $T = \{t_1, t_2, t_3, t_4\}$. Let G be obtained from H by identifying vertices t_1 and t_2 and by identifying vertices t_3 and t_4. Denote by a the vertex of G corresponding to $t_1 = t_2$ and by b the vertex of G corresponding to $t_3 = t_4$. Let $\Sigma = \delta_H(t_1) \triangle \delta_H(t_3)$. [2] Then (G,Σ) is a signed graph with blocking pair a, b. We say that (G,Σ) is obtained from (H,T) by *folding* and that (H,T) is obtained from (G,Σ) by *unfolding*. Pinch-graphic and pinch-cographic matroids are duals [13], page 26. Namely, we have

Proposition 18. *Let (G,Σ) be a signed graph with a blocking pair and let (H,T) be obtained from (G,Σ) by unfolding. Let M be the pinch-graphic matroid with representation (G,Σ) and let N be the pinch-cographic matroid with representation (H,T). Then $M^* = N$.*

Note that it follows from the previous proposition that our algorithm to recognize pinch-graphic matroid can be used to recognize pinch-cographic matroids as it is equivalent to recognizing if the dual of a given matroid is pinch-graphic.

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[2] For a graph H and vertex v, $\delta_H(v)$ denotes the set of non-loop edges incident to v.

5.2. **Equivalent grafts.** Consider a graph G with a partition X, Y of its edge set where $G|X$ and $G|Y$ are connected and where $\partial(X)$ consists of two vertices v_1 and v_2. Let G' be obtained from G by identifying, for $i = 1, 2$, vertex v_i of $G|X$ with with vertex v_{3-i} of $G|Y$. We say that G' is obtained from G by a 2-*flip* on the set X (resp. Y). We call a 1-*flip* the identification of two vertices in distinct components or splitting two blocks into different components. Two graphs are *equivalent* if they are related by a sequence of 1-flips and 2-flips.

We will make use of the following Theorem of Whitney [19],

Theorem 19. *Two graphs are equivalent if and only if they have the same set of cycles.*[3]

As cycles and cuts are orthogonal subspaces this last theorem says that graphs are equivalent exactly when they have the same cuts. A pair of *grafts* (H, T) and (H', T') are said to be *equivalent* if they have the same even cuts and H and H' are equivalent. Note that Theorem 19 implies readily that,

Remark 20 ([13], Remark 8.9)**.** *If a pair of grafts have the same even cuts and one common odd cut then they have the same cuts, in particular they are equivalent.*

5.3. **Minors.** A subset S of edges of a graph G is a T-*join* if T is the set of vertices of $G|S$ that have odd degree. Consider a graft (G, T) and $I, J \subseteq EG$ where $I \cap J = \emptyset$. The minor $(G, T) \setminus I/J$ is the graft defined as follows: If there exists an odd cut of (G, T) contained in I then $(G, T) \setminus I/J = (G \setminus I/J, \emptyset)$, otherwise there exists a T-join S of (G, T) where $S \cap I = \emptyset$ and $(G, T) \setminus I/J = (G \setminus I/J, R)$ where R is the set of vertices of odd degree of $G \setminus I/J|(S - J)$. Consider an even-cut matroid M with a representation (G, T). Then $(H, R) = (G, T) \setminus I/J$ is a representation of the minor $N = M/I \setminus J$ [13], page 23. In particular, the class of even-cut matroids is minor-closed. We say that the representation (H, R) of N *extends* to the representation (G, T) of M. Hence, every graft-representation of M extends some graft-representations of the minor N. Note that if $|T| \leq 4$, then $|R| \leq 4$. It follows in that case that (G, T) is a T_4-representation of M and that (H, R) is a T_4-representation of N. Hence, the class of pinch-cographic matroid is also minor-closed. Moreover, every T_4-representation of M extends some T_4-representation of the minor N.

5.4. **Extending graft representations.** An *equivalence class of graft-representations* of M is a set of pairwise-wise equivalent graft-representations of M that is inclusion-wise maximal.

[3]Here cycles are viewed as sets of edges.

Proposition 21 ([7] Lemma 9.4). *Let N be an even-cut matroid and let $\mathcal{F}$ be an equivalence class of graft-representations of N. Let M be a matroid with a non-coloop $e \in EM$ for which $N = M \setminus e$. Then the set of extensions of $\mathcal{F}$ to M is a (possibly empty) equivalence class of graft-representations.*

Proposition 22 ([7] Lemma 9.12). *Let N be a non-cographic, even-cut matroid and let $\mathcal{F}$ be an equivalence class of graft-representations of N. Let M be a matroid with a non-loop $e \in EM$ for which $N = M/e$. Then the set of extensions of $\mathcal{F}$ to M is either a (possibly empty) equivalence class of graft-representations or the union of two equivalence classes of graft-representations.*

6. THE PROOF OF THEOREM 7

To prove this theorem we require the following two results which we will prove in Section 7,

Proposition 23. *Let M be a $(4,6)$-connected pinch-graphic matroid with a BP-representation (G, Σ). Then the number of graphs H equivalent to G for which (H, Σ) is a BP-representation of M is in $\mathcal{O}(|EV|)$.*

Note that this result is asymptotically tight. Indeed consider the following example. Construct a graph G as follows: pick a polygon with vertices $v_1, \ldots, v_n$ and add a new vertex c and for all $i \in [n]$ add a pair of edges f_i, g_i with ends c and v_i and add a loop Ω. Let $\Sigma = \{f_i : i \in [n]\} \cup \{v_1v_2, \Omega\}$. Then $\mathrm{ecycle}(G, \Sigma)$ is internally 4-connected. Moreover, for every $i \in [n]$ we can place the loop Ω to be incident to vertex v_i or to c. This yields $|VG|$ distinct equivalent signed-graphs each with a blocking-pair.

Proposition 24. *Let M be a $(4,6)$-connected non-cographic pinch-cographic matroid with a T_4-representation (H,T). Then the number of grafts equivalent to (H,T) is in $\mathcal{O}(1)$.*

6.1. Nice and special representations.

We say that a graft (H,T) is *special* if $|T| = 4$, properties (a)-(e) of Proposition 16 hold, and there exists an odd cut of cardinality smaller or equal to 3. We say that a graft (H,T) is *nice* if $|T| = 4$, properties (a)-(e) of Proposition 16 hold, and every odd cut has cardinality at least 4.

Proposition 25. *The number of special representations of a pinch-cographic $(4,6)$-connected matroid M is in $\mathcal{O}(|EM|^3)$.*

Proof. Let (H,T) be a special graft-representation of M. Then for some set $B \subset EH$ with $|B| \le 3$, B is an odd cut of (H,T). By Remark 20, all special representations with a fixed odd cut B are equivalent. Hence, by Proposition 24, the number of such representations is in $\mathcal{O}(1)$. As the number of possible choices for a set $B \subset EH$ with $|B| \leq 3$ is in $\mathcal{O}(|EM|^3)$ the result follows. □

For a graph G and $v \in VG$ we denote the degree of v by $d_G(v)$.

Next we show that nice grafts are indeed nice.

Proposition 26. *If (H, T) is a nice graft, then H is 3-connected.*

Proof. (H, T) satisfies properties (a)-(e) of Proposition 16. It follows from (e) that,

Claim. *Cuts of (H, T) have cardinality at least 3 and odd cuts cardinality at least 4.*

In particular, (H, T) has no pin. Then by (a) H is 2-connected. Suppose for a contradiction that H is not 3-connected. Then there exists a partition X, Y of the edges of H where $H|X$ and $H|Y$ are connected, $\partial(X) = \{u_1, u_2\}$ for some distinct $u_1, u_2 \in VH$, and $\mathcal{I}(X), \mathcal{I}(Y) \neq \emptyset$. By (b), there exists $z \in \mathcal{I}(X) \cap T$ and by the Claim, $d_H(z) \geq 4$. By (d), there are no parallel edges. Hence, z has at least 4 neighbours and $|\mathcal{I}(X)| \geq 3$. By the Claim vertices $v \in \mathcal{I}(X)$ satisfy $d_H(v) \geq 3$. Thus,

$$\sum_{v \in \mathcal{I}(X)} d_H(v) \geq 3|\mathcal{I}(X)| + 1. \tag{1}$$

Let L denote the edges with one end in $\mathcal{I}(X)$ and one end in $\{u_1, u_2\}$. The Claim implies that $|L| \geq 3$. Let G be the graph induced by vertices $\mathcal{I}(X)$. Then (1) implies that,

$$\sum_{v \in \mathcal{I}(X)} d_G(v) \geq 3|\mathcal{I}(X)| + 1 - |L|. \tag{2}$$

Then X consists of edges of G, edges in L and possibly an edge with ends u_1, u_2. By (2),

$$|X| \geq \frac{1}{2} \sum_{v \in \mathcal{I}(X)} d_G(v) + |L| \geq \frac{1}{2}(3|\mathcal{I}(X)| + 1 - |L|) + |L|.$$

As $|\mathcal{I}(X)|, |L| \geq 3$ we have $|X| \geq \frac{1}{2}(3 \times 3 + 1 - 3) + 3 > 6$, a contradiction to (c). □

6.2. **Unstable sets of grafts.**

Proposition 27. *Suppose* $\mathrm{ecycle}(G, \Sigma)$ *is $(4, 6)$-connected and let a, b be a blocking pair of (G, Σ). Then the number of signatures of (G, Σ) with all edges incident to a or b is in $\mathcal{O}(1)$.*

Proof. Let $\mathcal{S}$ be the set of signatures of (G, Σ) with all edges incident to a or b. We will show $|\mathcal{S}| \in \mathcal{O}(1)$. We may assume after resigning that $\Sigma \in \mathcal{S}$. Pick $\Gamma \in \mathcal{S}$ where $\Gamma \neq \Sigma$. Since $\Sigma \,\Delta\, \Gamma$ intersects every cycle of G with even parity, it is a cut $\delta(U)$ of G. We say that Γ is *skewed* if exactly one of a, b is in U. Let $\Gamma_1, \ldots, \Gamma_\ell$ denote the signatures of $\mathcal{S} - \Sigma$ that are not skewed. Observe that if $\Gamma \in \mathcal{S}$ is skewed, then $\Gamma \,\Delta\, \delta(a)$ is not skewed, thus $\Gamma \,\Delta\, \delta(a) = \Gamma_i$ for some $i \in [\ell]$. It follows that $|\mathcal{S}| \leq 2\ell + 1$. For all $i \in [\ell]$, $\Sigma \,\Delta\, \Gamma_i = \delta(U_i)$ and we may assume that $a, b \notin U_i$. Let $H_1, \ldots, H_k$ denote the components of the graph obtained from G by deleting vertices a, b. Let $i \in [\ell]$ and $j \in [k]$. Since $\delta(U_i)$ is a cut either (i) $VH_k \subseteq U_i$ or (ii) $VH_k \cap U_i = \emptyset$. Define for $i \in [\ell]$, the set $J(i) = \{j \in [k] : VH_k \subseteq U_i\}$, i.e. $J(i)$ indicates what components of $H \setminus \{a, b\}$ are contained in the shore U_i of cut $\Sigma \,\Delta\, \Gamma_i$. Now observe that $\delta(U_i)$ are exactly the edges between vertices of H_j and $\{a, b\}$ for all $j \in J(i)$. It follows that $J(i)$ determines Γ_i uniquely. Hence, ℓ is bounded by the number of choices for $J(i)$ thus $\ell \leq 2^k$. However, as $\mathrm{ecycle}(G, \Sigma)$ is $(4, 6)$-connected, Proposition 13(c) implies that $k \in \mathcal{O}(1)$. Thus $|\mathcal{S}| \leq 2\ell + 1 \in \mathcal{O}(2^k) = \mathcal{O}(1)$. $\square$

Let $\mathcal{S} = \{(H_i, T_i) : i \in [n]\}$ where (H_i, T_i) are nice grafts. We say that the set $\mathcal{S}$ is *unstable* if there exists a pinch-cographic matroid M and for all $i \in [n]$ there exists a graph H_i' obtained from H_i by adding an edge Ω with both ends in T_i, so that (H_i', T_i) is a representation of M. Observe that this implies that $\mathcal{S}$ are all T_4-representations of M/Ω.

Proposition 28. *Let $\mathcal{S}$ be an unstable set of representations of a matroid M. Then $|\mathcal{S}| \in \mathcal{O}(|EM|^3)$.*

Proof. Then $\mathcal{S} = \{(H_i, T_i) : i \in [n]\}$. For each $i \in [n]$ denote by x_i, y_i, w_i, z_i the vertices in T_i. We may assume that H_i' is obtained from H_i by adding edge Ω with ends x_i, y_i. Let G_i be obtained from H_i by identifying x_i with y_i and by identifying w_i with z_i. Denote by a_i the vertex of G_i corresponding to $x_i = y_i$ and denote by b_i the vertex of G_i corresponding to $w_i = z_i$. Let $\Sigma_i = \delta_{H_i}(x_i) \,\Delta\, \delta_{H_i}(w_i)$. Observe that (G_i, Σ_i) is obtained from (H_i, T_i) by folding and that a_i, b_i is a blocking pair of (G_i, Σ_i). Then let $\mathcal{R} = \{(G_i, \Sigma_i) : i \in [n]\}$.

Claim. *The signed-graphs in $\mathcal{R}$ are all pairwise equivalent.*

Subproof. Pick $i, j \in [n]$. We will show that (G_i, Σ_i) and (G_j, Σ_j) are equivalent. Since $\mathcal{S}$ is unstable $\mathrm{ecut}(H_i', T_i) = \mathrm{ecut}(H_j', T_j)$. Hence, $\mathrm{ecut}(H_i', T_i) \setminus \Omega = \mathrm{ecut}(H_j', T_j) \setminus \Omega$. Note that

$$\mathrm{ecut}(H_i', T_i) \setminus \Omega = \mathrm{ecut}(H_i'/\Omega, \{w_i, z_i\}) = \mathrm{cut}(G_i).$$

Similarly, $\mathrm{ecut}(H_j', T_j) \setminus \Omega = \mathrm{cut}(G_j)$. Thus $\mathrm{cut}(G_i) = \mathrm{cut}(G_j)$. It follows from Theorem 19 that G_i and G_j are equivalent. Furthermore, by Proposition 18, (G_i, Σ_i) and (G_j, Σ_j) are BP-representations of M^*. In particular, $\mathrm{ecycle}(G_i, \Sigma_i) = \mathrm{ecycle}(G_j, \Sigma_j)$. Hence, (G_i, Σ_i) and (G_j, Σ_j) are equivalent as required. $\Diamond$

Proposition 18 we know that for each $i \in [n]$, (G_i, Σ_i) is a BP-representation of the $(4,6)$-connected matroid M^*. By the Claim the signed graphs in $\mathcal{R}$ are equivalent. It follows from Proposition 23, that the number of distinct graphs among $G_1, \ldots, G_n$ is in $\mathcal{O}(|EM|)$. Let $K \subseteq [n]$ such for all $k \in K$, $G_k = G$ for some fixed graph G. There are at most $|VG|^2 \in \mathcal{O}(|EM|^2)$ distinct blocking pairs $\{a_k, b_k\}$ of (G, Σ_k) among all $k \in K$. Moreover, by Proposition 27 there are at most $\mathcal{O}(1)$ different signatures Σ_ℓ, $\ell \in K$ for a given blocking pair $\{a_k, b_k\}$ of G. It follows that $|K| \in \mathcal{O}(|EM|^2)$ and hence that $|\mathcal{S}| = |\mathcal{R}| \in \mathcal{O}(|EM|^3)$ as required. $\square$

Proposition 29. *Let M be a binary matroid, $\Omega \in EM$ that is not a loop of M and let $N = M/\Omega$ where N is $(4,6)$-connected. Consider a nice representation (H,T) of N that does not extend uniquely to M. Then (H,T) extends to exactly two representations (H_1,T) and (H_2,T) of M where H_1 is obtained from H by adding edge Ω between t_1, t_2 and where H_2 is obtained from H by adding edge Ω between t_3, t_4 for some labeling t_1, t_2, t_3, t_4 of the vertices of T.*

Proof. Since Ω is not a loop of M there exists a cocircuit D of M with $\Omega \in D$. Cocircuits of even-cut matroids are polygons or inclusion-wise minimal T-joins of the graft representation [13], Remark 2.2. After possibly replacing D with $D \,\Delta\, J$ for some T-join J of H, we may assume that D is a polygon of H_1. If D is a cycle of H_2 then H_1 and H_2 have the same set of cycles, and thus by Theorem 19 are equivalent. But then as H is 3-connected by Proposition 26, so are H_1, H_2 which implies that $H_1 = H_2$, a contradiction. Thus D is a T-join of H_2. Let $P = D - \Omega$. Since D is a polygon of H_1, P is a path of H. As $T = \{t_1, t_2, t_3, t_4\}$ is the set of odd degree vertices in $P \cup \Omega$ of H_2 it follows that Ω has ends say t_3, t_4 in H_2, and that P has end t_1, t_2 in H. But then Ω has ends t_1, t_2 in H_1 and H_1, H_2 are as required. $\square$

This last result implies immediately the following,

Proposition 30. *Let M be a binary matroid, $\Omega \in EM$ that is not a loop of M and let $N = M/\Omega$ where N is $(4,6)$-connected. Then the set of nice representations of N that do not extend uniquely to M form an unstable set.*

6.3. **The proof of the main result.**

Proof of Theorem 7. For all $i \in [k]$ let $N_i = M_i^*$. Then $N_1, \ldots, N_k$ denote $(4,6)$-connected binary matroids where for each $i \in [k-1]$, N_{i+1} is a single element contraction or deletion of N_i and where $|EN_k| \in \mathcal{O}(1)$.

Claim 1.

a. for all $i \in [k]$, representations of N_i have at least four terminals;

b. for all $i \in [k]$, T_4-representations of N_i are nice or special.

Subproof. By hypothesis, M_k is not graphic, hence M_i is not graphic, or equivalently, N_i is not cographic. Then (a) follows by Remark 17. Since M_i is $(4,6)$-connected so is N_i. Then (a) and Proposition 16 implies (b). $\diamondsuit$

Denote by $f(i)$ the number of nice representations of N_i.

Claim 2. *For all $i \in [k-1]$, $f(i) \leq f(i+1) + \mathcal{O}(|EN_i|^3)$.*

Subproof. By Claim 1, every T_4-representation of N_{i+1} is nice or special. Hence, every T_4- representation of N_{i+1} extends either: (i) a nice representation of N_{i+1}, or (ii) a special representation of N_{i+1}. A nice representation of N_i is of *Type I* if it arises as in (i) and of *Type II* if it it arises as in (ii). Note, the definition allows N_i to be of both Type I and Type II.

Case 1. $N_{i+1} = N_i \setminus \Omega$.

Let (H', T') be a T_4-representation of N_i that extends some T_4-representation (H, T) of N_{i+1}. Then (H', T') is obtained from (H, T) by uncontracting edge Ω. If (H, T) is special it has an odd cut B with $|B| \leq 3$. But then B is an odd cut of (H', T') and (H', T') is not nice. Hence, nice representations of N_i are only of Type I. Then by Proposition 26 the equivalence classes of nice representations of N_i, N_{i+1} have cardinality one. By Proposition 21, every equivalence class of graft-representations of N_{i+1} extends to one (possibly empty) equivalence class of graft-representations of N_i. It follows that every nice representation of N_{i+1} extends to at most one nice representation of N_i. Therefore, $f(i) \leq f(i+1)$.

Case 2. $N_{i+1} = N_i / \Omega$.

Let (H, T) be a representation of N_{i+1} and let (H', T') be a representation of N_i that extends (H, T). Then (H', T') is obtained from (H, T) by adding edge Ω. By Proposition 22, every equivalence class of graft-representation of N_{i+1} extends to at most two equivalence classes of graft-representations of N_i. By Proposition 26 the equivalence classes of nice representations of N_i have cardinality one. Proposition 25 implies that the number of special representations of N_{i+1} is in $\mathcal{O}(|EN_{i+1}|^3)$. Therefore, the number of nice Type II representations of N_i is in $\mathcal{O}(|EN_i|^3)$. Let $\mathcal{L}$ be the set of all nice representations of N_{i+1} that extend non-uniquely to N_i. Proposition 30 implies $\mathcal{L}$ is an unstable set. Therefore by Proposition 28, $|\mathcal{L}| \in \mathcal{O}(|EN_i|^3)$. Thus the number of Type I nice representations of N_i is at most, $f(i+1) + |\mathcal{L}| \in f(i+1) + \mathcal{O}(|EN_i|^3)$. Hence, $f(i) \leq f(i+1) + \mathcal{O}(|EN_i|^3)$. $\Diamond$

As $|EN_k| \in \mathcal{O}(1)$, $f(k) \in \mathcal{O}(1)$. It then follows from Claim 1 that for all $i \in [k]$,

$$(\star) \qquad f(i) \in \mathcal{O}(|EN_i|^4).$$

Pick $\hat{\imath} \in [k]$. Let $\mathcal{R}$ denote the set of all T_4-representations of $N_{\hat{\imath}}$. By Claim 1 every graft in $\mathcal{R}$ is special or nice. By Proposition 25, the number of special representations of $N_{\hat{\imath}}$ is in $\mathcal{O}(|EN_{\hat{\imath}}|^3)$. Together with ($\star$) this implies that $|\mathcal{R}| \in \mathcal{O}(|EN_{\hat{\imath}}|^4)$. Let $\mathcal{S}$ denote the set of all BP-representations of $M_{\hat{\imath}}$. Pick an arbitrary representation $(G, \Sigma) \in \mathcal{S}$ and unfold it to get a graft (H, T). By Proposition 18, (H, T) is a T_4-representation of $N_{\hat{\imath}}$, i.e. $(H, T) \in \mathcal{R}$. Moreover, there are at most 12 ways of folding (H, T) to get a BP-representation in $\mathcal{R}$, [4] i.e. at most 12 BP-representations of $\mathcal{S}$ get mapped to the same T_4-representation of $\mathcal{R}$. It follows that $|\mathcal{S}| \leq 12|\mathcal{R}| \in \mathcal{O}(|EN_{\hat{\imath}}|^4)$ as required. $\square$

7. Size of equivalence classes

The goal of this section is to prove Propositions 23 and 24. For the former, we consider BP-representations of a $(4, 6)$-connected pinch-graphic matroid, and for the latter, we consider T_4-representations of a $(4, 6)$-connected pinch-cographic matroids. In both cases these representations have the property that for every 2-separation one side has cardinality at most 6. This leads to the notion of well connected-graphs that we study next.

7.1. **Well connected-graphs.** A graph G is *well-connected* if the following conditions hold:

- (w1) $|EG| \geq 25$;
- (w2) G is loopless and 2-connected;
- (w3) for every 2-separation X of G, we have $\min\{|X|, |EG - X|\} \leq 6$;
- (w4) parallel classes have cardinality at most two.

Let X be a 2-separation of a well-connected graph G. We say that X is *small* if $|X| \leq 6$. A small 2-separation X is *maximal* if it is inclusion maximal among all small 2-separations.

The following is the motivation for considering maximal small 2-separation,

Proposition 31. *In a well-connected graph any two maximal small* 2*-separations are disjoint. In particular, every small* 2*-separation is contained in a unique maximal small* 2*-separation.*

[4] 3 choices for decide which pairs of terminals get identified, and 2×2 choices for the signature.

Before we present the proof we require some definitions. A pair of 2-separations X and Y *cross* if all of the following are non-empty,

$$X \cap Y, \quad X \cap (EG - Y), \quad (EG - X) \cap Y \quad \text{and} \quad (EG - X) \cap (EG - Y).$$

A *necklace* is a graph obtained from a polygon C with at least 4 edges by replacing each edge by a connected graph. The graphs replacing edges of C are the *beads* of the necklace.

Proof of Proposition 31. Let G be a well-connected graph G and consider an arbitrary pair of maximal small 2-separations X and Y. We need to show that $X \cap Y = \emptyset$. Let $\bar{X} = EG - X$ and let $\bar{Y} = EG - Y$. Note that since X, Y are maximal $X \cap \bar{Y}, \bar{X} \cap Y \neq \emptyset$. Moreover, since X, Y are small and $|EG| \geq 25$, $X \cup Y \neq VG$, or equivalently $\bar{X} \cap \bar{Y} \neq \emptyset$. Thus it suffice to show that X and Y are not crossing. Suppose otherwise. As X, Y cross, one of the cases occur [13],

i. $\partial(X \cap Y) = \partial(X \cap \bar{Y}) = \partial(\bar{X} \cap Y) = \partial(\bar{X} \cap \bar{Y})$; or
ii. G is a necklace with beads $X \cap Y, X \cap \bar{Y}, \bar{X} \cap Y, \bar{X} \cap \bar{Y}$.

In both cases, $\bar{X} \cap \bar{Y}$ is either a 2-separation of G, a one edge set, or a set of two parallel edges. Let $Z = X \cup Y$ and observe that $EG - Z = \bar{X} \cap \bar{Y}$. Since $|X|, |Y| \leq 6$ and $|EG| \geq 25$ we have $|EG - Z| \geq 7$. Clearly, Z is not an edge or a pair of parallel edges as $X, Y \subseteq Z$. Thus Z is a small 2-separation. But this contradicts our assumption that X was a maximal small 2-separation. □

We are working in this paper with edge-labeled graphs. At this juncture we need to concern ourselves with vertex labels as well. Consider a pair of graphs G, G' with the same set of (labeled) edges. Then a bijection $f : VG \to VG'$ is an *isomorphism* from G to G' if for every *labeled edge* e: e has ends u, v in G if and only if e has ends $f(u), f(v)$ in G'.

Proposition 32. *Consider a well-connected graph G. Let $X_1, \ldots, X_k$ denote the maximal small 2-separations of G and let $Y = EG - (X_1 \cup \ldots \cup X_k)$. Suppose that G' is equivalent from G. Then*

a. $X_1, \ldots, X_k$ are precisely the maximal small 2-separations of G'; and
b. there is an isomorphism f from $G|Y$ to $G'|Y$ such that for every $i \in [k]$: f maps $\partial_G(X_i)$ to $\partial_{G'}(X_i)$.

Proof. Since G is 2-connected, G and G' are related by a sequence of 2-flips. It suffices to prove (a) and (b) when G' is obtained from G by a single 2-flip on a set Z as we can then iterate the result. After possibly replacing Z by $EG - Z$ we may assume that Z is a small 2-separation of G. For some $j \in [k]$, X_j is the maximal small 2-separation of G containing Z. Let us prove (a). Pick $i \in [k]$. By Proposition 31 $i = j$ or $X_i \cap X_j = \emptyset$. In particular, $Z \subseteq X_i$ or $Z \cap X_i = \emptyset$ and it follows that X_i

is also a 2-separation of G'. Suppose for contradiction X_i is not maximal in G'. Then there exists a maximal small 2-separation X_ℓ of G' strictly containing X_i. Note that G can be obtained from G' by a 2-flip on Z. But then by the previous argument, X_ℓ is a small 2-separation of G, a contradiction as X_i is maximal in G. Finally, note that (b) follows from the fact that $Y \cap Z \subseteq Y \cap X_j = \emptyset$. □

7.2. **The proof of Proposition 24.** Throughout this section M will denote a $(4,6)$-connected pinch-cographic matroid that is not cographic. Also throughout this section (H,T) will denote a T_4-representation of M. We will need to show that the number of T_4-representations equivalent to (H,T) is in $\mathcal{O}(1)$. Note that by Proposition 14, (H,T) has at most one pin.

Claim 1. *Suppose that $e = uv$ is a pin of (H,T) where $v \in T$ has degree 1. Then*

a. $u \notin T$, and

b. there are exactly 4 equivalent T_4-representations that can be obtained from (H,T) by moving e.

Proof. Let (H',T') be a graft equivalent to (H,T) obtained by moving the end u of e to some new vertex w. Using the same labeling of the vertices in $H \setminus e$ and $H' \setminus e$ we have $T' = T \Delta \{u,w\}$. In particular, $u \notin T$ for otherwise we can pick $w \in T$ and this yields $|T'| = 2$, which implies by Remark 17 that M is cographic, a contradiction. Moreover, if $w \notin T$ then $|T'| = 6$ and (H',T') is not a T_4-representation. It follows that there are exactly 4 possible T_4 that can be obtained from (H,T) by moving the pin e, namely move the end u of the pin to each terminal T. □

If (H,T) has no pin then we let $(G,R) = (H,T)$. If (H,T) has pin $e = uv$ then we let $G = H/e$ and $R = T - v \cup \{w\}$ where w is the vertex of G that corresponds to edge e of H.

Claim 2. *We may assume the following hold for the graft (G,R),*

a. G is well-connected;

b. for any 2-separation X of G we have $\mathcal{I}_G(X) \cap R \neq \emptyset$.

Proof. We may assume $|EG| \geq 25$ for otherwise trivially the number of grafts equivalent to (H,T) is in $\mathcal{O}(1)$. By Proposition 16(a), H is 2-connected except for a unique possible pin. It follows that G is 2-connected. By Proposition 16(c), if X is a 2-separation of H then $|X| \leq 6$. By Proposition 16(d), H and hence G have no parallel edges. This implies that (w1)-(w4) hold and G is well-connected, i.e. (a) holds. Finally, (b) follows from Proposition 16(b). □

Let $\mathcal{S}$ be the set of T_4-representation that are equivalent to (G,R). Note that it suffices to show that $|\mathcal{S}| \in \mathcal{O}(1)$ since by Claim 1(b), every graft in $\mathcal{S}$ corresponds to at most 4 grafts equivalent to (H,T).

Let $X_1, \dots, X_k$ denote the maximal small 2-separations of G. By Proposition 31 $X_1, \dots, X_k$ are pairwise disjoint, in particular, for all distinct $i, j \in [k]$, $\mathcal{I}_G(X_i) \cap \mathcal{I}_G(X_j) = \emptyset$. Because of Claim 2(b), $R \cap \mathcal{I}_G(X_j) \neq \emptyset$ for all $j \in [k]$. As $|R| = 4$ it follows that $k \leq 4$. Let $Y := EG - (X_1 \cup \ldots \cup X_k)$. Pick an arbitrary graft (G', R') from $\mathcal{S}$. By Proposition 32 there is an isomorphism f from $G|Y$ to $G'|Y$. Hence, G and G' only differ in the subgraphs induced by $X_1, \dots, X_k$. As $k \leq 4$ and $|X_i| \leq 6$, G' is obtained from G by a sequence of 2-flips that is bounded by a constant. Finally, observe that G' determines the set of terminals R' uniquely (as an R-join of G must be an R'-join of G' [13], page 11). Hence, $|\mathcal{S}| \in \mathcal{O}(1)$ as required.

7.3. **The proof of Proposition 23.** Throughout this section M will denote a $(4,6)$-connected pinch-graphic matroid. Also throughout this section (H, Γ) will denote a BP-representation of M. We will need to show that the number of graph H' equivalent to H for which (H', Γ) has a blocking pair is in $\mathcal{O}(|VH|)$. Note that by Proposition 11, (H, T) has at most one loop (that is odd). If (H, Γ) has no loop then we let $(G, \Sigma) = (H, \Gamma)$. If (H, Γ) has loop e then we let $G = H \setminus e$ and $\Sigma = \Gamma - e$.

Claim 1. *We may assume the following hold for the signed graph* (G, Σ)*,*

a. G is well-connected;

b. if X is a 2-separation of H then X contains an odd polygon.

Proof. We may assume $|EG| \geq 25$ for otherwise trivially the number of graphs equivalent to H is in $\mathcal{O}(1)$. By Proposition 13(a), H is 2-connected except for a unique possible loop. It follows that G is 2-connected. By Proposition 13(c), if X is a 2-separation of H then $\min\{|X|, |EH - X|\} \leq 6$. By Proposition 13(d), H and hence G have no parallel edges of the same parity, in particular, every parallel class has at most two edges. This implies that (w1)-(w4) hold and G is well-connected, i.e. (a) holds. Finally, (b) follows from Proposition 13(b). $\square$

Let $\mathcal{S}$ be the set of pairs (G', v) where

i. G' is equivalent to G;

ii. (G', Σ) has a blocking pair v, w for some $w \in VG'$.

We claim that it suffices to show that $|\mathcal{S}| \in \mathcal{O}(|VG|)$. Let $\mathcal{S}'$ denote the set of graphs H' equivalent to H for which (H', Γ) has a blocking pair. If (H, Γ) has no loop, then $\mathcal{S}' = \{G : (G, v) \in \mathcal{S}\}$. If (H, Γ) has a loop e then $\mathcal{S}'$ is the set of all graphs obtained by adding loop e at vertex v for graph G for all $(G, v) \in \mathcal{S}$. In both case $|\mathcal{S}'| = |\mathcal{S}| \in \mathcal{O}(|VH|)$ as required.

In the proof of Proposition 24 we could bound the number of maximal small 2-separations. Alas this is not possible in this case. Tackling this more complex situation requires additional tools.

7.3.1. *The blocking vertex lemma.* First an observation,

Claim 2. *Let G_1, G_2 be equivalent graphs where P is a path of both G_1 and G_2. For $i = 1, 2$ let H_i be the graph obtained from G_i by adding an edge e between the ends of P. Then, H_1, H_2 are equivalent.*

Proof. Note, that $P \cup e$ is a polygon of H_1 and H_2. The cycle space of H_i is generated by cycles not containing e and one cycle containing e. Since $H_i \setminus e = G_i$ and G_1, G_2 are equivalent, H_1, H_2 have the same cycle space and the result follows from Theorem 19. □

A vertex of a signed graph is a *blocking vertex* if it intersects every odd polygon.

Claim 3. *Let G' be equivalent to G and let X be a maximal small 2-separation of G and thus of G'. Suppose that $G|EG - X$ and $G'|EG - X$ are isomorphic and have the same vertex labelling. Let $\{v, w\} = \partial_G(X) = \partial_{G'}(X)$. If v is a blocking vertex of both $(G|X, \Sigma \cap X)$ and $(G'|X, \Sigma \cap X)$ then G and G' are isomorphic.*

Proof of Claim 3. Let $X' \subseteq X$ be an (inclusion-wise) minimal 2-separation of G. Consider first the case where there exists two internally disjoint paths in $G|X'$ between $\partial_G(X')$. Then $G|X'$ and $G'|X'$ are isomorphic and we may use the same vertex labelling for $\mathcal{I}_G(X')$ and $\mathcal{I}_{G'}(X')$. Let $s \in \mathcal{I}_G(X')$. By Claim 1 there exists an odd polygon $C \subseteq X'$. Since v is a blocking vertex of $(G|, \Sigma \cap X)$ and $(G'|X, \Sigma \cap X)$, C contains v in both G and G'. It follows that there exists an sv-path P_1 contained in X' in both G and G'. Consider now the case where there does not exists two internally disjoint paths in $G|X'$ between $\partial_G(X')$. Then since X' is minimal and since v is a blocking vertex of $(G|X, \Sigma \cap X)$, $X' = \{f, g, h\}$ where $\{f, g\}$ is an odd polygon, both f, g are incident to v and $\{f, g, h\}$ is a cut. Since $\{f, g\}$ is an odd polygon, and since v is a blocking vertex of $(G'|X, \Sigma \cap X)$, f, g are incident to v in G'. Then define P_1 as the path that consists of edge f. In both cases P_1 is a path of G and G' contained in X with an end in $\mathcal{I}_G(X')$ and an end v. Let t be a vertex in $\mathcal{I}_G(EG - X) = \mathcal{I}_{G'}(EG - X)$. Let P_2 be an vt-path in $G|EG - X$ and hence of $G'|EG - X$. Then $P_1 \cup P_2$ is an st-path of both G and G'. Let H, H' be the graphs obtained from G, G' by adding an edge $e_{X'}$ joining s, t. By Claim 2, H, H' are equivalent. Let J, J' be the graphs obtained from G, G' by adding repeatedly $e_{X'}$ for every minimal 2-separation $X' \subseteq X$ of G. Then, J, J' are equivalent and 3-connected. Hence, J, J' are isomorphic and thus so are G and G'. □

Denote by $X_1, \ldots, X_k$ the maximal small 2-separations of G. By Proposition 31, $X_1, \ldots, X_k$ are pairwise disjoint. Let $Y = EG - (X_1 \cup \ldots \cup X_k)$. Here is our key result about blocking vertices,

Claim 4. *Let G' be equivalent to G and assume because of Proposition 32 that $G|Y$ and $G'|Y$ have the same vertex labeling. Let $i \in [k]$ and denote by v, w the vertices in $\partial_G(X_i) = \partial_{G'}(X_i)$. If v is a blocking vertex of both $(G|X_i, \Sigma \cap X_i)$ and $(G'|X_i, \Sigma \cap X_i)$, then $G|Y \cup X_i$ and $G'|Y \cup X_i$ are isomorphic.*

Proof. G' is obtained from G by a sequence of 2-flips. Since $X_1, \ldots, X_k$ are disjoint we can assume that all 2-flips on sets contained in X_i are done first. Then apply Claim 3 after all these 2-flips to deduce that $G|Y \cup X_i$ and $G'|Y \cup X_i$ are isomorphic as required. □

7.3.2. *Case analysis.* Observe that for distinct $i, j \in [k]$, $|\partial_G(X_i) \cap \partial_G(X_j)| \leq 1$ for otherwise this would contradict the maximality of the sets X_i, X_j. We say that a vertex v of G is *special* if there exists distinct $i, j, \ell \in [k]$ such that $v \in \partial_G(X_i) \cap \partial_G(X_j) \cap \partial(X_\ell)$. Similarly, we define special vertices of G'.

Claim 5. *If v is a special vertex of G then every blocking pair of (G, Σ) contains v.*

Proof. As v is special, there exists distinct $i, j, \ell \in [k]$ such that $v \in \partial_G(X_i) \cap \partial_G(X_j) \cap \partial(X_\ell)$. By Claim 1 each of X_i, X_j, X_ℓ contains an odd polygon C_i, C_j, C_ℓ, respectively. Since $VC_i - v, VC_j - v, VC_\ell - v$ are pairwise disjoint the result follows. □

It follows from the previous claim that there are at most two special vertices. We will thus consider three cases, namely, (1) there are two special vertices, (2) there is exactly one special vertex and (3) there is no special vertex.

Case 1. G has exactly two special vertices, say v and w.

We will prove that $|\mathcal{S}| \in \mathcal{O}(1)$ in this case. By Claim 5, $\{v, w\}$ is the unique blocking pair of (G, Σ). Note that there are three possibility for each $i \in [k]$,

i. $\partial_G(X_i) \cap \{v, w\} = \{v\}$,
ii. $\partial_G(X_i) \cap \{v, w\} = \{w\}$, or
iii. $\partial_G(X_i) \cap \{v, w\} = \{v, w\}$.

Pick an arbitrary pair $(G', v') \in \mathcal{S}$. For cases (i) and (ii), by Claim 4, $G|Y \cup X_i$ and $G'|Y \cup X_i$ are isomorphic. There is at most one $i \in [k]$ for case (iii). Thus the graph G' is obtained by a sequence of 2-flips on sets contained in X_i. Since $|X_i| \leq 6$, there are $\mathcal{O}(1)$ such graphs G'. Finally, observe that v, w are also special vertices of G'. In particular, v, w is the unique blocking pair of (G', Σ). It follows that $v' = v$ or $v' = w$. Thus $|\mathcal{S}| \in \mathcal{O}(1)$ as claimed.

Case 2. G has exactly one special vertex, say v.

Pick an arbitrary pair $(G', v') \in \mathcal{S}$. Let us partition the set $[k]$ as follows,

$$A_1 = \{i \in [k] : v \notin \partial_G(X_i)\}$$

$$A_2 = \{i \in [k] : v \in \partial_G(X_i) \text{ and } v \text{ is a blocking vertex of } (G'|X_i, \Sigma \cap X_i)\}$$

$$A_3 = \{i \in [k] : v \in \partial_G(X_i) \text{ and } v \text{ is not a blocking vertex of } (G'|X_i, \Sigma \cap X_i)\}$$

Claim 6. $|A_1| \le 2$.

Proof. By Claim 5, every blocking pair of (G, Σ) contains v. Let v, w be an arbitrary blocking pair of (G, Σ). Let $i \in A_1$. By Claim 1(b), X_i contains an odd polygon. Since $v \notin \partial_G(X_i)$ either: (i) $w \in \mathcal{I}_G(X_i)$ or (ii) $w \in \partial_G(X_i)$. If we have $i \in A_1$ with outcome (i) then there is no $j \in A_1$, $j \neq i$ with either outcomes (i) or (ii). Since v is the unique special vertex of G, there are at most two $i \in [k]$ for which outcome (ii) holds. Hence, $|A_1| \le 2$. □

Claim 7. $|A_3| \le 1$.

Proof. Suppose for a contradiction there exists distinct $i, j \in A_3$. As v is special, there exists ℓ with $\delta_G(X_\ell) \ni v$. Then there exists odd polygons $C_i \subseteq X_i$ and $C_j \subseteq X_j$ of G' avoiding v. Moreover, there exists an odd polygon $C_\ell \subseteq X_\ell$. But then C_i, C_j, C_ℓ are vertex disjoint in G' which contradicts the fact that (G', Σ) has a blocking pair. □

For $i = 1, 2, 3$ let $Z_i = \bigcup(X_i : i \in A_i)$. It follows by Claim 4 that $G|Y \cup Z_2$ and $G'|Y \cup Z_2$ are isomorphic. Consider first the case where $A_3 = \emptyset$ and let $(G', v') \in \mathcal{S}$. Then G' is obtained from G by a sequence of 2-flips that are contained in Z_1. Since G is well-connected, $|X_i| \le 6$ for all $i \in [k]$ and by Claim 6, $|Z_1| \le 12$. Hence, there are $\mathcal{O}(1)$ such graphs G'. Trivially, there are $|VG|$ choices for the vertex v', thus $|\mathcal{S}| \in \mathcal{O}(|VG|)$ in this case. Consider now the case where $A_3 \neq \emptyset$ and let $(G', v') \in \mathcal{S}$. Then by Claim 7 there is a unique element $\hat{\imath} \in A_3$. Then G' is obtained from G by a sequence of 2-flips that are contained in $Z_1 \cup Z_3$. Since $|Z_1 \cup Z_3| \le 18$ and since there are at most $|A_3| \le |VG|$ choices to pick the element $\hat{\imath}$ in A_3 the number of such possible graphs G' is in $\mathcal{O}(|VG|)$. Observe that every blocking pair of (G', Σ) consists of v and a vertex of $G'|X_{\hat{\imath}}$. Thus, there are $\mathcal{O}(1)$ choices for the vertex v' and $|\mathcal{S}| \in \mathcal{O}(|VG|)$ in this case as well.

Case 3. G has no special vertex.

Let v, w denote a blocking pair of (G, Σ). For every $i \in [k]$, X_i contains an odd polygon. It follows that for all $i \in [k]$ either (i) $\{v, w\} \cap \partial_G(X_i) \neq \emptyset$ or (ii) $\{v, w\} \cap \mathcal{I}_G(X_i) \neq \emptyset$. Since neither v nor w are special there are at most 4 elements in $[k]$ for which (i) holds. Trivially, (ii) can hold for at most 2 elements in $[k]$. Thus $k \leq 6$. Let $(G', v') \in \mathcal{S}$. Let $Z = \bigcup(X_i : i \in [k])$ then $|Z| \leq 6k = 36$. Then G' is obtained from G by a sequence of 2-flips that are contained in Z. Hence, there are $\mathcal{O}(1)$ such graphs G'. Trivially, there are $|VG|$ choices for the vertex v', thus $|\mathcal{S}| \in \mathcal{O}(|VG|)$ in this case.

In all cases we have $|\mathcal{S}| \in \mathcal{O}(|VG|)$ which completes the proof of Proposition 23.

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