

# Differentiation of Monotone Functions

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# Derivatives of Monotonic Functions

**Recall:** We saw that if  $F \in AC(\mathbb{R}) \cap NBV(\mathbb{R})$ , then we could find an  $f \in L_1(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$  such that

$$F(x) = \int_{-\infty}^x f \, dm.$$

**Remark:** We will show that the function  $F(x)$  satisfies a variant of the Fundamental Theorem of Calculus, in that  $F(x)$  is differentiable  $m$ -almost everywhere with  $F'(x) = f(x)$ ,  $m$ -almost everywhere. In particular,

$$F(x) = \int_{-\infty}^x F' \, dm.$$

In addition, we will show that any monotonic function is also differentiable  $m$ -almost everywhere and then use this to characterize the absolutely continuous functions as those for which this variant of the Fundamental Theorem holds.

**Definition:** Let  $\mathcal{J}$  be a collection of non-degenerate intervals. We say that  $\mathcal{J}$  is a Vitali cover of  $E \subseteq \mathbb{R}$  if for all  $\epsilon > 0$  and any  $x \in E$  there exists  $I \in \mathcal{J}$  such that  $x \in I$  and  $\ell(I) < \epsilon$ .

**Theorem: [Vitali's Lemma]**

Let  $E \subseteq \mathbb{R}$  be such that  $m^*(E) < \infty$  and let  $\mathcal{J}$  be a Vitali cover of  $E$ . Then there exists for each  $\epsilon > 0$  a finite pairwise disjoint collection  $\{I_1, I_2, \dots, I_n\} \subseteq \mathcal{J}$  such that  $m^*(E \setminus \bigcup_{j=1}^n I_j) < \epsilon$ .

# Vitali's Lemma

**Proof:** We may assume that the intervals in  $\mathcal{J}$  are closed since finitely many endpoints will contribute only a set of measure zero.

Let  $E \subseteq U \subseteq \mathbb{R}$  be an open set with  $m(U) < \infty$ . Since  $\mathcal{J}$  is a Vitali cover, we may assume that if  $I \in \mathcal{J}$ , then

$$I \subseteq U.$$

Let  $I_1 \in \mathcal{J}$ . Suppose that  $\{I_1, I_2, \dots, I_n\} \subseteq \mathcal{J}$  have been chosen to be pairwise disjoint. If  $E \subseteq \bigcup_{j=1}^n I_j$ , then stop. Otherwise let

$$x \in E \setminus \bigcup_{j=1}^n I_j.$$

Then we can find  $I \in \mathcal{J}$  such that  $x \in I$  and

$$I \cap \left( \bigcup_{j=1}^n I_j \right) = \emptyset.$$

# Vitali's Lemma

**Proof Cont'd:** Let  $k_n$  be the supremum of the lengths of the intervals in  $\mathcal{J}$  that are disjoint from  $\bigcup_{j=1}^n I_j$ . Since  $E \not\subseteq \bigcup_{j=1}^n I_j$ , we can find  $I_{n+1} \in \mathcal{J}$  such that  $\ell(I_{n+1}) > \frac{k_n}{2}$  and  $I_{n+1}$  is disjoint from  $\{I_1, I_2, \dots, I_n\}$ .

Proceeding inductively gives us a possibly finite sequence  $\{I_j\}$  of pairwise disjoint intervals in  $\mathcal{J}$  such that  $\bigcup_{j=1}^{\infty} I_j \subseteq U$  and as such

$$m\left(\bigcup_{j=1}^{\infty} I_j\right) \leq m(U) < \infty.$$

From this it follows that we can find an  $N \in \mathbb{N}$  so that

$$\sum_{j=N+1}^{\infty} \ell(I_j) < \frac{\epsilon}{5}.$$

# Vitali's Lemma

**Proof Cont'd:** Let  $S = E \setminus \bigcup_{j=1}^N I_j$ . Let  $x \in S$ . Since  $x \in (\bigcup_{j=1}^N I_j)^c$  which is open, there is an interval  $I \in \mathcal{J}$  that is disjoint from  $\bigcup_{j=1}^N I_j$ . It follows that

$$\ell(I) \leq k_n < 2\ell(I_{n+1})$$

if  $I \cap \bigcup_{j=1}^n I_j = \emptyset$ . Since

$$\lim_{n \rightarrow \infty} \ell(I_n) = 0,$$

$I$  must meet at least one of the  $I_n$ 's. Let  $n_0$  be the smallest  $n$  so that

$$I_n \cap I \neq \emptyset.$$

Then  $n_0 > N$  and

$$\ell(I) \leq k_{n_0-1} < 2\ell(I_{n_0}).$$

# Vitali's Lemma

**Proof Cont'd:** Since  $I_{n_0} \cap I \neq \emptyset$  and  $x \in I$ , the distance from  $x$  to the midpoint of  $I_{n_0}$  is at most

$$\ell(I) + \frac{1}{2}\ell(I_{n_0}) \leq \frac{5}{2}\ell(I_{n_0}).$$

Therefore  $x \in J_{n_0}$ , where  $J_{n_0}$  is the interval with the same center as  $I_{n_0}$  and five times the length.

For each  $n \geq N + 1$  let  $J_n$  be the interval with the same center as  $I_n$  but with five times the length. Then

$$S \subseteq \bigcup_{n=N+1}^{\infty} J_n,$$

and hence

$$m^*(S) \leq 5 \sum_{n=N+1}^{\infty} \ell(I_n) < \epsilon.$$

**Definition:** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . For any  $x \in \mathbb{R}$ , let

$$D^+ f(x) = \limsup_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h} \quad (\text{upper right hand})$$

$$D_+ f(x) = \liminf_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h} \quad (\text{lower right hand})$$

$$D^- f(x) = \limsup_{h \rightarrow 0^+} \frac{f(x) - f(x-h)}{h} \quad (\text{upper left hand})$$

$$D_- f(x) = \liminf_{h \rightarrow 0^+} \frac{f(x) - f(x-h)}{h} \quad (\text{lower left hand})$$

These four quantities are called the *Dini derivatives* of  $f(x)$  at  $x$ .

## Remarks:

- 1)  $f$  is differentiable at  $x$  if and only if

$$D^+f(x) = D^-f(x) = D_+f(x) = D_-f(x) < \infty.$$

In this case,

$$f'(x) = D^+f(x).$$

- 2) In general

$$D^+f(x) \geq D_+f(x) \quad \text{and} \quad D^-f(x) \geq D_-f(x)$$

- 3) If  $D^+f(x) = D_+f(x)$ , we denote the common value by  $f(x^+)$ . If both are finite, then  $f(x^+)$  is called the *right hand derivative*.  $f(x^-)$  is defined in a similar fashion and is called the *left hand derivative* if it exists.

## Remarks:

4) If

$$E_1 = \{x \in [a, b] \mid D^+f(x) > D_-f(x)\}$$

and

$$E_2 = \{x \in [a, b] \mid D^-f(x) > D_+f(x)\}$$

and if

$$m(E_1 \cup E_2) = 0,$$

then

$$D^+f(x) = D^-f(x) = D_+f(x) = D_-f(x)$$

$m$ -a.e. (Exercise)

# Lebesgue's Differentiation Theorem:

**Theorem [Lebesgue's Differentiation Theorem]:** Let  $f : [a, b] \rightarrow \mathbb{R}$  be increasing. Then  $f$  is differentiable almost everywhere on  $[a, b]$ ,  $f'$  is measurable, integrable and

$$\int_{[a,b]} f' dm \leq f(b) - f(a).$$

**Corollary:** If  $f$  is of bounded variation on  $[a, b]$ , then it is differentiable almost everywhere. In particular, if  $f$  is absolutely continuous on  $[a, b]$ , then it is differentiable almost everywhere.