Modes of Convergence: Introduction

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Modes of Convergence

Note: Given a sequence $\{f_n\}$ of measurable real-valued functions defined on a measure space $(X, \mathcal{A}, \mu)$ we have various modes in which such a sequence may be deemed to converge to a function f. Notably

1) Pointwise convergence
2) Uniform convergence
3) Convergence almost everywhere
4) Convergence in L_p

Goal: We will introduce several other important modes of convergence and establish the known links between them.

Modes of Convergence

Definition: Let $\{f_n\}$ be a sequence of measurable real-valued functions defined on a measure space $(X, \mathcal{A}, \mu)$.

1) We say that $\{f_n\}$ converges pointwise to f if $f_n(x) \to f(x)$ for all $x \in X$.
2) We say that $\{f_n\}$ converges almost everywhere to f if $f_n(x) \to f(x)$ for almost all $x \in X$.
3) We say that $\{f_n\}$ converges uniformly to f if for evey $\epsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, then
$$|f_n(x) - f(x)| < \epsilon$$
for all $x \in X$.
4) If in addition $\{f_n\} \subset L_p(X, \mathcal{A}, \mu)$, then we say that $f_n \to f$ in L_p if
$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

Note: We are abusing notation by viewing the sequence $\{f_n\}$ as an explicit sequence of real valued functions rather than as equivalence classes.

Remark:

1) Uniform convergence $\Rightarrow$ pointwise convergence $\Rightarrow$ almost everywhere convergence, while the converse implications fail in general.
2) For $1 \leq p < \infty$, convergence in L_p does not even imply convergence at a single point. However, it does imply that there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f_0$ almost everywhere.
3) If μ is finite and if $f_n \to f_0$ uniformly, then $f_n \to f_0$ in L_p for all $1 \leq p \leq \infty$ since

$$\|f_n - f_0\|_p = (\int_X |f_n - f_o|^p \, d\mu)^{\frac{1}{p}} \leq \|f_n - f_0\|_\infty \mu(X)^{\frac{1}{p}} \to 0.$$

However, $\{\frac{1}{n^{\frac{1}{p}}} \cdot \chi_{[0,n]}\}$ converges uniformly to 0 on $\mathbb{R}$ but does not converge in $L_p(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ for any $1 \leq p < \infty$.

4) The sequence $\{n^{\frac{1}{p}} \cdot \chi_{(0,\frac{1}{n}]}\}$ converges pointwise on $[0,1]$ to the function $f_0(x) = 0$. However, the sequence fails to converge in any of the p-norms, $\|\cdot\|_p$.

Modes of Convergence

Definition: Let $\{f_n\}$ be a sequence of measurable real-valued functions defined on a measure space $(X, \mathcal{A}, \mu)$. We say that $\{f_n\}$ is *dominated* if there exists a measurable real valued function g such that

$$|f_n(x)| \leq g(x)$$

for all $x \in X$ and $n \in \mathbb{N}$.

Recall:

Theorem: [Lebesgue Dominated Convergence Theorem] Let $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(X, \mathcal{A}, \mu)$. Assume that $f = \lim\limits_{n\to\infty} f_n$ μ-a.e. If there exists an integrable function $g \in \mathcal{L}(X, \mathcal{A}, \mu)$ such that $|f_n| \leq g$ for all n, then f is integrable and

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu.$$

Theorem: Let $\{f_n\} \subseteq L_p(X, \mathcal{A}, \mu)$ which converges almost everywhere to a function $f_0(x)$. If there exists a $g \in L_p$ such that

$$|f_n(x)| \leq g(x)$$

for every $x \in X$ and $n \in \mathbb{N}$, then $f_0 \in L_p$ and $f_n \to f_0$ in L_p.

Proof: We can assume that $|f_0(x)| \leq g(x)$ for each $x \in X$, and hence that $f_0 \in L_p$. It also follows that

$$|f_n(x) - f_0(x)|^p \leq (2g(x))^p$$

for every $x \in X$. Now $\lim\limits_{n\to\infty} |f_n(x) - f_0(x)|^p \to 0$ almost everywhere. By the Lebesgue Dominated Convergence Theorem we get

$$\lim_{n\to\infty} \int_X |f_n - f_0|^p \, d\mu = 0.$$

That is $f_n \to f_0$ in L_p.

Convergence in Measure

Definition: A sequence $\{f_n\}$ of measurable real-valued functions defined on a measure space $(X, \mathcal{A}, \mu)$ is said to converge in measure to a real-valued measurable function f_0 if

$$\lim_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f_0(x)| \geq \alpha\}) = 0$$

for every $\alpha > 0$.

The sequence $\{f_n\}$ is said to be Cauchy in measure if for every $\epsilon > 0$ and every $\alpha > 0$, there exists an $N_0 \in \mathbb{N}$ such that if $n, m \geq N_0$, then

$$\mu(\{x \in X \mid |f_n(x) - f_0(x)| \geq \alpha\}) < \epsilon.$$

Convergence in Measure

Remark:

1) Uniform convergence implies convergence in measure and a sequence that is convergent in measure is also Cauchy in measure.
2) The sequence $\{\chi_{[n,n+1]}\}$ converges pointwise on $\mathbb{R}$ to 0, but it does not converge in measure.
3) Suppose that $f_n \to f_0$ in L_p. Let $\alpha > 0$. If

$$A_\alpha = \{x \in X \mid |f_n(x) - f_0(x)| \geq \alpha\},$$

then

$$\int_X |f_n(x) - f_0(x)|^p \, d\mu \geq \int_{A_\alpha} |f_n(x) - f_0(x)|^p \, d\mu \geq \alpha^p \mu(A_\alpha) \to 0.$$

From this it follows that $\mu(A_\alpha) \to 0$. That is, $f_n \to f_0$ in measure.

Theorem: [F. Riesz]
Let $\{f_n\}$ be a sequence of measurable real-valued functions which is Cauchy in measure. Then there is a subsequence which converges almost everywhere and in measure to a real-valued function f_0.

Proof: Step 1) We build the limit function.

Choose a subsequence $g_k = f_{n_k}$ such that for each $k \in \mathbb{N}$ if

$$E_k = \{x \in X \mid |g_{k+1}(x) - g_k(x_0)| \geq 2^{-k}, \}$$

then $\mu(E_k) < 2^{-k}$.

If

$$F_k = \bigcup_{j=k}^{\infty} E_j,$$

then

$$\mu(F_k) \leq \sum_{j=k}^{\infty} \mu(E_j) < \sum_{j=k}^{\infty} 2^{-j} = 2^{-(k-1)}.$$

Cont'd: Let $i \geq j \geq k$ and $x \notin F_k$. Then

$$\begin{aligned} |g_i(x) - g_j(x)| &\leq |g_i(x) - g_{i-1}(x)| + |g_{i-1}(x) - g_{i-2}(x)| + \cdots + |g_{j+1}(x) - g \\ &\leq \frac{1}{2^{i-1}} + \frac{1}{2^{i-2}} + \cdots + \frac{1}{2^j} \\ &< \frac{1}{2^{j-1}} \end{aligned}$$

Let $F = \bigcap\limits_{k=1}^{\infty} F_k$. Then $F \in \mathcal{A}$ and $\mu(F) = 0$. Moreover, $\{g_k\}$ is pointwise Cauchy, and hence convergent on $X \setminus F$. Let

$$f_0(x) = \begin{cases} \lim\limits_{k \to \infty} g_k(x) & \text{if } x \notin F, \\ 0 & \text{if } x \in F. \end{cases}$$

Convergence in Measure

Cont'd: Step 2) Show that $g_j \to f_0$ in measure.

Next we see that if $j \geq k$ and $x \notin F$, then

$$|f_0(x) - g_j(x)| < \frac{1}{2^{j-1}} \leq \frac{1}{2^{k-1}}.$$

Let $\alpha, \epsilon > 0$. Choose k large enough so that

$$\mu(F_k) < \frac{1}{2^{k-1}} < \min\{\alpha, \epsilon\}.$$

If $j \geq k$, then

$$\begin{aligned} \{x \in X \mid |f_0(x) - g_j(x)| \geq \alpha\} &\subseteq \{x \in X \mid |f_0(x) - g_j(x)| \geq \frac{1}{2^{k-1}}\} \\ &\subseteq F_k \end{aligned}$$

Consequently,

$$\mu(\{x \in X \mid |f_0(x) - g_j(x)| \geq \alpha\}) \leq \mu(F_k) < \epsilon$$

for all $j \geq k$.

Corollary: Let $\{f_n\}$ be a sequence of measurable real-valued functions which is Cauchy in measure. Then there is a measurable real-valued function f to which this sequence converges in measure. Moreover, this function is uniquely determined almost everywhere.

Proof: We know that there is a subsequence $\{f_{n_k}\}$ that converges in measure to a real-valued function f. Since

$$|f_n - f(x)| \leq |f_n - f_{n_k}| + |f_{n_k}(x) - f(x)|.$$

we have

$$\{x \in X \mid |f_n(x) - f(x)| \geq \alpha\} \subseteq A \cup B$$

where

$$A = \{x \in X \mid |f_n(x) - f_{n_k}(x)| \geq \frac{\alpha}{2}\} \quad \text{and} \quad \{x \in X \mid |f_{n_k}(x) - f(x)| \geq \frac{\alpha}{2}\}.$$

However, if $\epsilon > 0$, we can find an $N_0 \in \mathbb{N}$ so that if $n, m > N_0$ then

$$\mu(\{x \in X \mid |f_n(x) - f_m(x)| \geq \frac{\alpha}{2}\}) < \frac{\epsilon}{2}.$$

Convergence in Measure

Cont'd: So if $n \geq N_0$, then by choosing n_k large enough we get

$$\mu(\{x \in X \mid |f_n(x) - f_{n_k}(x)| \geq \frac{\alpha}{2}\}) < \frac{\epsilon}{2}$$

and

$$\mu(\{x \in X \mid |f_{n_k}(x) - f(x)| \geq \frac{\alpha}{2}\}) < \frac{\epsilon}{2}.$$

Hence if $n \geq N_0$, then

$$\mu(\{x \in X \mid |f_n(x) - f(x)| \geq \alpha\}) < \epsilon.$$

To establish the uniqueness assume that $f_n \to f$ and $f_n \to g$ in measure. Suppose that $\alpha > 0$ and that

$$\mu(\{x \in X \mid |f(x) - g(x)| \geq 2\alpha\}) = 2\epsilon > 0.$$

Then for every $n \in \mathbb{N}$ we have either

$$\mu(\{x \in X \mid |f(x) - f_n(x)| \geq \alpha\}) \geq \epsilon$$

or

$$\mu(\{x \in X \mid |f_n(x) - g(x)| \geq \alpha\}) \geq \epsilon$$

which is impossible if $f_n \to f$ and $f_n \to g$ in measure. Thus

$$\mu(\{x \in X \mid |f(x) - g(x)| > 0\}) = 0.$$

Convergence in Measure

Recall: We know that convergence in measure does not imply convergence in L_p. However, it does if the convergence is dominated.

Theorem: Let $\{f_n\}$ be a sequence of measurable real-valued functions which converges to f in measure. Let $1 \leq p < \infty$ and let $g \in L_p(X, \mathcal{A}, \mu)$ such that

$$|f_n(x)| \leq g(x) \quad a.e.$$

Then $f \in L_p(X, \mathcal{A}, \mu)$ and $f_n \to f$ in L_p.

Proof: If $\{f_n\}$ does not converge to f in L_p, then there exists an $\epsilon > 0$ and a subsequence $\{f_{n_k}\}$ such that

$$\|f_{n_k} - f\|_p \geq \epsilon$$

for every $k \in \mathbb{N}$. But $f_{n_k} \to f$ in measure as well. So by passing to a new subsequence we can assume that $f_{n_k} \to f$ almost everywhere. But then since $\{f_{n_k}\}$ is dominated, we also have $f_{n_k} \to f$ in L_p which is a contradiction.

Convergence in Measure

Remark: In case $f_n \to f$ in measure, a close glance at the proof that $\{f_n\}$ has a subsequence converging almost everywhere to f shows that we were actually able to construct this subsequence with the property that for any $\epsilon > 0$ we have a set $F \in \mathcal{A}$ such that $\mu(X \setminus F) < \epsilon$ and $f_n \to f$ uniformly on F.