Convergence in Measure

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Convergence in Measure

Definition: A sequence $\{f_n\}$ of measurable real-valued functions defined on a measure space $(X, \mathcal{A}, \mu)$ is said to converge in measure to a real-valued measurable function f_0 if

$$\lim_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f_0(x)| \geq \alpha\}) = 0$$

for every $\alpha > 0$.

The sequence $\{f_n\}$ is said to be Cauchy in measure if for every $\epsilon > 0$ and every $\alpha > 0$, there exists an $N_0 \in \mathbb{N}$ such that if $n, m \geq N_0$, then

$$\mu(\{x \in X \mid |f_n(x) - f_m(x)| \geq \alpha\}) < \epsilon.$$

Remark:

1) Uniform convergence implies convergence in measure and a sequence that is convergent in measure is also Cauchy in measure.
2) The sequence $\{\chi_{[n,n+1]}\}$ converges pointwise on $\mathbb{R}$ to 0, but it does not converge in measure.
3) Suppose that $f_n \to f_0$ in L_p. Let $\alpha > 0$. If

$$A_n = \{x \in X \mid |f_n(x) - f_0(x)| \geq \alpha\},$$

then

$$\int_X |f_n(x)-f_0(x)|^p \, d\mu \geq \int_{A_n} |f_n(x)-f_0(x)|^p \, d\mu \geq \alpha^p \mu(A_n) \to 0.$$

From this it follows that $\mu(A_n) \to 0$. That is, $f_n \to f_0$ in measure.

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Theorem: [F. Riesz]
Let $\{f_n\}$ be a sequence of measurable real-valued functions which is Cauchy in measure. Then there is a subsequence which converges almost everywhere and in measure to a real-valued function f_0.

Proof: Step 1) We build the limit function.

Choose a subsequence $g_k = f_{n_k}$ such that for each $k \in \mathbb{N}$ if

$$E_k = \{x \in X \mid |g_{k+1}(x) - g_k(x)| \geq 2^{-k}\},$$

then $\mu(E_k) < 2^{-k}$.

If

$$F_k = \bigcup_{j=k}^{\infty} E_j,$$

then

$$\mu(F_k) \leq \sum_{j=k}^{\infty} \mu(E_j) < \sum_{j=k}^{\infty} 2^{-j} = 2^{-(k-1)}.$$

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Cont'd: Let $i > j \geq k$ and $x \notin F_k$. Then

$$\begin{aligned} |g_i(x) - g_j(x)| &\leq |g_i(x) - g_{i-1}(x)| + |g_{i-1}(x) - g_{i-2}(x)| + \cdots + |g_{j+1}(x) - g_j(x)| \\ &\leq \frac{1}{2^{i-1}} + \frac{1}{2^{i-2}} + \cdots + \frac{1}{2^j} \\ &< \frac{1}{2^{j-1}} \end{aligned}$$

Let $F = \bigcap\limits_{k=1}^{\infty} F_k$. Then $F \in \mathcal{A}$ and $\mu(F) = 0$. Moreover, $\{g_k\}$ is pointwise Cauchy, and hence convergent on $X \setminus F$. Let

$$f_0(x) = \begin{cases} \lim\limits_{k\to\infty} g_k(x) & \text{if } x \notin F, \\ 0 & \text{if } x \in F. \end{cases}$$

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Cont'd: Step 2) Show that $g_j \to f_0$ in measure.

Next we see that if $j \geq k$ and $x \notin F$, then

$$|f_0(x) - g_j(x)| < \frac{1}{2^{j-1}} \leq \frac{1}{2^{k-1}}.$$

Let $\alpha, \epsilon > 0$. Choose k large enough so that

$$\mu(F_k) < \frac{1}{2^{k-1}} < \min\{\alpha, \epsilon\}.$$

If $j \geq k$, then

$$\begin{aligned}\{x \in X \mid |f_0(x) - g_j(x)| \geq \alpha\} &\subseteq \{x \in X \mid |f_0(x) - g_j(x)| \geq \frac{1}{2^{k-1}}\} \\ &\subseteq F_k\end{aligned}$$

Consequently,

$$\mu(\{x \in X \mid |f_0(x) - g_j(x)| \geq \alpha\}) \leq \mu(F_k) < \epsilon$$

for all $j \geq k$.

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Corollary: Let $\{f_n\}$ be a sequence of measurable real-valued functions which is Cauchy in measure. Then there is a measurable real-valued function f to which this sequence converges in measure. Moreover, this function is uniquely determined almost everywhere.

Proof: We know that there is a subsequence $\{f_{n_k}\}$ that converges in measure to a real-valued function f. Since

$$|f_n(x) - f(x)| \leq |f_n(x) - f_{n_k}(x)| + |f_{n_k}(x) - f(x)|.$$

we have

$$\{x \in X \mid |f_n(x) - f(x)| \geq \alpha\} \subseteq A \cup B$$

where

$$A = \{x \in X \mid |f_n(x) - f_{n_k}(x)| \geq \frac{\alpha}{2}\} \text{ and } B = \{x \in X \mid |f_{n_k}(x) - f(x)| \geq \frac{\alpha}{2}\}.$$

However, if $\epsilon > 0$, we can find an $N_0 \in \mathbb{N}$ so that if $n, m > N_0$ then

$$\mu(\{x \in X \mid |f_n(x) - f_m(x)| \geq \frac{\alpha}{2}\}) < \frac{\epsilon}{2}.$$

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Cont'd: So if $n \geq N_0$, then by choosing n_k large enough we get

$$\mu(\{x \in X \mid |f_n(x) - f_{n_k}(x)| \geq \frac{\alpha}{2}\}) < \frac{\epsilon}{2}$$

and

$$\mu(\{x \in X \mid |f_{n_k}(x) - f(x)| \geq \frac{\alpha}{2}\}) < \frac{\epsilon}{2}.$$

Hence if $n \geq N_0$, then

$$\mu(\{x \in X \mid |f_n(x) - f(x)| \geq \alpha\}) < \epsilon.$$

To establish the uniqueness assume that $f_n \to f$ and $f_n \to g$ in measure. Suppose that $\alpha > 0$ and that

$$\mu(\{x \in X \mid |f(x) - g(x)| \geq 2\alpha\}) = 2\epsilon > 0.$$

Then for every $n \in \mathbb{N}$ we have either

$$\mu(\{x \in X \mid |f(x) - f_n(x)| \geq \alpha\}) \geq \epsilon$$

or

$$\mu(\{x \in X \mid |f_n(x) - g(x)| \geq \alpha\}) \geq \epsilon$$

which is impossible if $f_n \to f$ and $f_n \to g$ in measure. Thus

$$\mu(\{x \in X \mid |f(x) - g(x)| > 0\}) = 0.$$

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Recall: We know that convergence in measure does not imply convergence in L_p. However, it does if the convergence is dominated.

Theorem: Let $\{f_n\}$ be a sequence of measurable real-valued functions which converges to f in measure. Let $1 \leq p < \infty$ and let $g \in L_p(X, \mathcal{A}, \mu)$ such that

$$|f_n(x)| \leq g(x) \quad a.e.$$

Then $f \in L_p(X, \mathcal{A}, \mu)$ and $f_n \to f$ in L_p.

Proof: If $\{f_n\}$ does not converge to f in L_p, then there exists an $\epsilon > 0$ and a subsequence $\{f_{n_k}\}$ such that

$$\|f_{n_k} - f\|_p \geq \epsilon$$

for every $k \in \mathbb{N}$. But $f_{n_k} \to f$ in measure as well. So by passing to a new subsequence we can assume that $f_{n_k} \to f$ almost everywhere. But then since $\{f_{n_k}\}$ is dominated, we also have $f_{n_k} \to f$ in L_p which is a contradiction.

Remark: In case $f_n \to f$ in measure, a close glance at the proof that $\{f_n\}$ has a subsequence converging almost everywhere to f shows that we were actually able to construct this subsequence with the property that for any $\epsilon > 0$ we have a set $F \in \mathcal{A}$ such that $\mu(X \setminus F) < \epsilon$ and $f_n \to f$ uniformly on F.