Chapter 7

Product Measures

7.1 The Product Measure Theorem

PROBLEM **7.1.1.** *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Is there a natural way to define a measure on the space $X \times Y$ which reflects the structure of the original measure space?*

DEFINITION **7.1.2.** Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. A *measurable rectangle* is a set of the form $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Let $Z = X \times Y$ and

$$\mathcal{Z}_0 = \{\coprod_{i=1}^{n} A_i \times B_i | A_i \in \mathcal{A}, B_i \in \mathcal{B}\}$$

LEMMA **7.1.3.** *$\mathcal{Z}_0$ is an algebra in $\mathcal{P}(Z)$.*

Let $\mathcal{Z}$ be the σ-algebra generated by $\mathcal{Z}_0$. We write $\mathcal{Z} = \mathcal{A} \times \mathcal{B}$. Assume that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \lambda)$ are measure spaces. A measure π on $(Z, \mathcal{Z})$ is called a *product measure* if $\pi(A \times B) = \mu(A)\lambda(B)$.

THEOREM **7.1.4** [PRODUCT MEASURE THEOREM]**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \lambda)$ be measure spaces. Then there exists a measure π on $(X \times Y, \mathcal{A} \times \mathcal{B})$ such that $\pi(A \times B) = \mu(A)\lambda(B)$. Moreover, if μ and λ are σ-finite, then π is unique and σ-finite.*

In the case where μ and λ are σ-finite, we denote the uniquely obtained measure by

$$\pi = \mu \times \lambda$$

and call the measure the product of μ and λ.

Proof. Suppose that $A \times B$ can be written as $\sum_{i=1}^{\infty} A_i \times B_i$, where each of the measurable rectangles $A_i \times B_i$ are disjoint. Then

$$\chi_A(x)\chi_B(y) = \chi_{A \times B}(x, y) = \sum_{i=1}^{\infty} \chi_{A_i}(x)\chi_{B_i}(y)$$

for all $x \in X$ and $y \in Y$. Fix x and integrate with respect to λ.

$$\begin{aligned}
\int_Y \chi_{A\times B}(x,y)\,d\lambda(y) &= \int_Y \sum_{i=1}^{\infty} \chi_{A_i}(x)\chi_{B_i}(y)\,d\lambda(y) \\
\int_Y \chi_A(x)\chi_B(y)\,d\lambda(y) &= \sum_{i=1}^{\infty} \chi_{A_i}(x)\int_Y \chi_{B_i}(y)\,d\lambda(y) && \text{by MCT} \\
\chi_A(x)\lambda(B) &= \sum_{i=1}^{\infty} \chi_{A_i}(x)\lambda(B_i)
\end{aligned}$$

Further integrating with respect to μ yields (again by MCT) $\mu(A)\lambda(B) = \sum_{i=1}^{\infty} \mu(A_i)\lambda(B_i) \quad (*)$.

Define π_0 on $\mathcal{Z}_0$ by $\pi_0(\cup_{i=1}^{n} A_i \times B_i) = \sum_{i=1}^{n} \mu(A_i)\lambda(B_i)$. Then π_0 is a measure on $\mathcal{Z}$ by $(*)$ (the only nontrivial issue to check was countable additivity). Caratheodory's Extension Theorem gives us a measure π defined on at least $\mathcal{Z}$ that extends π_0. If μ and λ are σ-finite, then π_0 is σ-finite, so Hahn's Extension Theorem tells us that π is unique. ∎

Definition **7.1.5.** Let $E \subseteq Z = X \times Y$. An *x-section* of E is the set $E_x = \{y \in Y | (x,y) \in E\}$. A *y-section* is $E^y = \{x \in X | (x,y) \in E\}$. Let $f : Z \to [-\infty, \infty]$ and $x \in X$. The *x-section of* f is $f_x(y) = f(x,y)$. For $y \in Y$, the *y-section of* f is $f^y(x) = f(x,y)$.

Lemma **7.1.6.** *If $E \subseteq Z$ is measurable in the product measurable space $(Z, \mathcal{Z} = \mathcal{A} \times \mathcal{B})$.*

i) E_x, E^y are measurable in the factors for each $x \in X$ and $y \in Y$.

ii) If $f : Z \to \mathbb{R}^*$ is $\mathcal{Z}$-measurable, then f_x, f^y are measurable in each factor for every $x \in X$, $y \in Y$.

Proof. *i*) First observe that it is a routine exercise to show that the set

$$\mathcal{S} = \{E \in \mathcal{Z} \,|\, E_x \text{ is measurable } \}$$

is a σ-algebra. However if $E = A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then

$$E_x = \begin{cases} B & \text{if } x \in A, \\ \emptyset & \text{if } x \notin A \end{cases}$$

so $A \times B \in \mathcal{S}$ and hence $\mathcal{S} = \mathcal{Z}$.

A similar argument works for the sections E^y.

Proof: *ii*) Let $f : Z \to \mathbb{R}^*$ be measurable, $x \in X$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}
\{y \in Y \,|\, f_x(y) > \alpha\} &= \{y \in Y \,|\, f(x,y) > \alpha\} \\
&= \{(w,y) \in X \times Y \,|\, f(w,y) > \alpha\}_x
\end{aligned}$$

which is measurable by *i*).

A similar argument shows that f^y is measurable.

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7.2 The Fubini's Theorem

Definition **7.2.1.** A *monotone class* is a non-empty collection $M \subseteq \mathcal{P}(X)$ such that

1. If $\{E_n\}_{n=1}^{\infty} \subseteq M$ with $E_n \subseteq E_{n+1}$, then $\bigcup_{n=1}^{\infty} \in M$.
2. If $\{E_n\}_{n=1}^{\infty} \subseteq M$ with $E_n \supseteq E_{n+1}$, then $\bigcap_{n=1}^{\infty} \in M$.

Every σ-algebra is a monotone class. If $\mathcal{A} \subseteq \mathcal{P}(X)$ is any collection of subsets, then there is a smallest monotone class $M(\mathcal{A})$ that contains $\mathcal{A}$. Simply take the intersection of all monotone classes that contain $\mathcal{A}$. With this in mind, it is clear that $M(\mathcal{A}) \subseteq \sigma(\mathcal{A})$, the smallest σ-algebra that contains $\mathcal{A}$. In fact, the reverse inclusion also holds when $\mathcal{A}$ is an algebra.

LEMMA **7.2.2** [MONOTONE CLASS LEMMA]. *If $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra, then $M(\mathcal{A}) = \sigma(\mathcal{A})$.*

Proof. We need only show that $M = M(\mathcal{A})$ is an algebra, since this combined with the fact that M is closed under countable ascending unions implies that M is closed under countable unions. For $E \in M$ define $M(E) = \{F \in M | E\backslash F, E \cup F, F\backslash E \in M\}$. Then $\emptyset \in M(E)$ and $E \in M(E)$. Further, if $F \in M(E)$ then $E \in M(F)$ by the symmetry of the definition. $M(E)$ is a monotone class since complementation, union, and intersection play nicely together.

Suppose that $E \in \mathcal{A}$. Then since $\mathcal{A}$ is an algebra, $\mathcal{A} \subseteq M(E)$. But $M(E)$ is also a monotone class, so $M \subseteq M(E) \subseteq M$ and $M = M(E)$. It follows that $\mathcal{A} \subseteq M(F)$ for every $F \in M$, and again we have $M = M(F)$. But $\emptyset, X \in \mathcal{A} \subseteq M(E)$, so this implies that M is closed under intersections and finite unions. ■

LEMMA **7.2.3.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \lambda)$ be σ-finite. If $E \in \mathcal{Z} = \mathcal{A} \times \mathcal{B}$, then $f(x) = \lambda(E_x)$ and $g(y) = \mu(E^y)$ are measurable and*

$$\int_X f\,d\mu = \pi(E) = \int_Y g\,d\lambda$$

Proof. **Proof: Case 1)** Assume that μ and λ are finite.

Let $\mathcal{M}$ denote the collection of all such E for which the lemma holds. We claim that $\mathcal{M}$ is a monotone class containing $\mathcal{Z}_0$ and that as such $\mathcal{M} = \mathcal{Z}$.

Let $E = A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then

$$f(x) = \chi_A(x)\lambda(B) \quad \text{and} \quad g(y) = \mu(A)\chi_B(y)$$

so

$$\int_X f\,d\mu = \mu(A)\lambda(B) = \int_Y g\,d\lambda.$$

Since $\mathcal{Z}_0$ consists of disjoint unions of such sets, $\mathcal{Z}_0 \subseteq \mathcal{M}$.

Let $\{E_n\} \subseteq \mathcal{M}$ with $E_n \subseteq E_{n+1}$ and let

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Then

$$f_n(x) = \lambda((E_n)_x) \quad \text{and} \quad g_n(y) = \mu((E_n)^y)$$

are measurable with

$$\int_X f_n\,d\mu = \pi(E_n) = \int_Y g_n\,d\lambda$$

If

$$f(x) = \lambda(E_x) \quad \text{and} \quad g(y) = \mu(E^y),$$

then $f_n \nearrow f$ and $g_n \nearrow g$ so the Monotone Convergence Theorem and continuity from below for π shows that

$$\int_X f\,d\mu = \pi(E) = \int_Y g\,d\lambda.$$

Given that π is finite, we can argue in much the same way using the continuity from above for π and the Lebesgue Dominated Convergence Theorem that if $\{E_n\} \subseteq \mathcal{M}$ with $E_{n+1} \subseteq E_n$ and

$$E = \bigcap_{n=1}^{\infty} E_n,$$

then $E \in \mathcal{M}$.

Case 2) If the measures are σ-finite, we let

$$Z = \bigcup_{n=1}^{\infty} Z_n$$

where $Z_n \subseteq Z_{n+1}$ and $\pi(Z_n) < \infty$. We then apply Case 1) to $E \cap Z_n$ and derive the final result from the MCT.

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THEOREM **7.2.4** [TONELLI'S THEOREM]. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \lambda)$ be σ-finite. Let $F : Z = X \times Y \to [0, \infty]$ be measurable. then the functions defined by $f(x) = \int_Y F_x d\lambda$ and $g(y) = \int_X F^y d\mu$ are mesurable and $\int_X f\, d\mu = \int_Z F\, d\pi = \int_Y g\, d\lambda$, where $\pi = \mu \times \lambda$. This is to say that*

$$\int_X \left(\int_Y F(x, y) d\lambda(y) \right) d\mu(x) = \int_Z F\, d\pi = \int_Y \left(\int_X F(x, y) d\mu(x) \right) d\lambda(y)$$

Proof. If $F = \chi_E$ for some $E \in \mathcal{Z} = \mathcal{A} \times \mathcal{B}$, then the theorem is exactly the previous lemma. It follows immediately that the theorem holds for all non-negative measurable simple functions. If F is arbitrary, we can find a sequence $\{\Phi_n\}_{n=1}^{\infty}$ of non-negative simple functions such that $\Phi_n \nearrow F$. Let $\varphi_n(x) = \int_Y (\Phi_n)_x\, d\lambda$ and $\psi_n(y) = \int_X (\Phi_n)^y\, d\mu$. then φ_n and ψ_n are measurable and monotonic in n. By the Monotone Convergence Theorem,

$$\lim_{n \to \infty} \varphi_n(x) = f(x) \qquad \text{and} \qquad \lim_{n \to \infty} \psi_n(y) = g(y)$$

Again, by the Monotone Convergence Theorem,

$$\int_X f\, d\mu = \lim_{n \to \infty} \int_X \varphi_n\, d\mu = \lim_{n \to \infty} \int_Z \Phi_n\, d\pi = \int_Z F\, d\pi$$

and similarily, $\int_Y g\, d\lambda = \int_Z F\, d\pi$. ∎

THEOREM **7.2.5** [FUBINI'S THEOREM]. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \lambda)$ be σ-finite and let $\pi = \mu \times \lambda$. If F is integrable with respect to π on $Z = X \times Y$, then the extended real valued functions defined almost everywhere by $f(x) = \int_Y F_x\, d\lambda$ and $g(y) = \int_X F^y\, d\mu$ have finie integrals and $\int_X f\, d\mu = \int_Z F\, d\pi = \int_Y g\, d\lambda$. That is to say,*

$$\int_X \left(\int_Y F(x, y) d\lambda(y) \right) d\mu(x) = \int_Z F\, d\pi = \int_Y \left(\int_X F(x, y) d\mu(x) \right) d\lambda(y)$$

Proof. Since F is π-integrable, so are F^+ and F^-. Apply Tonelli's Theorem to establsh that f^+ and f^- have finite integrals and hence are finite almost everywhere. Therefore $f = f^+ - f^-$ is defined almost everywhere and $\int_X f\, d\mu = \int_Z F\, d\pi$. Similarily, we can show that $g = g^+ - g^-$ is defined almost everywhere and $\int_Y f\, d\lambda = \int_Z F\, d\pi$. ∎