Chapter 5

The Space $Meas(X, \mathcal{A})$

In this chapter, we will study the properties of the space of finite signed measures on a measurable space $(X, \mathcal{A})$. In particular we will show that with respect to a very natural norm that this space is in fact a Banach space. We will then investigate the nature of this space when $X = \mathbb{R}$ and $\mathcal{A} = \mathcal{B}(\mathbb{R})$.

5.1 The Space $Meas(X, \mathcal{A})$

Definition **5.1.1.** Let $(X, \mathcal{A})$ be a measurable space. We let

$$Meas(X, \mathcal{A}) = \{\mu \mid \mu \text{ is a finite signed measure on } (X, \mathcal{A})\}$$

It is clear that $Meas(X, \mathcal{A})$ is a vector space over $\mathbb{R}$.

For each $\mu \in Meas(X, \mathcal{A})$ define

$$\|\mu\|_{meas} = |\mu|(X).$$

Observe that if $\mu, \nu \in Meas(X, \mathcal{A})$ and if

$$\mu = \mu^+ - \mu^- \qquad \text{and} \qquad \nu = \nu^+ - \nu^-$$

are the respective Jordan decomposiitions, then

$$\begin{aligned}
\|\mu + \nu\|_{meas} &= |\mu + \nu|(X) \\
&= (\mu + \nu)^+(X) + (\mu + \nu)^-(X) \\
&\leq (\mu^+(X) + \nu^+(X)) + (\mu^-(X) + \nu^-(X)) \\
&= (\mu^+(X) + \mu^-(X)) + (\nu^+(X) + \nu^-(X)) \\
&= |\mu|(X) + |\nu|(X) \\
&= \|\mu\|_{meas} + \|\nu\|_{meas}
\end{aligned}$$

From here it is easy to see that $(Meas(X, \mathcal{A}), \|\cdot\|_{meas})$ is a normed linear space.

Theorem **5.1.2.** *Let $(X, \mathcal{A})$ be a measurable space. Then $(Meas(X, \mathcal{A}), \|\cdot\|_{meas})$ is Banach space.*

Proof. Let $\{\mu_n\}$ be a Cauchy sequence in $(Meas(X, \mathcal{A}), \|\cdot\|_{meas})$. Let $E \in \mathcal{A}$. Since

$$|\mu_n(E) - \mu_m(E)| \leq |\mu_n - \mu_m|(E) \leq \|\mu_n - \mu_m\|_{meas}$$

we see that $\{\mu_n(E)\}$ is also Cauchy in $\mathbb{R}$. We define for $E \in \mathcal{A}$,

$$\mu(E) = \lim_{n\to\infty} \mu_n(E).$$

Note that the convergence above is actually uniform on $\mathcal{A}$.

It is easy to see that μ satisfies the first two conditions on a signed measure. To see that μ is actually a signed measure, let $\{E_n\} \subseteq \mathcal{A}$ be a sequence of pairwise disjoint measurable sets and let $E = \bigcup_{n=1}^{\infty} E_n$. First observe that

$$\begin{aligned} \mu(\bigcup_{n=1}^{k} E_n) &= \lim_{j\to\infty} \mu_j(\bigcup_{n=1}^{k} E_n) \\ &= \lim_{j\to\infty} \sum_{n=1}^{k} \mu_j(E_n) \\ &= \sum_{n=1}^{k} \lim_{j\to\infty} \mu_j(E_n) \\ &= \sum_{n=1}^{k} \mu(E_n) \end{aligned}$$

for every $k \in \mathbb{N}$. In particular, μ is finitely additive.

Let $\epsilon > 0$. Choose an $N \in \mathbb{N}$ so that if $n \geq N$, then

$$|\mu(A) - \mu_n(A)| < \frac{\epsilon}{3}$$

for every $A \in \mathcal{A}$.

We can choose a K_0 such that if $k \geq K_0$, then

$$|\mu_N(E) - \mu_N(\bigcup_{n=1}^{k} E_n)| = |\mu_N(E) - \sum_{n=1}^{k} \mu_N(E_n)| < \frac{\epsilon}{3}.$$

Now assume that $k \geq K_0$. Then

$$\begin{aligned} |\mu(E) - \sum_{n=1}^{k} \mu(E_n)| &= |\mu(E) - \mu(\bigcup_{n=1}^{k} E_n)| \\ &\leq |\mu(E) - \mu_N(E)| + |\mu_N(E) - \mu_N(\bigcup_{n=1}^{k} E_n)| + |\mu_N(\bigcup_{n=1}^{k} E_n) - \mu(\bigcup_{n=1}^{k} E_n)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

This shows that

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n).$$

∎

5.2 Functions of Bounded Variation

In this section, we introduct the space of functions of bounded variation.

Definition **5.2.1.** Let $f : [a, b] \to \mathbb{R}$. For a partion $\Pi = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ of $[a, b]$, define

$$V_a^b(f, \Pi) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

Note that if $\Pi_1 \subseteq \Pi_2$, then $V_a^b(f,\Pi_1) \le V_a^b(f,\Pi_2)$.
The variation of f on $[a,b]$ is

$$V_a^b(f) = \sup_{\Pi} V_a^b(f,\Pi).$$

We say that f is of bounded variation on $[a,b]$ if $V_a^b(f) < \infty$. Let

$$BV[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ is of bounded variation on } [a,b]\}.$$

Example **5.2.2.** 1) Let $f : [a,b] \to \mathbb{R}$ be increasing. Then $f \in BV[a,b]$ with

$$V_a^b(f) = f(b) - f(a).$$

If $f = f_1 - f_2$ where both f_1 and f_2 are increasing then $f \in BV[a,b]$ with

$$V_a^b(f) \le (f_1(b) - f_1(a)) + (f_2(b) - f_2(a)).$$

2) Let $f : [a,b] \to \mathbb{R}$ be differentiable with $|f'(x)| \le M$ for all $x \in [a,b]$. Then $f \in BV([a,b])$ and since by the Mean Value Theorem given $\Pi = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$,

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le \sum_{i=1}^{n} M(x_i - x_{i-1}) = M(b-a)$$

we have

$$V_a^b(f) \le M(b-a).$$

Remark **5.2.3.** $BV([a,b])$ is a vector space and it is easy to see that

$$\|f\|_{BV} = V_a^b(f)$$

defines a semi-norm on $BV([a,b])$. Moreover, $\|f\|_{BV} = 0$ if and only if f is constant on $[a,b]$.
If we let

$$NBV([a,b]) = \{f \in BV([a,b]) |\ f \text{ is right continuous on } [a,b].\},$$

then

$$\|f\|_{NBV} = V_a^b(f) + |f(a)|$$

makes $NBV([a,b])$ into a normed linear space.

Definition **5.2.4.** Let $f : \mathbb{R} \to \mathbb{R}$. Define

$$V_f(x) = \lim_{a \to -\infty} V_a^x(f).$$

V_f is called the variation function of f.

We say that f is of bounded variation on $\mathbb{R}$ if $Var(f) = \lim_{x\to\infty} V_f(x) < \infty$.

Let

$$BV(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is of bounded variation on } \mathbb{R}\}.$$

If $f \in BV(\mathbb{R})$, then $\lim_{x\to\infty} V_f(x)$ is called the total variation of f.

Lemma **5.2.5.** *Let $f \in BV(\mathbb{R})$. Then*

$$V_f(x) = sup\{\ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \mid -\infty < x_0 < x_1 < x_2 < \cdots < x_n = x\}.$$

Proof. Observe that if $a_1 < a_2 \leq x$, then $V_{a_2}^x(f) \leq V_{a_1}^x(f)$. Moreover, if $\Pi = \{x_0 < x_1 < x_2 < \cdots < x_n = x\}$, then

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq V_{x_0}^x(f)$$

hence

$$sup\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \mid -\infty < x_0 < x_1 < x_2 < \cdots < x_n = x\} \leq V_f(x).$$

Now let $\epsilon > 0$ and choose $x_0 < x$ such that

$$V_f(x) \leq V_{x_0}^x(f) + \frac{\epsilon}{2}.$$

We can choose $\Pi = \{x_0 < x_1 < x_2 < \cdots < x_n = x\}$ so that

$$V_{x_0}^x(f) \leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \frac{\epsilon}{2}.$$

It follows that

$$V_f(x) \leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \epsilon$$

which establishes the lemma. ∎

The following statements are all easy to establish. In each case, the proof is left to the reader.

Remark **5.2.6.** 1) If $F : \mathbb{R} \to \mathbb{R}$ is bounded and increasing, then $F \in BV(\mathbb{R})$ and

$$V_F(x) = F(x) - \lim_{y \to -\infty} F(y).$$

2) If $F, G \in BV(\mathbb{R})$, then for each $\alpha, \beta \in \mathbb{R}$, we have that $\alpha F + \beta G \in BV(\mathbb{R})$ and

$$Var(\alpha F + \beta G) \leq |\alpha| Var(F) + |\beta| Var(G).$$

In particular,

$$\|F\|_{BV} = Var(F)$$

defines a seminorm on $BV(\mathbb{R})$ with $\|F - G\|_{BV} = 0$ If and only if $F = G + c$ for some $c \in \mathbb{R}$.

3) If $F \in BV(\mathbb{R})$, then $F_{|_{[a,b]}} \in BV([a,b])$ and

$$V_a^b(f) = V_F(b) - V_F(a).$$

Conversely, if $F \in BV([a,b])$ and

$$G(x) = \begin{cases} F(x) & \text{if } x \in [a,b], \\ F(a) & \text{if } x < a, \\ F(b) & \text{if } x > b, \end{cases}$$

then $G \in BV(\mathbb{R})$ and $Var(G) = V_a^b(F)$.

As a consequence of the previous observation, much of what we say about $BV(\mathbb{R})$ will carry over to $BV([a,b])$.

4) Assume that $F \in BV(\mathbb{R})$, then F is bounded.

The following theorem characterizes functions of bounded variation.

THEOREM **5.2.7.** *Let $F \in BV(\mathbb{R})$, then both $V_F + F$ and $V_F - F$ are bounded and increasing. In particular,*

$$F = \frac{V_F + F}{2} - \frac{V_F - F}{2}$$

so F is the difference of two increasing functions.

Conversely, assume that $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ are both increasing an bounded. Then if

$$H = F - G$$

we have that $H \in BV(\mathbb{R})$ and

$$Var(H) \leq Var(F) + Var(G) = (\lim_{x\to\infty} F(x) - \lim_{x\to-\infty} F(x)) + (\lim_{x\to\infty} G(x) - \lim_{x\to-\infty} G(x)).$$

Proof. Let $F \in BV(\mathbb{R})$. Then the first statement follows immediately from the observation that if $x < y$, then

$$V_F(y) - V_F(x) = V_x^y(F) \geq |F(y) - F(x)|.$$

The second statement follows from the fact that $H = F - G \in BV([a, b])$ for all $a < b$ and

$$V_a^b(H) \leq (F(b) - F(a)) + (G(a) - G(b)).$$

∎

DEFINITION **5.2.8.** Let $F \in BV(\mathbb{R})$. The decomposition

$$F = \frac{V_F + F}{2} - \frac{V_F - F}{2}$$

is called the Jordan Decomposition of F. $\frac{V_F+F}{2}$ and $\frac{V_F-F}{2}$ are called the positive and negative variation functions of F.

REMARK **5.2.9.** Let $F \in BV(\mathbb{R})$, Then since F is the difference of two monotonic functions the following statements hold.

1) The set of discontinuities of F is countable.

2) We have that $\lim_{x\to a^+} F(x)$ exists for each $a \in \mathbb{R}$.

5.3 The Spaces $Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $Meas([a, b], \mathcal{B}([a, b]))$

In this section we will use what we know about functions of bounded variation to charaterize both $Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $Meas([a, b], \mathcal{B}([a, b]))$.

We will need the following technical proposition.

PROPOSITION **5.3.1.** *If $F \in BV(\mathbb{R})$, then*

$$\lim_{x\to-\infty} V_F(x) = 0.$$

Moreover, if $F(x)$ is right continuous, then so is $V_F(x)$.

Proof. Let $\epsilon > 0$. Given $x \in \mathbb{R}$ we can find

$$\Pi = \{x_0 < x_1 < x_2 < \cdots < x_n = x\}$$

so that

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| > V_F(x) - \epsilon.$$

We also note that

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \leq V_{x_0}^{x}(F) = V_F(x) - V_F(x_0).$$

From this it follows that

$$V_F(x) - V_F(x_0) \geq V_F(x) - \epsilon$$

and hence $V_F(x_0) \leq \epsilon$. Finally this shows that $V_F(z) \leq \epsilon$ for any $z \leq x_0$.

Assume the F is right continuous. Let $x \in \mathbb{R}$. Let

$$L = \lim_{y \to x^+} V_F(y).$$

Since $V_F(x)$ is increasing we have $L - V_F(x) \geq 0$. To show that V_F is right continuous at x we must show that $\alpha = L - V_F(x) = 0$.

Let $x \in \mathbb{R}$ and let $\epsilon > 0$. We can find a $\delta > 0$ such that if $0 < h < \delta$, then

$$|F(x+h) - F(x)| < \epsilon$$

and

$$V_F(x+h) - L < \epsilon.$$

Fix an h with $0 < h < \delta$. Then we can find $\{x = x_0 < x_1 < \cdots < x_n = x+h\}$ such that

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \geq \frac{3}{4}[V_F(x+h) - V_F(x)] \geq \frac{3}{4}\alpha.$$

It also follows that

$$\sum_{i=2}^{n} |F(x_i) - F(x_{i-1})| \geq \frac{3}{4}\alpha - |F(x_1) - F(x)| \geq \frac{3}{4}\alpha - \epsilon.$$

Next we note that $x < x_1 < x + \delta$, so as above we can find a new partition $x = x_0 = t_0 < t_1 < \cdots < t_m = x_1$ with

$$\sum_{j=1}^{m} |F(t_j) - F(t_{j-1}| \geq \frac{3}{4}\alpha.$$

Putting this all together with $x = x_0 = t_0 < t_1 < \cdots < t_m = x_1 < x_2 < \cdots < x_n = x+h$, we get

$$\begin{aligned} \alpha + \epsilon &> V_F(x+h) - V_F(x) \\ &\geq \sum_{j=1}^{m} |F(t_j) - F(t_{j-1}| + \sum_{i=2}^{n} |F(x_i) - F(x_{i-1})| \\ &\geq \frac{3}{4}\alpha + (\frac{3}{4}\alpha - \epsilon) \\ &= \frac{3}{2}\alpha - \epsilon \end{aligned}$$

This gives us that

$$0 \leq \alpha \leq 4\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have $\alpha = 0$.

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DEFINITION **5.3.2.** We let

$$NBV(\mathbb{R}) = \{F : \mathbb{R} \to \mathbb{R} \mid F \in BV(\mathbb{R}), F \text{ is right continuous}, \lim_{x\to-\infty} F(x) = 0.\}$$

Then $NBV(\mathbb{R})$ is a subspace of $BV(\mathbb{R})$. Moreover, if we let

$$\|F\|_{NBV} = Var(F)$$

then $(NBV(\mathbb{R}), \|\cdot\|_{NBV})$ is a normed linear space.

REMARK **5.3.3.** If $F \in NBV(\mathbb{R})$, then $f = F_{|_{[a,b]}} \in NBV([a,b])$.

Conversely, if $f \in NBV([a,b])$, then if

$$F(x) = \begin{cases} f(x) & \text{if } x \in [a,b], \\ 0 & \text{if } x < a, \\ f(b) & \text{if } x > b, \end{cases}$$

then $F \in NBV(\mathbb{R})$. Moreover, in this case,

$$\|f\|_{NBV} = V_a^b(f) + |f(a)| = Var(F) = \|F\|_{NBV}$$

We are now able to complete the link between $Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $NBV(\mathbb{R})$. But before we do so we will need he following observation.

PROPOSITION **5.3.4.** *Let μ be a signed measure on $(X, \mathcal{A})$. Let $E \in \mathcal{A}$. Then*

$$|\mu|(E) = \sup\{\sum_{i=1}^n |\mu(E_i)| \mid E = \bigcup_{i=1}^n E_i, \text{ with } E_1, E_2, \ldots, E_n \text{ pairwise disjoint}\}.$$

Proof. Let $\{P, N\}$ be a Hahn Decomposition for μ and let $\{\mu^+, \mu^-\}$ where

$$\mu^+(A) = \mu(A \cap P) \qquad and \qquad \mu^-(A) = -\mu(A \cap N)$$

be the Jordan decomposition of μ arising from $\{P, N\}$. Let $E \in \mathcal{A}$.

Let $E \in \mathcal{A}$. If $E_1 = E \cap P$ and $E_2 = E \cap N$, then $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$. Moreover,

$$|\mu|(E) = \mu^+(E) + \mu^-(E) = |\mu(E_1)| + |\mu(E_2)|$$

so

$$|\mu|(E) \leq \sup\{\sum_{i=1}^n |\mu(E_i)| \mid E = \bigcup_{i=1}^n E_i, \text{ with } E_1, E_2, \ldots, E_n \text{ pairwise disjoint}\}.$$

Next assume that $\{E_1, E_2, \ldots, E_n\}$ is pairwise disjoint collection of measurable sets with $E = \bigcup_{i=1}^n E_i$. Then for each $i = 1, 2, \ldots, n$, we have that

$$|\mu|(E_i) = \mu^+(E_i) + \mu^-(E_i) \geq |\mu^+(E_i) - \mu^-(E_i)| = |\mu(E_i)|.$$

Hence,

$$|\mu|(E)| = \sum_{i=1}^n |\mu|(E_i) \geq \sum_{i=1}^n |\mu(E_i)|.$$

This completes the proof.

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THEOREM **5.3.5.** *If μ is a finite regular signed measure on $\mathcal{B}(\mathbb{R})$ and $F(x) = \mu((-\infty, x])$, then $F \in NBV(\mathbb{R})$.*

Conversely, if $F \in NBV(\mathbb{R})$, then there is a unique signed measure $\mu_F \in Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $F(x) = \mu_F((-\infty, x])$. Moreover, in this case, $|\mu_F| = \mu_{V_F}$. In particular,

$$\|\mu_F\|_{meas} = \|F\|_{NBV}.$$

Proof. Assume that $\mu \in Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let

$$\mu = \mu^+ - \mu^-$$

be the Jordan decomposition of μ arising from the Hahn decomposition $\{P, N\}$. Let

$$F^+(x) = \mu^+((-\infty, x])$$

and

$$F^-(x) = \mu^-((-\infty, x])$$

.

Then $F(x) = F^+(x) - F^-(x)$ and both $F^+(x), F^-(x)$ are right continuous and increasing. Moreover,

$$\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} F^+(x) - \lim_{x\to-\infty} F^+(x) = 0.$$

Finally

$$\lim_{x\to\infty} F^+(x) = \lim_{x\to-\infty} \mu^+((-\infty, x]) = \mu^+(\mathbb{R}) < \infty,$$

and

$$\lim_{x\to-\infty} F^-(x) = \lim_{x\to-\infty} \mu^-((-\infty, x]) = \mu^-(\mathbb{R}) < \infty.$$

This shows that $F \in NBV(\mathbb{R})$.

Conversely, if $F \in NBV(\mathbb{R})$, then

$$F = F_1 - F_2$$

where

$$F_1 = \frac{V_F + F}{2} \qquad \text{and} \qquad F_2 = \frac{V_F - F}{2}$$

are the positive and negative variation functions respectvely. It follows that if

$$\mu_1 = \mu_{F_1} \qquad \text{and} \qquad \mu_2 = \mu_{F_2}$$

respectively, then

$$\mu_F = \mu_1 - \mu_2$$

and

$$\mu_{V_F} = \mu_1 + \mu_2$$

It then follows immediately, that

$$|\mu_F| \le \mu_{V_F}.$$

Now let $x \in \mathbb{R}$ and let $x_0 < x$. Let $\Pi = \{x_0 < x_1 < \cdots < x_n = x\}$. Then

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{n} |\mu_F((x_{i-1}, x_i])| \le |\mu_F|((x_0, x])$$

by the previous proposition. Moreover, it follows immediately that

$$\mu_{V_F}((x_0, x]) = V_{x_0}^x(F) \le |\mu_F|((x_0, x])$$

and hence that

$$\mu_{V_F} \le |\mu_F|.$$

Finally, the last statement follows since

$$\|\mu_F\|_{meas} = |\mu_F|(\mathbb{R}) = \mu_{V_F}(\mathbb{R}) = Var(F) = \|F\|_{NBV}.$$

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Remark **5.3.6.** Let $F \in NBV(\mathbb{R})$. We have just seen that

$$\mu_{V_F} \leq |\mu_F|.$$

Moreover, we know that if

$$F_1 = \frac{V_F + F}{2} \qquad \text{and} \qquad F_2 = \frac{V_F - F}{2}$$

are the positive and negative variation functions respectvely, then

$$\mu(F) = \mu_{F_1} - \mu_{F_2}$$

and

$$|\mu_F| = \mu_{V_F} = \mu_{F_1} + \mu_{F_2}.$$

From this we can conclude that $\{\mu_{F_1}, \mu_{F_2}\}$ is also the Jordan Decomposition of μ_F.

The next theorem establishes the broad extent of the link between $(NBV(\mathbb{R}), \|\cdot\|_{NBV})$ and $Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Theorem **5.3.7.** *Let* $\Gamma : (NBV(\mathbb{R}), \|\cdot\|_{NBV}) \to (Meas(\mathbb{R}, \mathcal{B}(\mathbb{R})), \|\cdot\|_{meas})$ *be given by*

$$\Gamma(F) = \mu_F.$$

Then Γ *is a linear isomorophism between* $NBV(\mathbb{R})$ *and* $Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. *Moreover, for each* $F \in NBV(\mathbb{R})$ *we have*

$$\|F\|_{NBV} = \|\Gamma(F)\|_{meas} = \|\mu_F\|_{meas}.$$

Next we turn our attention to the space of finite signed measures on $([a,b], \mathcal{B}(\mathbb{R}))$.

Remark **5.3.8.** Observe that given any measure $\mu \in Meas([a,b], \mathcal{B}([a,b]))$ we can define a function $f \in NBV([a,b])$ by

$$f(x) = \mu([a,x])$$

.

Conversely, given any $f \in NBV([a,b])$ we have a finite signed measure μ_f on $\mathcal{B}([a,b])$ given by

$$\mu_f(\{a\}) = f(a)$$

and

$$\mu_f((x,y]) = f(y) - f(x)$$

for any $a \leq x < y \leq b$.

Note that this is precisely the same measure that we would obtain if we first extend this f to a function $F \in NBV(\mathbb{R})$ by

$$F(x) = \begin{cases} f(x) & \text{if } x \in [a,b], \\ 0 & \text{if } x < a, \\ F(b) & \text{if } x > b, \end{cases}$$

and then define

$$\mu_f = (\mu_F)_{|\mathcal{B}([a,b])}.$$

Moreover, in this case,

$$\|\mu_f\|_{meas} = \|f\|_{NBV} = V_a^b(f) + |f(a)| = Var(F) = \|F\|_{NBV} = \|\mu_F\|_{meas}.$$

The previous remark can be summarized as follows:

Theorem **5.3.9.** *Let* $\Gamma : (NBV([a,b]), \|\cdot\|_{NBV}) \to (Meas([a,b], \mathcal{B}([a,b])), \|\cdot\|_{meas})$ *be given by*

$$\Gamma(f) = \mu_f.$$

Then Γ *is a linear isomorophism between* $NBV([a,b]))$ *and* $Meas(\mathbb{R}, \mathcal{B}([a,b]))$. *Moreover, for each* $f \in NBV([a,b]))$ *we have*

$$\|f\|_{NBV} = \|\Gamma(f)\|_{meas} = \|\mu_f\|_{meas}.$$

5.4 Discrete and Continuous Measures on $\mathbb{R}$

In this section we will introduce the spaces of *discrete and continuous measures* on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

DEFINITION **5.4.1.** Let $x \in \mathbb{R}$. We can define the measure $\mu_x \in Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by

$$\mu_x(E) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E, \end{cases}$$

The measure μ_x is called the point mass at x.

We denote by $M_d(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ the closed span of $\{\mu_x | x \in \mathbb{R}\}$ in $Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The measures in $M_d(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are called discrete measures.

REMARK **5.4.2.** Let γ denote the counting measure on $(\mathbb{R}.\mathcal{P}(\mathbb{R}))$. We denote the space $L_1(\mathbb{R}, \mathcal{P}(\mathbb{R}), \gamma)$ by $(\ell_1(\mathbb{R}), \|\cdot\|_1)$. For each function $f \in (\ell_1(\mathbb{R}), \|\cdot\|_1)$ there are at most countably many $x \in \mathbb{R}$ so that $f(x) \neq 0$. Moreover, the Weierstrass M-Test shows that

$$\sum_{x\in\mathbb{R}} f(x)\mu_x$$

converges to some measure $\mu \in Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. In fact the following is easy to show:

THEOREM **5.4.3.** *The map* $\Gamma : (\ell_1(\mathbb{R}), \|\cdot\|_1) \to (Meas(\mathbb{R}, \mathcal{B}(\mathbb{R})), \|\cdot\|_{meas})$ *given by*

$$\Gamma(f) = \sum_{x\in\mathbb{R}} f(x)\mu_x$$

is a linear isometry onto $M_d(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

PROBLEM **5.4.4.** *We know that every* $\mu \in Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *corresponds to a unique function* $F \in NBV(\mathbb{R})$. *Which functions correspond to discrete measures?*

To answer this question we first observe that if $F = \chi_{[x,\infty)}$, *then* $F \in NBV(\mathbb{R})$ *and* $\mu_F = \mu_x$. *It follows that* $\mu_F \in M_d(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *if and only if*

$$F = \sum_{x\in\mathbb{R}} f(x)\chi_{[x,\infty)},$$

where $f \in (\ell_1(\mathbb{R}), \|\cdot\|_1)$. *Moreover in this case, we have*

$$\|F\|_{NBV} = \|f\|_1 = \|\mu_F\|_{meas}.$$

REMARK **5.4.5.** Let $F \in NBV(\mathbb{R})$. Then F has countably many points of discontinuity. Moreover, for each point $x \in \mathbb{R}$ if we let

$$f(x) = F(x) - \lim_{z\to x^-} F(z),$$

then $f \in (\ell_1(\mathbb{R}), \|\cdot\|_1)$ and $\|f\|_1 \leq \|F\|_{NBV}$. In fact, if

$$G(x) = \sum_{x\in\mathbb{R}} f(x)\chi_{[x,\infty)}$$

then

$$H(x) = F(x) - G(x)$$

is a continuous function in $NBV(\mathbb{R})$, with

$$\|F\|_{NBV} = \|G\|_{NBV} + \|H\|_{NBV}.$$

DEFINITION **5.4.6.** A positive measure $\mu \in Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is said to be continuous if whenever $A \in \mathcal{B}(\mathbb{R})$ is such that $\mu(A) > 0$, there exists $B \subset A$ in $\mathcal{B}$ so that

$$0 < \mu(B) < \mu(A).$$

A signed measure $\mu \in Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is continuous if both μ^+ and μ^- are continuous. Let $M_c(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ denote the space of all continuous measures in $Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

REMARK **5.4.7.** 1) Let $\mu = \mu_F$ be a positive measure in $Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. If μ is continuous, then so is F. To see why observe that if F has a discontinuity at x_0, then $\mu_F(\{x_0\}) \neq 0$. But this is impossible if μ_F is continuous.

Conversely, assume that $F \in NBV(\mathbb{R})$ is continuous and increasing. Then we claim that μ_F is continuous. To see why we observe that $F(X) = \mu_F((-\infty, x])$. Now assume that $\mu_F(A) > 0$. Let $G(X) = \mu_F(A \cap (-\infty, x]$. Let $x_0 \in \mathbb{R}$ and let $e > o$. Then there exists a $\delta > 0$ so that $F(x_0 + \delta) - F(x_0 - \delta) < \epsilon$. But then $G(x_0 + \delta) - G(x_0 - \delta) < \epsilon$. Since G is increasing this is enough to show that G is also continuous. Form here we can apply the Intermediate Value Theorem to find an $x_0 \in \mathbb{R}$ so that $G(x_0) = \mu_F(A \cap (-\infty, x_0]) = \frac{\mu_F(A)}{2}$.

It follows that the continuous measures correspond exactly to the continuous functions in $NBV(\mathbb{R})$.

2) $M_c(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is closed in $Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. To see why observe that for $F \in NBV(\mathbb{R})$, we have

$$\|F\|_\infty \leq \|F\|_{NBV}.$$

Consequently if $\{\mu_{F_n}\}$ is a sequence of continuous measures with $\mu_{F_n} \to \mu_F$, then each F_n is continuous and since $F_n \to F$ uniformly, F is continuous.

We have essentially established the following decomposition for $Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

THEOREM **5.4.8.** *We have*

$$Meas(\mathbb{R}, \mathcal{B}(\mathbb{R})) = M_c(\mathbb{R}, \mathcal{B}(\mathbb{R})) \oplus_1 M_d(\mathbb{R}, \mathcal{B}(\mathbb{R})).$$

That is for every $\mu \in Meas(\mathbb{R}), \mathcal{B}(\mathbb{R})$ there exists unique measures $\mu_c \in M_c(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $\mu_d \in M_d(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\mu = \mu_c + \mu_d$$

and

$$\|\mu\|_{meas} = \|\mu_c\|_{meas} + \|\mu_d\|_{meas}.$$

REMARK **5.4.9.** In the previous theorem we have obtained a decomposition of a measure $\mu \in Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ into its discrete and continuous parts

$$\mu = \mu_c + \mu_d.$$

We also observe that if m is the Lebesgue measure than since $m(A) = 0$ for any countable set we get that m and μ_d are mutually singular. We also know that by applying the Lebesgue Decomposition Theorem to μ_c^+ and μ_c^- respectively that we can further decompose μ_c into two measures one of which is absolutely continuous with respect to m and the other being singular to m.

5.5 Absolutely Continuous Functions

In this section, we will introduce the concept of Absolute Continuity for functions on $\mathbb{R}$.

REMARK **5.5.1.** Assume that $\mu = \mu_F \in Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $\mu \ll m$. Then given $\epsilon > 0$, we can find a $\delta > 0$ such that if $E \in \mathcal{B}(\mathbb{R})$ with $\mu_F(E) < \delta$, then $|m_F|(E) < \epsilon$.. In particular, whenever $\{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\}$ is a finite collection of pairwise disjoint intervals such that

$$m(\bigcup_{i=1}^{n}(a_i, b_i)) = \sum_{i=1}^{n} b_i - a_1 < \delta,$$

we have

$$\begin{aligned}\sum_{i=1}^{n}|F(b_i) - F(a_i)| &= \sum_{i=1}^{n}|\mu_F((a_i, b_i))| \\ &\leq \sum_{i=1}^{n}|\mu_F|((a_i, b_i)) \\ &= |\mu_F|(\bigcup_{i=1}^{n}(a_i, b_i)) \\ &< \epsilon.\end{aligned}$$

The previous remark leads us to propose the following definition:

DEFINITION **5.5.2.** We say that a function $F : \mathbb{R} \to \mathbb{R}$ is *absolutely continuous on* $\mathbb{R}$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $\{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\}$ is a finite collection of pairwise disjoint intervals such that

$$\sum_{i=1}^{n} b_i - a_1 < \delta,$$

we have

$$\sum_{i=1}^{n}|F(b_i) - F(a_i)| < \epsilon.$$

We denote the collection of all absolutely continuous functions on $\mathbb{R}$ by $AC(\mathbb{R})$.

We say that $F : [a, b] \to \mathbb{R}$ is *absolutely continuous on* $[a, b]$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $\{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\}$ is a finite collection of pairwise disjoint intervals such that $(a_i, b_i) \subseteq [a, b]$ for each $i = 1, 2, \ldots, n$ and

$$\sum_{i=1}^{n} b_i - a_1 < \delta,$$

we have

$$\sum_{i=1}^{n}|F(b_i) - F(a_i)| < \epsilon.$$

We denote the collection of all absolutely continuous functions on $[a, b]$ by $AC([a, b])$.

We will now present some basic observations about absolutely continuous functions.

REMARK **5.5.3.**

1) If $F \in AC(\mathbb{R})$ or $F \in AC([a, b])$ then it follows immediately from the definition that F is continuous on $\mathbb{R}$ and $[a, b]$ respectively.

2) Let $F(x)$ be such that it is differentiable on (a, b) and continuous on $[a, b]$ with $|F'(x)| \leq M$ for each $x \in (a, b)$. Then a simple application of the Mean Value Theorem shows that $F \in AC([a, b])$. In fact given $\epsilon > 0$ we can choose $\delta = \frac{\epsilon}{M}$. The details are left to the reader.

3) If F is absolutely continuous on $\mathbb{R}$, then it is absolutely continuous on each closed interval $[a,b]$. In contrast the function $F(x) = x^2$ is absolutely continuous on each interval $[a,b]$ but not on all of $\mathbb{R}$.

4) In the previous section we showed that if $\mu = \mu_F \in Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is such that $\mu_F \ll m$, then $F \in AC(\mathbb{R})$. A straight forward modification of this argument shows that if $\mu = \mu_F \in Meas([a,b], \mathcal{B}([a,b]))$ is such that $\mu_F \ll m$, then $F \in AC([a,b])$ and $F(a) = 0$.

PROPOSITION **5.5.4.** *If $F \in AC([a,b])$, then $F \in BV([a,b])$*

Proof. Assume that $F \in AC([a,b])$ and that $\epsilon = 1$. Choose a $\delta < 0$ which satisfies the definition of absolute continuity for $\epsilon = 1$. Now let $\Pi = \{a = c_0 < c_1 < c_2 < \cdots < c_n = b\}$ be a partition of $[a,b]$ with $|c_i - c_{i-1}| < \delta$ for each $i = 1, 2, \ldots, n$. Then it is clear that

$$V_{c_{i-1}}^{c_i}(F) \leq 1$$

for each $i = 1, 2, \ldots, n$. From this it follows that

$$V_a^b(F) \leq n.$$

■

We already know that if $\mu = \mu_F \in Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $\mu = \mu_F \in Meas([a,b], \mathcal{B}([a,b]))$ is such that $\mu_F \ll m$, then $F \in AC(\mathbb{R})$ or $F \in AC([a,b])$ respectively. We now establish the converse for $F \in NBV(\mathbb{R})$ and $F \in NBV(a,b])$ respectively.

THEOREM **5.5.5.** *We have the following:*

1) Suppose that $F \in NBV(R)$. If $F \in AC(\mathbb{R})$, then $\mu_F \ll m$.

2) Suppose that $F \in NBV([a,b])$. If $F \in AC([a,b])$ and $F(a) = 0$, then $\mu_F \ll m$.

Proof. We will prove 1). The proof of 2) is nearly identical.

Asume that $F \in AC(\mathbb{R})$. Let $E \in \mathcal{B}$ be such that $m(E) = 0$. Let $\epsilon > 0$ and let $\delta > 0$ be chosen to satisfy the definition of absolute continuity for F. Then we can choose a sequence $\{U_k\}$ of open set in $\mathbb{R}$ so that $m(U_1) < \delta$,

$$U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots \supseteq E$$

with

$$\lim_{k \to \infty} \nu(U_k) = \nu(E).$$

Now each U_k is the countable union of a disjoint set of open intervals $\{(a_i^k, b_i^k)\}_{i=1}^{\infty}$. Moreover, since $m(U_k) < \delta$, we have

$$\sum_{i=1}^{N} |\mu_F((a_i^k, b_i^k))| \leq \sum_{i=1}^{N} |F(b_i^k) - F(a_i^k)| < \epsilon$$

for each $N \in \mathbb{N}$. From this it follows that

$$|\mu_F(U_k)| = |\sum_{i=1}^{\infty} \mu_F((a_i^k, b_i^k))| \leq \epsilon.$$

Hence $|\mu_F(E)| \leq \epsilon$ as $|\mu_F(U_k)| \to |\mu_F(E)|$. And finally since $\epsilon > 0$ was arbitrary, we have $\mu_F(E) = 0$. ■

DEFINITION **5.5.6.** Let

$$M_{ca}(\mathbb{R}, \mathcal{B}(\mathbb{R})) = \{\mu \in Meas(\mathbb{R}, \mathcal{B}(\mathbb{R})) \mid \mu \ll m\}$$

and

$$M_{cs}(\mathbb{R}, \mathcal{B}(\mathbb{R})) = \{\mu \in M_{ca}(\mathbb{R}, \mathcal{B}(\mathbb{R})) \mid \mu \perp m\}.$$

In general, we let

$$M_s(\mathbb{R}, \mathcal{B}(\mathbb{R})) = \{\mu \in Meas(\mathbb{R}, \mathcal{B}(\mathbb{R})) \mid \mu \perp m\}$$

Let

$$M_{cs}([a,b], \mathcal{B}([a,b])) = \{\mu \in Meas([a,b], \mathcal{B}([a,b])) \mid \mu \ll m\}$$

and

$$M_{cs}([a,b], \mathcal{B}([a,b])) = \{\mu \in M_{ca}([a,b], \mathcal{B}([a,b]) \mid \mu \perp m\}.$$

In general, we let

$$M_s([a,b], \mathcal{B}([a,b])) = \{\mu \in Meas([a,b], \mathcal{B}([a,b])) \mid \mu \perp m\}.$$

Remark **5.5.7.** 1) Let $\{\mu_n\} \subseteq M_{ca}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be such that $\mu_n \to \mu$. Let $E \in \mathcal{A}$ be such that $m(E) = 0$. Since $\mu_n(E) = 0$ for all $n \in \mathbb{N}$ we can conclude that $\mu(E) = 0$ as well. Using a similar argument for $\{\mu_n\} \subseteq M_{ca}([a,b], \mathcal{B}([a,b]))$, we can conclude that $M_{ca}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $M_{ca}([a,b], \mathcal{B}([a,b]))$ are closed subspaces of $Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $Meas([a,b], \mathcal{B}([a,b]))$ respectively.

2) Let $\{\mu_n\} \subseteq M_s(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be such that $\mu_n \to \mu$. For each $n \in \mathbb{N}$ choose disjoint sets $A_n, B_n \in \mathcal{A}$ such that $\mathbb{R} = A_n \cup B_n$, $\mu_n(E) = 0$ for any $E \subseteq A_n, E \in \mathcal{A}$, and $m(B_n) = 0$. Then let $A = \bigcap\limits_{n=1}^{\infty} A_n$ and $B = \bigcup\limits_{n=1}^{\infty} B_n$. Then clearly $X = A \cup B$ and $m(B) = 0$.

   Let $E \in \mathcal{A}$ be such that $E \subseteq A$. Then $E \subseteq A_n$ for every $n \in \mathbb{N}$. It follows that $\mu_n(E) = 0$ for every $n \in \mathbb{N}$. From here we get immediately, that $\mu(E) = 0$ and hence that $\mu \perp m$. This shows that $M_s(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is closed in $Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Again, a similar argument shows that $M_s([a,b], \mathcal{B}([a,b]))$ is closed in $Meas([a,b], \mathcal{B}([a,b]))$.

   If $\mu \in M_s(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then it is easy to see that $\alpha\mu \in M_s(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for any $\alpha \in \mathbb{R}$.

   Let $\mu_1, \mu_2 \in M_s(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and choose $A_i, B_i \in \mathcal{A}$ such that $\mathbb{R} = A_i \cup B_i$, $\mu_i(E) = 0$ for any $E \subseteq A_i, E \in \mathcal{A}$, and $m(B_i) = 0$ where $i = 1, 2$. Let $A = A_1 \cap A_2$ and $B = B_1 \cup B_2$. Then $\mathbb{R} = A \cup B$, $\mu_1(E) + \mu_2(E) = 0$ for any $E \subseteq A, E \in \mathcal{A}$, and $m(B) = 0$. That is $\mu_1 + \mu_2 \in M_s(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. That is $M_s(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a closed subspace of $Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

   A similar argument shows that $M_s([a,b], \mathcal{B}([a,b]))$ is a closed subspace of $Meas([a,b], \mathcal{B}([a,b]))$.

3) Assume that $\mu = \mu_1 + \mu_2$ where $\mu_1 \in M_{ca}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $\mu_2 \in M_s(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then

$$\|\mu\|_{meas} = \|\mu_1\|_{meas} + \|\mu_2\|_{meas}.$$

   We know that

$$\|\mu\|_{meas} \leq \|\mu_1\|_{meas} + \|\mu_2\|_{meas}.$$

   Let $A, B \in \mathcal{A}$ be disjoint sets such that $\mathbb{R} = A \cup B$, $\mu_2(E) = 0$ for any $E \subseteq A, E \in \mathcal{A}$, and $m(B) = 0$.

   Let $\epsilon > 0$. We can choose pairwise disjoint collections $\{E_1, E_2, \ldots, E_n\}$ and $\{F_1, F_2, \ldots, F_k\}$ such that

$$\bigcup_{i=1}^{n} E_i = X = \bigcup_{j=1}^{k} F_j$$

   and

$$|\mu_1|(X) < \sum_{i=1}^{n} |\mu_1(E_i)| + \frac{\epsilon}{2}$$

and

$$|\mu_2|(X) < \sum_{j=1}^{k} |\mu_2(F_j)| + \frac{\epsilon}{2}$$

Let $A_i = E_i \cap A$ and $B_j = F_j \cap B$. Then $\{A_1, A_2, \ldots, A_n\} \cup \{B_1, B_2, \ldots, B_k\}$ is a pairwise disjoint collection with

$$X = \bigcup_{i=1}^{n} A_i \cup \bigcup_{j=1}^{k} B_j$$

Now

$$\begin{aligned}
|\mu|(X)| &\geq \sum_{i=1}^{n} |\mu(A_i)| + \sum_{j=1}^{k} |\mu(B_j)| \\
&= \sum_{i=1}^{n} |(\mu_1 + \mu_2)(A_i)| + \sum_{j=1}^{k} |(\mu_1 + \mu_2)(B_j)| \\
&= \sum_{i=1}^{n} |\mu_1(A_i)| + \sum_{j=1}^{k} |\mu_2(B_j)| \\
&= \sum_{i=1}^{n} |\mu_1(E_i)| + \sum_{j=1}^{k} |\mu_2(F_j)| \\
&\geq (|\mu_1|(X) - \frac{\epsilon}{2}) + (|\mu_2|(X) - \frac{\epsilon}{2}) \\
&= \|\mu_1\|_{meas} + \|\mu_2\|_{meas} - \epsilon
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we have

$$\|\mu\|_{meas} \geq \|\mu_1\|_{meas} + \|\mu_2\|_{meas}.$$

Putting together everything we know so far we have the following theorem:

THEOREM **5.5.8.** *We have*

$$Meas(\mathbb{R}, \mathcal{B}(\mathbb{R})) = M_{ca}(\mathbb{R}, \mathcal{B}(\mathbb{R})) \oplus_1 M_{cs}(\mathbb{R}, \mathcal{B}(\mathbb{R})) \oplus_1 M_d(\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

That is, for every $\mu \in Meas(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ there exists unique signed measures $\mu_1 \in M_{ca}(\mathbb{R}, \mathcal{B}(\mathbb{R})), \mu_2 \in M_{cs}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $\mu_3 \in M_d(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\mu = \mu_1 + \mu_2 + \mu_3$$

and

$$\|\mu\|_{meas} = \|\mu_1\|_{meas} + \|\mu_2\|_{meas} + \|\mu_3\|_{meas}.$$

Moreover, there is an isometric linear isomorphism $\Gamma_1 : L_1(\mathbb{R}, \mathcal{B}(\mathbb{R}), m) \to M_{ca}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by

$$\Gamma_1(f)(E) = \int_E f dm.$$

In this case $\Gamma_1(f) = \mu_F$ where

$$F(x) = \int_{(-\infty,x]} f dm.$$

We also have an isometric linear isomorphism $\Gamma_2 : \ell_1(\mathbb{R}) \to M_d(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by

$$\Gamma_2(f)(E) = \sum_{x \in E} f(x)\mu_x.$$

Consequently

$$Meas(\mathbb{R}, \mathcal{B}(\mathbb{R})) \cong L_1(\mathbb{R}, \mathcal{B}(\mathbb{R}), m) \oplus_1 M_{cs}(\mathbb{R}, \mathcal{B}(\mathbb{R})) \oplus_1 \ell_1(\mathbb{R})$$

5.6 Differentiation of Monotone Function

In the previous section we saw that if $F \in AC(\mathbb{R}) \cap NBV(\mathbb{R})$, then we could find an $f \in L_1(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ such that

$$F(x) = \int_{-\infty}^{x} f\, dm.$$

Moreover, we will show that the function $F(x)$ satisfies a variant of the Fundamental Theorem of Calculus, in that $F(x)$ is differentiable m-almost everywhere with $F'(x) = f(x)$, m-almost everywhere. In fact, we will show that any monotonic function is also differentiable m-almost everywhere and then use this to characterize the absolutely continuous functions as those for which this variant of the Fundamental Theorem does indeed hold.

As claimed above we will show that monotonic functions are almost everywhere differentiable. To do so we begin with the following definition:

Definition **5.6.1.** Let $\mathcal{J}$ be a collection of non-degenerate intervals. We say that $\mathcal{J}$ is a Vitali cover of $E \subseteq \mathbb{R}$ if for all $\epsilon > 0$ and any $x \in E$ there exists $I \in \mathcal{J}$ such that $x \in I$ and $\ell(I) < \epsilon$.

Lemma **5.6.2.** *[Vitali]*

Let $E \subseteq \mathbb{R}$ be such that $m^*(E) < \infty$ and let $\mathcal{J}$ be a Vitali cover of E. Then there exists for each $\epsilon > 0$ a finite pairwise disjoint collection $\{I_1, I_2, \dots, I_n\} \subseteq \mathcal{J}$ such that $m^*(E \setminus \bigcup_{i=1}^{n} I_i) < \epsilon$.

Proof. We may assume that the intervals in J are closed since infinitely many endpoints will contribute only a set of measure zero. Let $E \subseteq U \subseteq \mathbb{R}$ be an open set with $m(U) < \infty$. Since J is a Vitali cover, we may assume that if $I \in \mathcal{J}$, then $I \subseteq U$.

Let $I_1 \in \mathcal{J}$. Suppose that $\{I_1, I_2, \dots, I_n\} \subseteq \mathcal{J}$ have been chosen to be pairwise disjoint. If $E \subseteq \bigcup_{i=1}^{n} I_i$, then stop. Otherwise let k_n be the supremum of the lengths of the intervals in $\mathcal{J}$ that are disjoint from $\bigcup_{i=1}^{n} I_i$. Since $E \not\subseteq \bigcup_{i=1}^{n} I_i$, we can find $I_{n+1} \in \mathcal{J}$ such that $\ell(I_{n+1}) > \frac{k_n}{2}$ and I_{n+1} is disjoint from $\{I_1, I_2, \dots, I_n\}$.

Proceeding inductively gives us a possibly finite sequence $\{I_i\}$ of pairwise disjoint intervals in $\mathcal{J}$ such that $\bigcup_{i=1}^{\infty} I_i \subseteq U$ and as such

$$m(\bigcup_{i=1}^{\infty} I_i) \leq m(U) < \infty.$$

From this it follows that we can find an $N \in \mathbb{N}$ so that

$$\sum_{i=N+1}^{\infty} \ell(I_i) < \frac{\epsilon}{5}.$$

Let $S = E \setminus \bigcup_{i=1}^{N} I_i$. Let $x \in S$. Since $x \in (\bigcup_{i=1}^{N} I_i)^c$ which is open, there is an interval $I \in \mathcal{J}$ that is disjoint from $\bigcup_{i=1}^{N} I_i$. It follows that

$$\ell(I) < k_n < 2\ell(I_{n+1})$$

if $I \cap I_n = \emptyset$.

∎

DEFINITION **5.6.3.** Let $f : \mathbb{R} \to \mathbb{R}$. For any $x \in \mathbb{R}$, let

$$\begin{aligned}
D^+f(x) &= \limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \\
D_+f(x) &= \liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \\
D^-f(x) &= \limsup_{h\to 0^+} \frac{f(x)-f(x-h)}{h} \\
D_-f(x) &= \liminf_{h\to 0^+} \frac{f(x)-f(x-h)}{h}
\end{aligned}$$

These four quantities are called the *Dini derivatives* of $f(x)$ at x. f is differentiable at x if and only if $D^+f(x) = D^-f(x) = D_+f(x) = D_-f(x) < \infty$. In this case, $f'(x) = D^+f(x)$. If $D^+f(x) = D_+f(x)$, we denote the common value by $f(x^+)$. If both are finite, then $f(x^+)$ is called the *right hand derivative.* $f(x^-)$ is defined in a similar fashion and is called the *left hand derivative* if it exists.

In general, $D^+f(x) \geq D_+f(x)$ and $D^-f(x) \geq D_-f(x)$.

PROPOSITION **5.6.4.** *If $f \in C[a,b]$ and one of its Dini derivatives is everywhere non-negative, then f is non-decreasing on $[a,b]$.*

Proof. Exercise. ∎

THEOREM **5.6.5.** *Let $f : [a,b] \to \mathbb{R}$ be increasing. Then f is differentiable almost everywhere on $[a,b]$, f' is measurable, and*

$$\int_{[a,b]} f'dm \leq f(b) - f(a)$$

Proof. Consider $E = \{x \in [a,b] \mid D^+f(x) > D_-f(x)\}$. Let $E_{r,s} = \{x \in [a,b] \mid D^+f(x) > r > s > D_-f(x)\}$, for $r, s \in \mathbb{Q}$. Then $E = \bigcup_{r,s\in\mathbb{Q}} E_{r,s}$. Let $\alpha = m^*(E_{r,s})$ and $\epsilon > 0$. Choose U open such that $m(U) < \alpha + \epsilon$ and $E_{r,s} \subseteq U$. for each $x \in E_{r,s}$, there is an arbitrarily small interval $[x-h, x] \subseteq U$ so that $\frac{f(x)-f(x-h)}{h} < s$. By Vitali's Lemma, there are finitely may disjoint intervals $I_1, \ldots, I_N$ of this type such that the interiors of the I_n's cover a subset of A of $E_{r,s}$ with $m^*(A) > \alpha - \epsilon$. We have

$$\sum_{n=1}^{N}[f(x_n) - f(x_n - h_n)] \leq s\sum_{n=1}^{N} h_n < sm(U) < s(\alpha + \epsilon)$$

Now for each point $y \in A$, there is an arbitrarily small interval $[y, y+k]$ that is contained in some I_n and is such that $\frac{f(y+k)-f(y)}{k}$. Therefore $f(y+k) - f(y) > rk$. Applying Vitali's Lemma again, we get intervals $J_1, \ldots, J_M$ disjoint and of the type $[y, y+k] \subseteq I_n$ such that $\bigcup_{j=1}^{M} J_j$ contains a subset of A with outer measure at least $\alpha - 2\epsilon$. Hence

$$\sum_{j=1}^{M}[f(y_j + k_j) - f(y_j)] > r\sum_{j=1}^{M} k_j \geq r(\alpha - 2\epsilon)$$

We know $J_j \subseteq I_n$. We have $\sum_{J_j\subseteq I_n}[f(y_j+k_j) - f(y_j)] \leq f(x_n) - f(x_n - h_n)$ since f is increasing. It follows that

$$r(\alpha - 2\epsilon) \leq \sum_{j=1}^{M}[f(y_j + k_j) - f(y_j)] \leq \sum_{n=1}^{N}[f(x_n) - f(x_n - h_n)] \leq s(\alpha - \epsilon)$$

Since $\epsilon > 0$ is arbitrary, $r\alpha \leq s\alpha$. Since $r > s$, this implies that $\alpha = 0$. Therefore $m^*(E_{r,s}) = 0$, so $m(E_{r,s}) = 0$ and so $m(E) = 0$.

Similarly, if $E_1 = \{x \in [a,b] \mid D^- f(x) > D_+ f(x)\}$ then $m(E_1) = 0$. From this we can deduce that $D^+ f(x) = D^- f(x) = D_+ f(x) = D_- f(x)$ almost everywhere. (Verify this.) This means that $g(x) = \lim_{h\to 0} \frac{f(x)-f(x+h)}{h}$ is defined as an extended real number almost everywhere on $[a,b]$. Let $g_n(x) = n[f(x+\frac{1}{n}) - f(x)]$, where $f(x) = f(b)$ for $x \geq b$. Then $g_n \to g$ almost everywhere on $[a,b]$. Since f is increasing, $g_n > 0$. By Fatou's Lemma,

$$\begin{aligned}
\int_{[a,b]} g\, dm &\leq \liminf_n \int_{[a,b]} g_n\, dm \\
&= \liminf_n\, n \int_{[a,b]} [f(x+\frac{1}{n}) - f(x)]\, dx \\
&= \liminf_n \left[n \int_{[b,b+\frac{1}{n}]} f\, dm - \int_{[a,a+\frac{1}{n}}]f\, dm \right] \\
&= \liminf_n \left[f(b) - \int_{[a,a+\frac{1}{n}]} f\, dm \right] \\
&\leq f(b) - f(a)
\end{aligned}$$

Therefore, g is integrable. Hence $g(x)$ is finite almost everywhere. This shows that $f(x)$ is differentiable almost everywhere and $f'(x) = g(x)$ is finite. It follows that $f' = g$ almost everywhere on $[a,b]$ and finally,

$$\int_{[a,b]} f'\, dm = \int_{[a,b]} g\, dm \leq f(b) - f(a)$$

∎

Corollary **5.6.6.** *If f is of bounded variation on $[a,b]$, then it is differentiable almost everywhere. In particular, if f is absolutely continuous on $[a,b]$, then it is differentiable almost everywhere.*

5.7 The Fundamental Theorem of Calculus Revisited

Proposition **5.7.1.** *Suppose that f is integrable on $[a,b]$. Let $F(x) = \int_{[a,x]} f\, dm$. Then F is continuous and of bounded variation on $[a,b]$.*

Proof. To see that F is continuous, let $\lambda(E) = \int_E |f| dm$ for $E \in \mathcal{B}([a,b])$. λ is a (positive) measure and $\lambda \ll m$. Thus, given $\epsilon > 0$, there is a $\delta > 0$ such that $m(E) < \delta$ implies that $\lambda(E) < \epsilon$. Hence if $|x-y| < \delta$, then

$$|f(x) - f(y)| = \left| \int_{[x,y]} f\, dm \right| \leq \int_{[x,y]} |f|\, dm < \epsilon$$

To see that F is of bounded variation, let Π be a partition of $[a,b]$. Then

$$\begin{aligned}
\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^{n} \left| \int_{[x_{i-1},x_i]} f\, dm \right| \\
&\leq \sum_{i=1}^{n} \int_{[x_{i-1},x_i]} |f|\, dm \\
&= \int_{[a,b]} |f|\, dm
\end{aligned}$$

Therefore, $V_a^b(f) \leq \int_{[a,b]} |f|\, dm < \infty$. ∎

Note: For the rest of this chapter we write $\int_a^x f(t)\, dt$ rather than $\int_{[a,x]} f\, dm$.

Lemma **5.7.2.** *If $f : [a,b] \to \mathbb{R}$ is integrable and $\int_a^x f(t)\,dt = 0$ for every $x \in [a,b]$, then $f(x) = 0$ almost everywhere.*

Proof. Let $E = \{x \in [a.b] \,|\, f(x) > 0\}$. Assume that $m(E) > 0$. Then there exists a closed set $F \subseteq E$ with $m(F) > 0$. Let $U = (a,b)\backslash F$, an open set. Since U is open, $U = \bigcup_{i=1}^{\infty}(a_i, b_i)$, where $\{(a_i,b_i)\}_{i=1}^{\infty}$ is pairwise disjoint. We also have that $0 = \int_a^b f(t)dt = \int_U f\,dm + \int_F f\,dm$, so $\int_U f\,dm = -\int_F f\,dm \neq 0$. Since $0 \neq \int_U f\,dm = \sum_{i=1}^{\infty}\int_{a_i}^{b_i} f(t)dt$, there is some n such that $\int_{a_n}^{b_n} f(t)dt \neq 0$. But then either $\int_a^{a_n} f(t)dt \neq 0$ or $\int_a^{b_n} f(t)dt \neq 0$. Since this is impossible, $m(E) = 0$. Similarly, $m(\{x \in [a,b] \,|\, f(x) < 0\}) = 0$. ∎

Lemma **5.7.3.** *If $f : [a,b] \to \mathbb{R}$ is bounded and measurable and $F(x) = \int_a^x f(t)dt + F(a)$ then $F'(x) = f(x)$ almost everywhere on $[a,b]$.*

Proof. We know that $F \in BV[a,b]$, and so F' exists almost everywhere on $[a,b]$. Suppose that $|f| \leq k$. Let $f_n(x) = n(F(x+\frac{1}{n}) - F(x))$, so that $f_n(x) = n\,\int_x^{x+\frac{1}{n}} f(t)dt$. Hence $|f_n(x)| \leq k$ for all $n \in \mathbb{N}$ and $x \in [a,b]$. Since $f_n \to F'$ almost everywhere, the Lebesgue Dominated Convergence Theorem show that

$$\begin{aligned}
\int_a^c F'(x)\,dx &= \lim_{n\to\infty}\int_a^c f_n(x)\,dx \\
&= \lim_{n\to\infty} n\int_a^c (F(x+\frac{1}{n}) - F(x))\,dx \\
&= \lim_{n\to\infty}\left[n\int_a^{c+\frac{1}{n}} F(x)\,dx - n\int_a^{a+\frac{1}{n}} F(x)\,dx\right] \\
&= F(c) - F(a) \qquad\qquad \text{by the FTC since } F \text{ is continuous} \\
&= \int_a^c f(t)\,dt
\end{aligned}$$

Hence $\int_a^c (F'(t) - f(t))dt = 0$ for all $c \in [a,b]$. Therefore $F'(x) = f(x)$ almost everywhere on $[a,b]$. ∎

Theorem **5.7.4.** *Let $f : [a,b] \to \mathbb{R}$ be integrable and $F(x) = F(a) + \int_a^x f(t)\,dt$. Then $F'(x) = f(x)$ almost everywhere.*

Proof. Assume without loss of generality that $f(x) \geq 0$ on $[a,b]$. Let

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n \\ n & \text{if } f(x) \geq n \end{cases}$$

Then $f - f_n \geq 0$. Hence $G_n(x) = \int_a^x (f(t) - f_n(t))dt$ is increasing on $[a,b]$. Therefore, G_n is differentiable and $G_n'(x) \geq 0$ almost everywhere. Moreover, $\frac{d}{dx}\int_a^x f_n(t)\,dt = f_n(x)$ almost everywhere. Hence $F'(x) = \frac{d}{dx}G_n + \frac{d}{dx}\int_a^x f_n(t)\,dt$, so $F'(x) \geq f_n(x)$ almost everywhere for all $n \in \mathbb{N}$. Therefore $F'(x) \geq f(x)$ almost everywhere. It follows that $\int_a^b F'(x)\,dx \geq \int_a^b f(x)\,dx = F(b) - F(a)$. However, $F(b) - F(a) \geq \int_a^b F'(x)\,dx$, so $\int_a^b F'(x)\,dx = F(b) - F(a) = \int_a^b f(x)\,dx$. Thus, $0 = \int_a^b (F'(x) - f(x))dx$, so $F'(x) = f(x)$ almost everywhere on $[a,b]$ since $F'(x) \geq f(x)$ almost everywhere on $[a,b]$. ∎

Lemma **5.7.5.** *If f is absolutely continuous on $[a,b]$ and $f'(x) = 0$ almost everywhere, then f is constant on $[a,b]$.*

Proof. Let $c \in [a,b]$ with $c > a$. Let $E \subseteq [a,c]$ be such that $m(E) = c - a$ and $f'(x) = 0$ for all $x \in E$. Let $\epsilon, \eta > 0$ be arbitrary. For each $x \in E$ we can find arbitrarily small h's such that $[x, x+h] \subseteq [a,c]$ and $|f(x+h) - f(x)| < \eta h$. Let $\delta > 0$ be chosen from ϵ in the definition of absolute continuity. By the Vitali Covering Lemma there is a finite disjoint collection $\{[x_k, y_k]\}$ of such intervals such that the union covers

E, except for a set of measure at most δ. Without loss of generality, we may assume that $x_k < x_{k+1}$. We have $y_0 = a \le x_1 < y_1 \le x_2 < y_2 \le \cdots \le x_n < y_n \le c = x_{n+1}$. Then $\sum_{k=0}^n |x_{k+1} - y_k| < \delta$. Hence $\sum_{k=0}^n |f(x_{k+1}) - f(y_k)| < \epsilon$. We also have $\sum_{k=1}^n |f(y_k) - f(x_k)| \le \eta \sum_{k=0}^n |y_k - x_k| \le \eta(c-a)$. But we can write this as a telescoping sum

$$|f(c) - f(a)| = \left| \sum_{k=0}^n [f(x_{k+1}) - f(y_k)] + \sum_{k=0}^n [f(y_k) - f(x_k)] \right| < \epsilon + \eta(c-a)$$

Since ϵ and η were arbitrary, $f(c) = f(a)$. Therefore f is constant on $[a,b]$. ∎

THEOREM **5.7.6.** *A function $F : [a,b] \to \mathbb{R}$ is of the form $F(x) = F(a) + \int_a^x f(t)dt$ for some integrable function f if and only if F is absolutely continuous.*

Proof. If F has this form then we have seen that F is absolutely continuous. Assume that F is absolutely continuous on $[a,b]$. Then F is of bounded variation, so $F(x) = F_1(x) - F_2(x)$, where each F_i is increasing on $[a,b]$. Therefore F is differentiable almost everywhere on $[a,b]$ and $|F'(x)| \le F_1'(x) + F_2'(x)$. It follows that

$$\int_a^b |F'(x)|dx \le \int_a^b F_1'(x)dx + \int_a^b F_2'(x)dx \le F_1(b) - F_1(a) + F_2(b) - F_2(a)$$

This shows that F' is integrable on $[a,b]$. Let $G(x) = \int_a^x F'(t)dt$. Then G is absolutely continuous and hence so is $f = F - G$. However, $f' = F' - G' = 0$ almost everywhere on $[a,b]$, so f is constant on $[a,b]$. But $f(a) = F(a) - G(a) = F(a)$. This shows that $F(x) = F(a) + G(x) = F(a) + \int_a^x F'(t)dt$.

∎