Chapter 3

Integration

3.1 Measurable Functions

DEFINITION **3.1.1.** Let $(X, \mathcal{A})$ be a measurable space. We say that $f : X \to \mathbb{R}$ is measurable if

$$f^{-1}((\alpha, \infty,)) \in \mathcal{A}$$

for every $\alpha \in \mathbb{R}$.

Let $\boldsymbol{M}(X, \mathcal{A})$ denote the set of measurable functions.

LEMMA **3.1.2.** *The following are equivalent:*

1) f *is measurable.*

2) $f^{-1}((-\infty, \alpha]) \in \mathcal{A}$ *for all* $\alpha \in \mathbb{R}$.

3) $f^{-1}([\alpha, \infty)) \in \mathcal{A}$ *for all* $\alpha \in \mathbb{R}$.

4) $f^{-1}((-\infty, \alpha)) \in \mathcal{A}$ *for all* $\alpha \in \mathbb{R}$.

Proof. 1) $\Leftrightarrow$ 2) This follows since

$$(f^{-1}((-\infty, \alpha]))^c = f^{-1}((\alpha, \infty)).$$

1) $\Rightarrow$ 3) Observe that

$$f^{-1}([\alpha, \infty)) = \bigcap_{n=1}^{\infty} f^{-1}((\alpha - \frac{1}{n}, \infty)) \in \mathcal{A}.$$

3) $\Rightarrow$ 1) Observe that

$$f^{-1}((\alpha, \infty)) = \bigcup_{n=1}^{\infty} f^{-1}([\alpha + \frac{1}{n}, \infty)) \in \mathcal{A}.$$

3) $\Leftrightarrow$ 4) This follows since

$$(f^{-1}((-\infty, \alpha)))^c = f^{-1}([\alpha, \infty)).$$

∎

EXAMPLE **3.1.3.** 1) Given any $(X, \mathcal{A})$, the constant function $f(x) = c$ for all $x \in X$ is measurable for any $c \in \mathbb{R}$.

2) If $A \subseteq X$, the characteristic function $\chi_A(x)$ of A is measurable if and only if $A \in \mathcal{A}$.

3) If $X = \mathbb{R}$ and $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{A}$, then the continuous functions are measurable since $f^{-1}((\alpha,\infty))$ is open and hence measurable.

PROPOSITION **3.1.4.** *[Arithmetic of Measurable Functions] Assume that* f, $g \in \boldsymbol{M}(X,\mathcal{A})$.

1) $cf \in \boldsymbol{M}(X,\mathcal{A})$ *for every* $c \in \mathbb{R}$

2) $f^2 \in \boldsymbol{M}(X,\mathcal{A})$

3) $f+g \in \boldsymbol{M}(X,\mathcal{A})$

4) $fg \in \boldsymbol{M}(X,\mathcal{A})$

5) $|f| \in \boldsymbol{M}(X,\mathcal{A})$

6) $\max\{f,g\} \in \boldsymbol{M}(X,\mathcal{A})$

7) $\min\{f,g\} \in \boldsymbol{M}(X,\mathcal{A})$

Proof. 1) Assume that $c = 0$. Then

$$(cf)^{-1}((\alpha,\infty)) = \begin{cases} \emptyset & \text{if } \alpha \geq 0 \\ X & \text{if } \alpha < 0 \end{cases}$$

Hence clearly f is measurable.

If $c \neq 0$, then

$$(cf)^{-1}((\alpha,\infty)) = \begin{cases} f^{-1}((\frac{\alpha}{c},\infty)) & \text{if } c \geq 0 \\ f^{-1}((-\infty,\frac{\alpha}{c})) & \text{if } c < 0. \end{cases}$$

2)

$$(f^2)^{-1}((\alpha,\infty)) = \begin{cases} f^{-1}((\sqrt{\alpha},\infty)) & \text{if } \alpha \geq 0 \\ X & \text{if } \alpha < 0. \end{cases}$$

3) Note that

$$\{x \in X | f+g(x) > \alpha\} = \bigcup_{r\in\mathbb{Q}} \{x \in X | f(x) > r\} \cap \{x \in X | g(x) > \alpha - r\}.$$

It follows that $f+g$ is measurable.

4) Note that

$$fg = \frac{(f+g)^2 - (f^2+g^2)}{2}.$$

5)

$$(|f|)^{-1}((\alpha,\infty)) = \begin{cases} f^{-1}((\alpha,\infty)) \cup f^{-1}((-\infty,-\alpha)) & \text{if } \alpha \geq 0 \\ X & \text{if } \alpha < 0. \end{cases}$$

6) Observe that

$$\max\{f,g\} = \frac{(f+g)+|f-g|}{2}.$$

7) We have that

$$\min\{f,g\} = -\max\{-f,-g\}.$$

∎

REMARK **3.1.5.** Let $(X, \mathcal{A})$ be a measure space and let $f : X \to \mathbb{R}$. Let $f^+(x) := \sup\{f(x), 0\}$ and $f^-(x) := \sup\{-f(x), 0\}$.

It follows that $f = f^+ - f^-$ and $|f| = f^+ + f^-$, so $f^+ = \frac{1}{2}(|f| + f)$ and $f^- = \frac{1}{2}(|f| - f)$. Hence f^+ and f^- are measurable if and only if f is measurable.

DEFINITION **3.1.6.** Let $(X, \mathcal{A})$ be a measure space. An extended real-valued function is a function $f : X \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$. It is *measurable* if $f^{-1}((\alpha, \infty]) \in \mathcal{A}$ for each $\alpha \in \mathbb{R}$.

Notice that if f is measurable, then both

$$f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} f^{-1}((n, \infty])$$

and

$$f^{-1}(\{-\infty\}) = \left(\bigcup_{n=1}^{\infty} f^{-1}((-n, \infty]) \right)^c$$

are in $\mathcal{A}$.

The following proposition is straight forward. The proof is left as an exercise.

PROPOSITION **3.1.7.** *An extended real-valued function $f : \mathbb{R} \to \mathbb{R}^*$ is measurable if and only if $f^{-1}(\{\infty\})$ and $f^{-1}(\{-\infty\})$ are in $\mathcal{A}$ and the real-valued function defined by*

$$f_1(x) = \begin{cases} 0 & x \in f^{-1}(\{\infty\}) \cup f^{-1}(\{-\infty\}) \\ f(x) & \textit{otherwise} \end{cases}$$

is measurable.

REMARK **3.1.8.** The collection of all extended real-valued measurable functions will be denoted $\mathcal{M}(X, \mathcal{A})$. With the convention that $0(\pm\infty) = 0$, cf, f^2, f^+, $f^- \in \mathcal{M}(X, \mathcal{A})$ if and only if $f \in \mathcal{M}(X, \mathcal{A})$.

For $f + g$, we use the convention that $\infty - \infty = 0$. Then $f + g \in \mathcal{M}(X, \mathcal{A})$ if $f, g \in \mathcal{M}(X, \mathcal{A})$.

DEFINITION **3.1.9.** Given $\{a_n\} \subseteq \mathbb{R}$, define

$$\limsup_{n\to\infty} a_n := \inf_n \sup_{k \geq n} \{a_k\} \text{ and } \liminf_{n\to\infty} a_n := \sup_n \inf_{k \geq n} \{a_k\}$$

Note: In general, $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ for any sequence $\{a_n\}$. Moreover $\lim_{n\to\infty} a_n = L$ if and only if $\liminf_{n\to\infty} a_n = L = \limsup_{n\to\infty} a_n$.

PROPOSITION **3.1.10.** *If $\{f_n\} \in \mathcal{M}(X, \mathcal{A})$ then let*

$$f(x) = \inf_n \{f_n(x)\} \qquad F(x) = \sup_n \{f_n(x)\}$$
$$f^*(x) = \liminf_{n\to\infty} \{f_n(x)\} \qquad F^*(x) = \limsup_{n\to\infty} \{f_n(x)\}$$

All of these functions are measurable.

Proof. We will show first that $F(x) = \sup_n \{f_n(x)\}$ is measurable. This follows since

$$\{x \in X | F(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x \in X | f_n(x) > \alpha\}.$$

That $f(x)$ is measurable follows since

$$\inf_n\{f_n(x)\} = -\sup_n\{-f_n(x)\}$$

It is now immediate that f^* and F^* are also measurable.

■

The following corollary is immediate from the previous proposition.

Corollary **3.1.11.** *Let $\{f_n\} \in \mathcal{M}(X,\mathcal{A})$. Assume also that $\{f_n\}$ converges pointwise on X to f_0. Then $f_0 \in \mathcal{M}(X,\mathcal{A})$.*

Corollary **3.1.12.** *Let $f, g \in \mathcal{M}(X,\mathcal{A})$. Then $fg \in \mathcal{M}(X,\mathcal{A})$.*

Proof. For each $n, m \in \mathbb{N}$ define

$$f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \le n \\ n & \text{if } f(x) > n \\ -n & \text{if } f(x) < -n \end{cases}$$

and

$$g_m(x) = \begin{cases} g(x) & \text{if } |g(x)| \le m \\ m & \text{if } g(x) > m \\ -m & \text{if } g(x) < -m \end{cases}$$

It is straight forward to show that each f_n and g_m is measurable. Consequently, we have that $f_n g_m$ is also measurable. From this we conclude that for each $m \in \mathbb{N}$ that

$$f(x)g_m(x) = \lim_{n\to\infty} f_n(x)g_m(x)$$

is measurable. And as such so is

$$f(x)g(x) = \lim_{m\to\infty} f(x)g_m(x).$$

■

Definition **3.1.13.** A *simple function* is a function $\varphi : X \to \mathbb{R}$ such that $\varphi(\mathbb{R})$ is finite. If the range of φ is $\{a_1, \ldots, a_n\}$ and $E_i = \varphi^{-1}(\{a_i\})$ then $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$. We say that φ is in *standard form* if the a_i's are distinct and the E_i's partition $\mathbb{R}$.

Up to order, the standard form of a simple function is unique. $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$ (in standard form) is measurable if and only if each $E_i \in \mathcal{A}$.

Theorem **3.1.14.** *Let $f \in \mathcal{M}^+(X,\mathcal{A})$. Then there exists $\{\varphi_n\} \subseteq \mathcal{M}^+(X,\mathcal{A})$ such that*

1) φ_n is simple for each n.

2) $0 \le \varphi_n(x) \le \varphi_{n+1}(x)$ for all $x \in \mathbb{R}$ and each n.

3) $f(x) = \lim_{n\to\infty} \varphi_n(x)$ for all $x \in \mathbb{R}$.

Proof. Let $n \in \mathbb{N}$. For $k = 0, 1, \ldots, n2^n - 1$, let $E_{n,k} := \{x \in \mathbb{R} \mid \frac{k}{2^n} \le f(x) < \frac{k+1}{2^n}\}$ and if $k = n2^n$ then let $E_{n,k} := \{x \in \mathbb{R} \mid f(x) \ge n\}$. Define $\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n}\chi_{E_{n,k}}$. It is easy to see that $\{\varphi_n\}$ is the desired sequence. ■

3.2 The Generalized Lebesgue Integral

In this section we will introduce the basic theory of integration for a measure space $(X, \mathcal{A}, \mu)$.

DEFINITION **3.2.1.** Let $(X, \mathcal{A}, \mu)$ be a measure space. Assume that $\varphi \in \mathcal{M}^+(X, \mathcal{A})$ is simple and suppose that $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$ is the standard form of φ. We define

$$\int \varphi \, d\mu := \sum_{i=1}^n a_i \mu(E_i).$$

REMARK **3.2.2.** It is often inconvenient to have to assume that a simple function is always in standard form. Indeed it is a relatively simple exercise to show that if that $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$, where $E_i \in \mathcal{A}$, but this is not the standard form for φ, then we still have

$$\int \varphi \, d\mu := \sum_{i=1}^n a_i \mu(E_i).$$

LEMMA **3.2.3.** *Let φ, $\psi \in \mathcal{M}^+(X, \mathcal{A})$ be simple functions and $c \geq 0$. Then*

1) $\int c\varphi \, d\mu = c \int \varphi \, d\mu$

2) $\int (\varphi + \psi) \, d\mu = \int \varphi \, d\mu + \int \psi \, d\mu$

Furthermore, if we define $\lambda : \mathcal{A} \to \mathbb{R}^*$ by $\lambda(E) = \int \varphi \chi_E \, d\mu$, then λ is a measure on $\mathcal{A}$.

Proof. Let $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$ and $\psi = \sum_{j=1}^m b_j \chi_{F_j}$. Since

$$c\varphi = \sum_{i=1}^n c a_i \chi_{E_i}$$

and

$$\varphi + \psi = \sum_{i=1}^n a_i \chi_{E_i} + \sum_{j=1}^m b_j \chi_{F_j},$$

then the previous remark allows us to immediately show that

$$c\varphi \, dm = c \int \varphi \, d\mu$$

and

$$\int (\varphi + \psi) \, d\mu = \int \varphi \, d\mu + \int \psi \, d\mu.$$

∎

DEFINITION **3.2.4.** If $f \in \mathcal{M}^+(X, \mathcal{A})$, we define

$$\int f \, d\mu := \sup_{0 \leq \varphi \leq f} \int \varphi \, d\mu$$

where the supremum is taken over all simple function $\varphi \in \mathcal{M}^+(\mathbb{R}, \mathcal{A})$ with $0 \leq \varphi \leq f$.

If $E \in \mathcal{A}$, then $\int_E f \, d\mu := \int f \chi_E \, d\mu$.

To establish the last statement observe that if $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$ and if we let $\gamma_i(E) = \mu(E \cap E_i)$, then

$$\lambda = \sum_{i=1}^n a_i \gamma_i.$$

The following proposition is immediate from the definition of the integral.

PROPOSITION 3.2.5. *1) If f, $g \in \mathcal{M}^+(X, \mathcal{A})$ and $f \leq g$ then*

$$\int f\, d\mu \leq \int g\, d\mu.$$

2) If $E \subseteq F \in \mathcal{A}$ then $\int_E f\, d\mu \leq \int_F f\, d\mu$ for all $f \in \mathcal{M}^+(X, \mathcal{A})$.

The next two results are fundamental tools in the abstract theory of integration. The first, is essentially the analog of countable additivity for integration.

THEOREM 3.2.6. *[Monotone Convergence Theorem]*
If $\{f_n\}_{n=1}^\infty \subseteq \mathcal{M}^+(X, \mathcal{A})$ is such that $f_n \leq f_{n+1}$ for all $n \geq 1$ and $\lim_{n\to\infty} f_n(x) = f(x)$ then

$$\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu$$

Proof. We know that $f \in \mathcal{M}^+(X, \mathcal{A})$ and

$$\int f_n\, d\mu \leq \int f_{n+1}\, d\mu \leq \int f\, d\mu$$

Hence $\lim_{n\to\infty} \int f_n\, d\mu \leq \int f\, d\mu$.

Let $0 < \alpha < 1$ and let $\varphi \in \mathcal{M}^+(X, \mathcal{A})$ be simple with $0 \leq \varphi \leq f$. For each n, let $A_n = \{x \in X \mid f_n(x) \geq \alpha\varphi(x)\}$. Then $A_n \in \mathcal{A}$, $A_n \subseteq A_{n+1}$, and $X = \bigcup_{n=1}^\infty A_n$. We have

$$\int_{A_n} \alpha\varphi\, d\mu \leq \int_{A_n} f_n\, d\mu \leq \int f_n d\mu$$

By the Monotone Convergence Theorem for measures, $\int \varphi\, d\mu = \lim_{n\to\infty} \int_{A_n} \varphi\, d\mu$ (since $\lambda(E) = \int_E \varphi\, d\mu$ is a measure). Therefore

$$\alpha \int \varphi\, d\mu = \lim_{n\to\infty} \int_{A_n} \alpha\varphi\, d\mu \leq \lim_{n\to\infty} \int f_n\, d\mu$$

Since $0 \leq \alpha < 1$ is arbitrary,

$$\int \varphi\, d\mu \leq \lim_{n\to\infty} \int f_n\, d\mu.$$

Thus

$$\int f\, d\mu = \sup_{0\leq\varphi\leq f} \int \varphi\, d\mu \leq \lim_{n\to\infty} \int f_n\, d\mu,$$

and the result is established. ∎

COROLLARY 3.2.7. *(Fatou's Lemma) If $\{f_n\}_{n=1}^\infty \subseteq \mathcal{M}^+(X, \mathcal{A})$ then*

$$\int \liminf_{n\to\infty} f_n\, d\mu \leq \liminf_{n\to\infty} \int f_n\, d\mu$$

Proof. For $j \in \mathbb{N}$, let $g_j = \inf_{n\geq j} \{f_n\}$. Then $g_j \leq f_n$ for all $n \geq j$. Hence $\int g_j\, d\mu \leq \int f_n\, d\mu$ for all $n \geq j$. This implies that $\int g_j\, d\mu \leq \liminf_{n\to\infty} \int f_n\, d\mu$. Note that $g_j \nearrow \liminf_{n\to\infty} f_n$. By the Monotone Convergence Theorem,

$$\int \liminf_{n\to\infty} f_n\, d\mu = \lim_{j\to\infty} \int g_j\, d\mu \leq \liminf_{n\to\infty} \int f_n\, d\mu$$

∎

Corollary **3.2.8.** *1) If $f \in \mathcal{M}^+(X, \mathcal{A})$ and $c \geq 0$ then $\int cf\, d\mu = c \int f\, d\mu$.*

2) If f, $g \in \mathcal{M}^+(X, \mathcal{A})$ then $\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu$.

Proof. Both results have been established for simple functions. Let $\{\varphi_n\}$ and $\{\psi\}$ be two sequences of simple functions in $\mathcal{M}^+(X, \mathcal{A})$ with $\varphi_n \nearrow f$ and $\psi_n \nearrow g$ respectively. Then $c\varphi_n \nearrow cf$ and $(\varphi_n + \psi_n) \nearrow (f+g)$. Consequently, the MCT shows that

$$\int cf\, d\mu = \lim_{n\to\infty} \int c\varphi_n d\mu = c \lim_{n\to\infty} \int \varphi_n d\mu = c \int f\, d\mu$$

and

$$\int (f+g)\, d\mu = \lim_{n\to\infty} \int \varphi_n + \psi_n d\mu = \lim_{n\to\infty} \int \varphi_n d\mu + \lim_{n\to\infty} \int \psi_n d\mu = \int f\, d\mu + \int g\, d\mu.$$

■

We are now in a position to construct a significant class of new measures on our σ-algebra $\mathcal{A}$.

Corollary **3.2.9.** *Let $f \in \mathcal{M}^+(X, \mathcal{A})$ and let $\lambda(E) = \int_E f\, d\mu$ for all $E \in \mathcal{A}$. Then λ is a measure.*

Proof. Since $f \geq 0$, $\lambda(E) \geq 0$ for all $E \in \mathcal{A}$, and clearly $\lambda(\emptyset) = 0$. Let $\{E_n\}_{n=1}^\infty \subseteq \mathcal{A}$ be pairwise disjoint and $E = \bigcup_{n=1}^\infty E_n$. Let $f_n = \sum_{k=1}^n f\chi_{E_k}$. Then

$$\int f_n\, d\mu = \sum_{k=1}^n \int_{E_k} f\, d\mu = \sum_{k=1}^n \lambda(E_k)$$

Notice that $f_n \nearrow f\chi_E$, so by the Monotone Convergence Theorem,

$$\lambda(E) = \int_E f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu = \lim_{n\to\infty} \sum_{k=1}^n \lambda(E_k) = \sum_{n=1}^\infty \lambda(E_n)$$

■

Problem **3.2.10.** *If we have a measure space $(X, \mathcal{A}, \mu)$ and another measure λ on $\mathcal{A}$, is $\lambda(E) = \int_E f\, d\mu$ for some $f \in \mathcal{M}^+(X, \mathcal{A})$? That is, does a converse to the above corollary hold?*

It turns out that as we shall soon see the answer to this question is: No. However, we shall see later in the course that under the proper conditions such a function f will indeed exist.

Definition **3.2.11.** Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that a statement P holds μ almost everywhere (μ-a.e.) if P holds for all $x \in X$ except possibly on a set E with $\mu(E) = 0$.

Proposition **3.2.12.** *Suppose that $f \in \mathcal{M}^+(X, \mathcal{A})$. Then $f(x) = 0$ μ-a.e. if and only if $\int f\, d\mu = 0$.*

Proof. Assume that $\int f\, d\mu = 0$. For each $n \in \mathbb{N}$, let $E_n = \{x \in X \mid f(x) > \frac{1}{n}\}$. Then $f \geq \frac{1}{n}\chi_{E_n}$ for each $n \in \mathbb{N}$, which implies that

$$0 = \int f\, d\mu \geq \int_{E_n} \frac{1}{n}\, d\mu = \frac{1}{n}\mu(E_n)$$

so $\mu(E_n) = 0$. If $E = \{x \in X \mid f(x) \neq 0\}$, then $E = \bigcup_{n=1}^\infty E_n$, so $\mu(E) \leq \sum_{n=1}^\infty \mu(E_n) = 0$.

Now suppose that $f(x) = 0$ μ-a.e. Let $E = \{x \in X \mid f(x) > 0\}$ and let $f_n = n\chi_E$. Since $f \leq \liminf_{n\to\infty} f_n$, by Fatou's Lemma

$$0 \leq \int f\, d\mu \leq \liminf_{n\to\infty} \int f_n\, d\mu = \liminf_{n\to\infty} n\mu(E) = 0$$

Therefore $\int f\, d\mu = 0$. ■

EXAMPLE **3.2.13.** Let $(\mathbb{R}, \boldsymbol{M}(\mathbb{R}), m)$ be the ususal Lebesgue measure space. Define a measure μ on $\boldsymbol{M}(\mathbb{R})$ by

$$\mu(E) = \begin{cases} 1 & \text{if } 0 \in E \\ 0 & \text{if } 0 \notin E \end{cases}$$

Then $\mu(\{0\}) = 1$. However, if $f \in \mathcal{M}^+(\mathbb{R}, \boldsymbol{M}(\mathbb{R}))$ and if

$$\lambda(E) = \int_E f\,dm,$$

then we have

$$\lambda(\{0\}) = \int_{\{0\}} f\,dm = \int f\chi_{\{0\}}\,dm = 0$$

since $f\chi_{\{0\}} = 0$ m-a.e.

DEFINITION **3.2.14.** Let $(X, \mathcal{A})$. Let μ, γ be two measures on $\mathcal{A}$. We say that γ is absolutely continuous with respect to μ if whenever $E \in \mathcal{A}$ is such that $\mu(E) = 0$, we have $\gamma(E) = 0$. In this case we write $\gamma \ll \mu$.

We say that μ and γ are mutually singular if there exists disjoint set $A, B \in \mathcal{A}$ such that $X = A \cup B$ and

$$\mu(A) = 0 = \gamma(B).$$

PROPOSITION **3.2.15.** *If $f \in \mathcal{M}^+(X, \mathcal{A})$ and $\lambda(E) = \int_E f\,d\mu$, then for every $E \in \mathcal{A}$ with $\mu(E) = 0$, then $\lambda(E) = 0$.*

Proof. If $\mu(E) = 0$, then $\chi_E f = 0$ μ-a.e. Hence $\lambda(E) = \int_E f\,d\mu = 0$. ∎

REMARK **3.2.16.** We had previously asked about when, given a measure space $(X, \mathcal{A}, \mu)$, and any measure λ on $\mathcal{A}$, does there exists an $f \in \mathcal{M}^+(X, \mathcal{A})$ with the property that

$$\lambda(E) = \int_E f\,d\mu,$$

for all $E \in \mathcal{A}$ We have just seen that if such an f exists then it must be the case that $\lambda \ll \mu$. Consequently, we can revise our question above to consider only absolutely continuous measures λ.

Finally, we will conclude this section with an upgrade on the MCT.

THEOREM **3.2.17.** *[Monotone Convergence Theorem, II] If $\{f_n\}_{n=1}^{\infty}$ is a monotonically increasing sequence in $\mathcal{M}^+(X, \mathcal{A})$ which converges μ-a.e. to $f \in \mathcal{M}^+(X, \mathcal{A})$, then*

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$$

Proof. Let $N \in \mathcal{A}$ be such that $\mu(N) = 0$ and $f_n \to f$ on $M = X \backslash N$. Then $f_n\chi_M \nearrow f\chi_M$. The Monotone Convergence Theorem shows that $\int f\chi_M\,d\mu = \lim\limits_{n\to\infty} \int f_n\chi_M\,d\mu$. But

$$\begin{aligned}
\int f\,d\mu &= \int f\chi_N\,d\mu + \int f\chi_M\,d\mu \\
&= 0 + \lim_{n\to\infty} \int f_n\chi_M\,d\mu \\
&= \lim_{n\to\infty} 0 + \int f_n\chi_M\,d\mu \\
&= \lim_{n\to\infty} \int f_n\chi_N\,d\mu + \int f_n\chi_M\,d\mu \\
&= \lim_{n\to\infty} \int f_n\,d\mu
\end{aligned}$$

and the result is established. ∎

Corollary **3.2.18.** *Let $\{g_n\}_{n=1}^{\infty} \subseteq \mathcal{M}^+(X, \mathcal{A})$. Then*

$$\int \sum_{n=1}^{\infty} g_n \, d\mu = \sum_{n=1}^{\infty} \int g_n \, d\mu.$$

3.3 Integrable Functions

In the previous section we defined the integral of a positive measurable function. In this section we extend the definition to functions of arbitray sign.

Definition **3.3.1.** Given $(X, \mathcal{A}, \mu)$, we say that an *extended real-valued function* f is integrable if $\int f^+ \, d\mu < \infty$ and $\int f^- \, d\mu < \infty$. In this case, we write

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

We let $\mathcal{L}(X, \mathcal{A}, \mu)$ denote the set of all integrable functions on $(X, \mathcal{A}, \mu)$.

If $E \in \mathcal{A}$, we define $\int_E f \, d\mu := \int f \chi_E \, d\mu$.

Remark **3.3.2.** If $f = f_1 - f_2$, where $f_1, f_2 \in \mathcal{M}^+(X, \mathcal{A})$, $\int f_1 \, d\mu < \infty$, and $\int f_2 \, d\mu < \infty$, then we would like $\int f \, d\mu = \int f_1 \, d\mu - \int f_2 \, d\mu$. Indeed, $f^+ - f^- = f_1 - f_2$, so $f^+ + f_2 = f_1 + f^-$. Consequently,

$$\int f^+ \, d\mu + \int f_2 \, d\mu = \int f^+ + f_2 \, d\mu = \int f_1 + f^- \, d\mu = \int f_1 \, d\mu + \int f^- \, d\mu.$$

From this it follows that

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu = \int f_1 \, d\mu - \int f_2 \, d\mu.$$

Proposition **3.3.3.** *If f is measurable, then $f \in \mathcal{L}(X, \mathcal{A}, \mu)$ if and only if $|f| \in \mathcal{L}(X, \mathcal{A}, \mu)$.*

Proof. We know $f \in \mathcal{L}(X, \mathcal{A}, \mu)$ if and only if $f^+, f^- \in \mathcal{L}(X, \mathcal{A}, \mu)$. Since $|f| = f^+ + f^-$, we get that $|f| \in \mathcal{L}(X, \mathcal{A}, \mu)$.

Conversely, if $|f| \in \mathcal{L}(X, \mathcal{A}, \mu)$, then $\int |f| \, d\mu < \infty$, so both $\int f^+ \, d\mu < \infty$ and $\int f^- \, d\mu < \infty$. This shows that $f \in \mathcal{L}(X, \mathcal{A}, \mu)$.

∎

Remark **3.3.4.** Notice that if $f \in \mathcal{L}(X, \mathcal{A}, \mu)$, then

$$\left| \int f \, d\mu \right| = \left| \int f^+ \, d\mu - \int f^- \, d\mu \right| \leq \int f^+ \, d\mu + \int f^- \, d\mu = \int |f| \, d\mu$$

Theorem **3.3.5.** *If $f, g \in \mathcal{L}(X, \mathcal{A}, \mu)$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g \in \mathcal{L}(X, \mathcal{A}, \mu)$ and*

$$\int (\alpha f + \beta g) \, d\mu = \alpha \int f \, d\mu + \beta \int g \, d\mu$$

.

Proof. Let $f = f^+ - f^-$ and let $\alpha \in \mathbb{R}$. Then $|\alpha f| = |\alpha| f^+ + |\alpha| f^-$ and

$$\int |\alpha f|\, d\mu = \int |\alpha| f^+\, d\mu + \int |\alpha| f^-\, d\mu < \infty.$$

This shows that $\alpha f \in \mathcal{L}(X, \mathcal{A}, \mu)$

If $\alpha \geq 0$, then $(\alpha f)^+ = \alpha \cdot f^+$ and $(\alpha f)^- = \alpha f^-$. Hence

$$\begin{aligned} \int \alpha f\, d\mu &= \int \alpha f^+\, d\mu - \int \alpha f^-\, d\mu \\ &= \alpha\Big(\int f^+\, d\mu - \int f^-\, d\mu\Big) \\ &= \alpha \int f\, d\mu \end{aligned}$$

If $\alpha < 0$, then $(\alpha f)^+ = -\alpha f^-$ and $(\alpha f)^- = -\alpha f^+$. Hence

$$\begin{aligned} \int \alpha f\, d\mu &= \int -\alpha f^-\, d\mu - \int -\alpha f^+\, d\mu \\ &= -\alpha\Big(\int f^-\, d\mu - \int f^+\, d\mu\Big) \\ &= \alpha \int f\, d\mu \end{aligned}$$

If $f,\, g \in \mathcal{L}(X, \mathcal{A}, \mu)$, since $|f + g| \leq |f| + |g|$, it follows that $f + g \in \mathcal{L}(X, \mathcal{A}, \mu)$. We also have that

$$f + g = (f^+ + g^+) - (f^- + g^-)$$

It follows from a previous remark that

$$\begin{aligned} \int f + g\, d\mu &= \int f^+ + g^+\, d\mu - \int f^- + g^-\, d\mu \\ &= \int f^+\, d\mu + \int g^+\, d\mu - \int f^-\, d\mu - \int g^-\, d\mu \\ &= \Big(\int f^+\, d\mu - \int f^-\, d\mu\Big) + \Big(\int g^+\, d\mu - \int g^-\, d\mu\Big) \\ &= \int f\, d\mu + \int g\, d\mu \end{aligned}$$

∎

Definition **3.3.6.** Two functions $f,\, g \in \mathcal{L}(X, \mathcal{A}, \mu)$ are said to be equal μ-a.e. if $\mu(\{x \mid |f(x) - g(x)| > 0\}) = 0$.

The next theorem is another of the fundamental convergence results in the theory of intergration.

Theorem **3.3.7.** *[Lebesgue Dominated Convergence Theorem] Let $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(X, \mathcal{A}, \mu)$. Assume that $f = \lim\limits_{n\to\infty} f_n$ μ-a.e. If there exists an integrable function $g \in \mathcal{L}(X, \mathcal{A}, \mu)$ such that $|f_n| \leq g$ for all n, then f is integrable and*

$$\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu.$$

Proof. By redefining f_n, f if necessary, we may assume that $f = \lim_{n\to\infty} f_n$ everywhere. This shows that f is measurable. We have $|f| \leq g$, so $|f|$ is integrable. Hence f is also integrable.

Notice that $g + f_n \geq 0$. By Fatou's Lemma

$$\begin{aligned}\int g\,d\mu + \int f\,d\mu &= \int (g+f)\,d\mu \\ &\leq \liminf_{n\to\infty} \int (g+f_n)\,d\mu \\ &= \liminf_{n\to\infty} \left(\int g\,d\mu + \int f_n\,d\mu\right) \\ &= \int g\,d\mu + \liminf_{n\to\infty} \int f_n\,d\mu.\end{aligned}$$

It follows that

$$\int f\,d\mu \leq \liminf_{n\to\infty} \int f_n\,d\mu.$$

On the other hand, $g - f_n \geq 0$, so by arguing as above we see that

$$-\int f\,d\mu \leq \liminf_{n\to\infty} \int -f_n\,d\mu = -\limsup_{n\to\infty} \int f_n\,d\mu$$

and as such

$$\limsup_{n\to\infty} \int f_n\,d\mu \leq \int f\,d\mu.$$

Therefore $\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$.

∎

3.4 L_p-spaces

In this section we will introduce the L_p spaces and outline some of their basic properties.

Definition **3.4.1.** 1) $L_1(X, \mathcal{A}, \mu)$:

We have seen that $\mathcal{L}(X, \mathcal{A}, \mu)$ is a vector space over $\mathbb{R}$. For $f \in \mathcal{L}(X, \mathcal{A}, \mu)$, define

$$\|f\|_1 = \int |f|\,d\mu.$$

Then $\|\cdot\|_1$ defines a seminorm on $\mathcal{L}(X, \mathcal{A}, \mu)$.

Let $\sim$ be the equivalence relation defined on $\mathcal{L}(X, \mathcal{A}, \mu)$ by $f \sim g \iff \|f - g\|_1 = 0$, or equivalently $f \sim g$ if and only $f = g$ μ-a.e.

Let $S = \{f \in \mathcal{L}(X, \mathcal{A}, \mu) \mid \|f\|_1 = 0\}$, the subspace of all integrable functions whose seminorm is zero. Let

$$L_1(X, \mathcal{A}, \mu) = L_1(X, \mu) \stackrel{def}{=} \mathcal{L}(X, \mathcal{A}, \mu)/\sim.$$

We define for $f, g \in \mathcal{L}(X, \mathcal{A}, \mu)$ and $c \in \mathbb{R}$,

1) $c[f] \stackrel{def}{=} [cf]$.
2) $[f] + [g] \stackrel{def}{=} [f + g]$,

Then it is easy to see that these operations are well defined and that with these operations, L_1 is a normed vector space with respect to the norm $\|[f]\| := \|f\|_1$.

Note: In an abuse of notation, we will normally write f to represent $[f]$.

2) $L_p(X, \mathcal{A}, \mu)$:

Let $1 < p < \infty$. Define

$$\mathcal{L}_p(X, \mathcal{A}, \mu) = \mathcal{L}_p(X, \mathcal{A}) \overset{def}{=} \{f \mid f \text{ is measurable and } |f|^p \text{ is integrable.}\}$$

Such an f is said to be p-integrable.

Let $\|f\|_p := \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}}$. As before, we let $f \sim g$ if and only $f = g$ μ-a.e. Finally, we let

$$L_p(X, \mathcal{A}, \mu) = L_p(X, \mathcal{A}) \overset{def}{=} \mathcal{L}_p(X, \mathcal{A}) / \sim .$$

3) $L_\infty(X, \mathcal{A}, \mu)$:

Let $f \in \mathcal{M}(X, \mathcal{A})$. We say that f is essentially bounded if there exists M such that $\mu(\{x \in X \mid |f(x)| > M\}) = 0$. Let

$$\mathcal{L}_\infty(X, \mathcal{A}, \mu) = \{f \in \mathcal{M}(X, \mathcal{A}) \mid f \text{ is essentially bounded}\}.$$

Let

$$\|f\|_\infty = \inf\{M \mid \mu(\{x \in X \mid |f(x)| > M\}) = 0\}.$$

Then $\|\cdot\|_\infty$ is a seminorm on $\mathcal{L}_\infty(X, \mathcal{A}, \mu)$ As before we let

$$L_\infty(X, \mathcal{A}, \mu) = L_\infty(X, \mathcal{A}) \overset{def}{=} \mathcal{L}_\infty(X, \mathcal{A}, \mu) / \sim .$$

Once again, $f, g \in L_\infty(X, \mathcal{A})$ and $c \in \mathbb{R}$, we define

1) $c[f] \overset{def}{=} [cf]$.
2) $[f] + [g] \overset{def}{=} [f + g]$,

Then it is again easy to see that these operations are well defined and that with these operations, L_∞ is a normed vector space with respect to the norm $\|[f]\|_\infty := \|f\|_\infty$.

Remark **3.4.2.** At this point we have yet to establish $L_p(X, \mathcal{A}, \mu)$ to be normed linear spaces for any $1 < p < \infty$. To do so we need to establish two fundamental inequalities, Hölders Inequality and Minkowski's Inequality.

Towards this end, if $1 \leq p \leq \infty$, we say that $1 \leq q \leq \infty$ is conjugate to p if $\frac{1}{p} + \frac{1}{q} = 1$ where by convention $\frac{1}{\infty} = 0$.

We also note that if $1 < p < \infty$ and if $\frac{1}{p} + \frac{1}{q} = 1$, then

$$1 + \frac{p}{q} = p$$

so

$$p - 1 = \frac{p}{q} \Rightarrow \frac{1}{p-1} = \frac{q}{p} = q - 1$$

and

$$(p-1)q = p.$$

Lemma **3.4.3.** *Let $\alpha, \beta > 0$, Let $1 < p < \infty$. Then if $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

Proof. Let $u = t^{p-1}$ and as such $t = u^{\frac{1}{p-1}} = u^{q-1}$. Then

$$\alpha\beta \leq \int_0^\alpha t^{p-1}\, dt + \int_0^\beta u^{q-1}\, dt = \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

∎

THEOREM **3.4.4.** *[Hölders Inequality]*

Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for each $f \in L_p(X, \mathcal{A}, \mu), g \in L_q(X, \mathcal{A}, \mu)$ we have $fg \in L_1(X, \mathcal{A}, \mu)$ with

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proof. **Case 1:** $p = 1$.

Let $M = \|g\|_\infty$. Then

$$|f(x)g(x)| \leq M|f(x)|$$

μ-a.e. It follows that

$$\int |\, fg \,|\, d\mu \leq \int M \,|\, f \,|\, d\mu = \|f\|_1 \|g\|_\infty.$$

Case 2: $1 < p < \infty$.

Clearly the result holds if either $f(x) = 0$ for almost every $x \in X$ or if $g(x) = 0$ for almost every $x \in X$. As such we may assume that this is not the case. Since for any $\alpha, \beta \geq 0$ we have

$$\int |\, (\alpha f)(\beta g) \,|\, d\mu = \alpha\beta \int |\, fg \,|\, d\mu,$$

$$(\int |\, \alpha f \,|^p\, d\mu)^{\frac{1}{p}} = \alpha(\int |\, f \,|\, d\mu)^{\frac{1}{p}},$$

and

$$(\int |\, \beta g) \,|^q\, d\mu)^{\frac{1}{q}} = \beta(\int |\, g \,|^q\, d\mu)^{\frac{1}{q}},$$

by scaling if necessary we see that we only need to prove $(*)$ under the additional assumption that

$$\int |\, f \,|^p\, d\mu = 1 = \int |\, g \,|^q\, d\mu,$$

Again, we have that for each $t \in X$,

$$|\, f(t)g(t) \,| \leq \frac{|\, f(t) \,|^p}{p} + \frac{|\, g(t) \,|^q}{q}.$$

Integrating, we get

$$\begin{aligned} \int |\, fg \,|\, dt &\leq \int (\frac{|\, f \,|^p}{p} + \frac{|\, g \,|^q}{q}) d\mu \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1 \\ &= \|f\|_p \|g\|_q. \end{aligned}$$

∎

LEMMA **3.4.5.** *If $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g \in \mathcal{L}_p(X, \mathcal{A}, \mu)$, then $f + g \in \mathcal{L}_p(X, \mathcal{A}, \mu)$ and $|f + g|^{p-1} \in \mathcal{L}_q(X, \mathcal{A}, \mu)$.*

Proof. Note that

$$|f(x)+g(x)|^p \leq (2\cdot max\{|f(x)|,|g(x)|\})^p \leq 2^p(|f(x)|^p+|g(x)|^p)$$

and

$$|f+g|^{(p-1)q} = |f+g|^p.$$

∎

THEOREM **3.4.6.** *[Minkowski's Inequality]*
Let $f,g \in L_p(X,\mathcal{A},\mu)$. Let $1<p<\infty$. Then $f+g \in L_p(X,\mathcal{A},\mu)$ and

$$(\int_a^b |\ (f+g)\ |^p\ d\mu)^{\frac{1}{p}} \leq (\int |\ f\ |^p\ d\mu)^{\frac{1}{p}} + (\int |\ g\ |^p\ d\mu)^{\frac{1}{p}}.$$

Proof. This result can be obtained from Hölders Inequality as follows:
We may assume that

$$\int |\ f\ |^p\ d\mu \neq 0 \neq \int |\ g\ |^p\ d\mu.$$

Then

$$\begin{aligned}\int |\ (f+g)\ |^p\ d\mu &= \int |\ f+g\ |\cdot|\ (f+g)\ |^{p-1}\ d\mu \\ &\leq \int |\ f\ |\cdot|\ (f+g)\ |^{p-1}\ d\mu + \int |\ g\ |\cdot|\ (f+g)\ |^{p-1}\ d\mu\end{aligned}$$

By Hölders Inequality

$$\begin{aligned}\int |\ f\ |\cdot|\ (f+g)\ |^{p-1}\ d\mu &\leq (\int |\ f\ |^p)d\mu)^{\frac{1}{p}}\cdot(\int |\ (f+g)\ |^{(p-1)q}\ d\mu)^{\frac{1}{q}} \\ &= (\int |\ f\ |^p)d\mu)^{\frac{1}{p}}\cdot(\int |\ (f+g)\ |^p\ d\mu)^{\frac{1}{q}}\end{aligned}$$

Similarly we get that

$$\int |\ g\ |\cdot|\ (f+g)\ |^{p-1}\ d\mu \leq (\int |\ g\ |^p)d\mu)^{\frac{1}{p}}\cdot(\int |\ (f+g)\ |^p\ d\mu)^{\frac{1}{q}}.$$

Putting everything together we get

$$\begin{aligned}\int |\ (f+g)\ |^p\ d\mu &\leq \int |\ f\ |\cdot|\ (f+g)\ |^{p-1}\ d\mu + \int |\ g\ |\cdot|\ (f+g)\ |^{p-1}\ d\mu \\ &\leq [(\int |\ f\ |^p\ d\mu)^{\frac{1}{p}} + (\int |\ g\ |^p\ d\mu)^{\frac{1}{p}}]\cdot(\int |\ (f+g)\ |^p\ d\mu)^{\frac{1}{q}}).\end{aligned}$$

Finally, dividing both sides of the above inequality by $(\int |\ (f+g)\ |^p\ dt)^{\frac{1}{q}}$ gives

$$\begin{aligned}(\int |\ (f+g)\ |^p\ d\mu)^{\frac{1}{p}} &= (\int |\ (f+g)\ |^p\ d\mu)^{1-\frac{1}{q}} \\ &\leq (\int |\ f\ |^p\ d\mu)^{\frac{1}{p}} + (\int |\ g\ |^p\ d\mu)^{\frac{1}{p}}.\end{aligned}$$

∎

REMARK **3.4.7.** Minkowski's inequality establishes that $L_p(X, \mathcal{A}, \mu)$ is a normed linear space for $1 < p < \infty$. As such we now know that this holds true for all p.

We now show that $L_p(X, \mathcal{A}, \mu)$ is a Banach space for each $1 \leq p \leq \infty$.

THEOREM **3.4.8.** *[Completeness of $L_p(X, \mathcal{A}, \mu)$]*
Let $1 \leq p \leq \infty$. Then $(L_p(X, \mathcal{A}, \mu), \|\cdot\|_p)$ is a Banach space.

Proof. **Case 1:** Assume that $1 \leq p < \infty$. Let $\{f_n\}_{n=1}^{\infty} \subseteq L_p(X, \mathcal{A}, \mu)$ be a Cauchy sequence. We can find a subsequence $\{g_k\}$ of $\{f_n\}$ such that

$$\|g_{k+1} - g_k\|_p < \frac{1}{2^k}.$$

Define, for all $x \in X$,

$$g(x) = |g_1(x)| + \sum_{k=1}^{\infty} |g_{k+1}(x) - g_k(x)|$$

Then $g \in \mathcal{M}^+(X, \mathcal{A})$, and by Fatou's Lemma,

$$\int |g|^p \, d\mu \leq \liminf_{n\to\infty} \int \left(|g_1| + \sum_{k=1}^{n} |g_{k+1} - g_k| \right)^p d\mu.$$

Next by Minkowski's Inequality we get

$$\begin{aligned} (\int |g|^p \, d\mu)^{\frac{1}{p}} &\leq \liminf_{n\to\infty} \|g_1\|_p + \sum_{k=1}^{n} \|g_{k+1} - g_k\|_p \\ &\leq \|g_1\|_p + 1 < \infty \end{aligned}$$

Let $E = \{x \in X \mid g(x) < \infty\}$. Then it follows from the above calculation that $\mu(X \setminus E) = 0$. Hence

$$|g_1(x)| + \sum_{k=1}^{\infty} |g_{k+1}(x) - g_k(x)|$$

converges to a finite number μ-a.e. Let

$$f(x) = \begin{cases} g_1(x) + \sum_{k=1}^{\infty} g_{k+1}(x) - g_k(x) & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Note that since the series is telescoping this actually shows that $g_k \to f$ μ-a.e. We also notice that $|g_k(x)| \leq g(x)$ for all $x \in X$. The Lebesgue Dominated Convergence Theorem shows us that

$$\int |f|^p \, d\mu = \lim_{k\to\infty} \int |g_k|^p \, d\mu \leq \int |g|^p \, d\mu < \infty$$

Therefore $f \in L_p(X, \mathcal{A}, \mu)$. Since $|f| \leq g$, we have $|f - g_k| \leq 2|g|$ and again by the LDCT,

$$0 = \lim_{k\to\infty} \int |f - g_k|^p \, d\mu$$

since $g_k \to f$ μ-a.e. Therefore the subsequence $\{g_k\}$ converges to f in $L_p(X, \mathcal{A}, \mu)$. It follows that $\{f_n\}$ converges to f in $L_p(X, \mathcal{A}, \mu)$

Case 2: Now let $\{f_n\}_{n=1}^{\infty} \subseteq L_\infty(X, \mathcal{A}, \mu)$ be a Cauchy sequence. Let $E \subseteq X$ be such that $\mu(E) = 0$ and if $x \notin E$ then $|f_n(x)| \leq \|f_n\|_\infty$ and $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$. Then $\{f_n(x)\}$ converges uniformly on $X \setminus E$ to some function $f(x)$. It follows that f is measurable and $\|f_n - f\|_\infty \to 0$. Hence $f \in L_\infty(X, \mathcal{A}, \mu)$. ∎

The proof of the previous theorem also establishes the following useful proposition:

COROLLARY **3.4.9.** *If $1 \leq p < \infty$ and if $\{f_n\}_{n=1}^{\infty} \subseteq L_p(X, \mathcal{A}, \mu)$ is such that $f_n \to f$ in $L_p(X, \mathcal{A}, \mu)$ then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}$ such that $f_{n_k}(x) \to f(x)$ μ-a.e.*

EXAMPLE **3.4.10.** The proof of the Completeness Theorem actually shows that if $\{f_n\}_{n=1}^{\infty} \subseteq L_\infty(X, \mathcal{A}, \mu)$ converges to f, then there exists a set $E \in \mathcal{A}$ with $\mu(E) = 0$ such that $f_n \to f$ uniformly on $X \setminus E$. In particular $f_n \to f$ pointwise μ-a.e. We may ask if the same thing happens when the convergence is in $L_p(X, \mathcal{A}, \mu)$. The following example shows that this is not the case.

Let $(X, \mathcal{A}, \mu) = ([0,1], \boldsymbol{M}([0,1]), m)$ be the restriction of the Lebesgue measure to the closed interval $[0,1]$. Let $f_1 = \chi_{[0,1]}, f_2 = \chi_{[0,\frac{1}{2}]}, f_3 = \chi_{[\frac{1}{2},\frac{2}{2}]}, f_4 = \chi_{[0,\frac{1}{4}]}, f_5 = \chi_{[\frac{1}{4},\frac{2}{4}]}, f_6 = \chi_{[\frac{2}{4},\frac{3}{4}]}, f_7 = \chi_{[\frac{3}{4},\frac{4}{4}]}, \ldots$.

In general, for $m = 0, 1, 2, \ldots$ and for $k = 0, 1, \ldots 2^m - 1$, we let

$$f_{2^m+k} = \chi_{[\frac{k}{2^m}, \frac{k+1}{2^m}]}.$$

Then for any $1 \leq p < \infty$, we have $f_j \to 0$ in $L_p([0,1], \boldsymbol{M}([0,1]), m)$. However the sequence $\{f_j(x)\}$ diverges for each $x \in [0,1]$.

REMARK **3.4.11.** If $X = \mathbb{N}$, $\mathcal{A} = \mathcal{P}(\mathbb{N})$ and if μ is the counting measure, then

$$L_p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu) = l_p(\mathbb{N}) = \{\{a_n\} | \sum_{n=1}^{\infty} |a_n| < \infty\}.$$

In this case, we know that if $1 < p < q < \infty$, then

$$l_1 \subsetneq l_p \subsetneq l_q \subsetneq l_\infty.$$

In contrast with the example above, if $(X, \mathcal{A}, \mu)$ is a finite measure space then it is easy to see that

$$L_\infty(X, \mathcal{A}, \mu) \subseteq L_p(X, \mathcal{A}, \mu)$$

for any $1 \leq p \leq \infty$ with

$$\|f\|_p \leq \|f\|_\infty \cdot \mu(X)^{\frac{1}{p}}.$$

Moreover, if $1 \leq p < q < \infty$ and if $f \in L_q(X, \mathcal{A}, \mu)$, then $|f|^p \in L_{\frac{q}{p}}(X, \mathcal{A}, \mu)$. From here Hölders Inequality shows that

$$\begin{aligned} \int |f|^p \, d\mu &= \int |f|^p \cdot 1 \, d\mu \\ &\leq (\int (|f|^p)^{\frac{q}{p}} \, d\mu)^{\frac{p}{q}} (\int 1^{\frac{q}{q-p}} \, d\mu)^{\frac{q-p}{q}} \\ &= \|f\|_q^p \cdot \mu(X)^{\frac{q-p}{q}} \end{aligned}$$

In particular, $f \in L_p(X, \mathcal{A}, \mu)$ and

$$\|f\|_p \leq \|f\|_q \cdot \mu(X)^{\frac{q-p}{pq}}.$$

Hence

$$L_\infty(X, \mathcal{A}, \mu) \subseteq L_q(X, \mathcal{A}, \mu) \subseteq L_p(X, \mathcal{A}, \mu) \subseteq L_1(X, \mathcal{A}, \mu)$$

In addition, if μ is a probability measure, that is if $\mu(X) = 1$, then if $1 \leq p < q \leq \infty$, we have

$$\|f\|_p \leq \|f\|_q.$$

Finally, if $\mu(X) = \infty$, then there are no general containment results for the L_p spaces.

3.5 Modes of Convergence

At this point, given a sequence $\{f_n\}$ of measurable real-valued functions defined on a measure space $(X, \mathcal{A}, \mu)$ we have various modes in which such a sequence may be deemed to converge to a function f. Most notably we have pointwise and uniform convergence, convergence almost everywhere, and convergence in one of the L_p-norms. In this section we will introduce several other important modes of convergence and establish the known links between them.

We begin by recalling the definitions of the modes we have listed above:

Definition **3.5.1.** Let $\{f_n\}$ be a sequence of measurable real-valued functions defined on a measure space $(X, \mathcal{A}, \mu)$.

1) We say that $\{f_n\}$ converges pointwise to f_0 if $f_n(x) \to f_0(x)$ for all $x \in X$.

2) We say that $\{f_n\}$ converges almost everywhere to f_0 if $f_n(x) \to f_0(x)$ for almost all $x \in X$.

3) We say that $\{f_n\}$ converges uniformly to f_0 if for every $\epsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, then
$$|f_n(x) - f_0(x)| < \epsilon$$
for all $x \in X$.

4) If in addition $\{f_n\} \subset L_p(X, \mathcal{A}, \mu)$, then we say that $f_n \to f_0$ in L_p if
$$\lim_{n\to\infty} \|f_n - f_0\|_p = 0.$$
Note: In this case we are abusing notation by viewing the sequence $\{f_n\}$ as an explicit sequence of real valued functions rather than as an equivalence class as would properly be the case for elements in $L_p(X, \mathcal{A}, \mu)$.

Remark **3.5.2.** 1) It is clear that uniform convergence $\Rightarrow$ pointwise convergent $\Rightarrow$ almost everywhere convergence, while the converse implications fail in general.

2) For $1 \leq p < \infty$ we have seen that convergence in L_p does not even imply convergence at a single point. However, it does imply that there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f_0$ almost everywhere.

3) If μ is finite and if $f_n \to f_0$ uniformly, then $f_n \to f_0$ in L_p for all $1 \leq p \leq \infty$. We simply observe that
$$\|f_n - f_0\|_p = (\int_X |f_n - f_o|^p \, d\mu)^{\frac{1}{p}} \leq \|f_n - f_0\|_\infty \mu(X)^{\frac{1}{p}} \to 0.$$
However, $\{\frac{1}{n^{\frac{1}{p}}} \cdot \chi_{[0,n]}\}$ converges uniformly to 0 on $\mathbb{R}$ but does not converge in $L_p(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ for any $1 \leq p < \infty$.

4) The sequence $\{n^{\frac{1}{p}} \cdot \chi_{(0,\frac{1}{n}]}\}$ converges pointwise on $[0,1]$ to the function $f_0(x) = 0$. However, the sequence fails to converge in any of the p-norms, $\|\cdot\|_p$.

We have seen that even for finite measure spaces, pointwise convergence does not imply convergence in L_p. The next theorem, which is really a modification of Lebesgue's Dominated Convergence Theorem shows that in the presence of an additional assumption, namely that the convergence is *dominated*, then indeed almost everywhere convergence will imply convergence in L_p for $1 \leq p < \infty$.

Theorem **3.5.3.** *Let $\{f_n\} \subseteq L_p(X, \mathcal{A}, \mu)$ which converges almost everywhere to a function $f_0(x)$. If there exists a $g \in L_p$ such that*
$$|f_n(x)| \leq g(x)$$
for every $x \in X$ and $n \in \mathbb{N}$, then $f_0 \in L_p$ and $f_n \to f_0$ in L_p.

Proof. We can assume that $|f_0(x)| \leq g(x)$ for each $x \in X$, and hence that $f_0 \in L_p$. It also follows that

$$|f_n(x) - f_0(x)|^p \leq (2g(x))^p$$

for every $x \in X$. Now $\lim_{n\to\infty} |f_n(x) - f_0(x)|^p \to 0$ almost everywhere. By the Lebesgue Dominated Convergence Theorem we get

$$\lim_{n\to\infty} \int_X |f_n - f_0|^p \, d\mu = 0.$$

That is $f_n \to f_0$ in L_p.

∎

Definition **3.5.4.** A sequence $\{f_n\}$ of measurable real-valued functions defined on a measure space $(X, \mathcal{A}, \mu)$ is said to converge in measure to a real-valued measurable function f_0 if

$$\lim_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f_0(x)| \geq \alpha\}) = 0$$

for every $\alpha > 0$.

The sequence $\{f_n\}$ is said to be Cauchy in measure if for every $\epsilon > 0$ and every $\alpha > 0$, there exists an $N_0 \in \mathbb{N}$ such that if $n, m \geq N_0$, then

$$\mu(\{x \in X \mid |f_n(x) - f_0(x)| \geq \alpha\}) < \epsilon.$$

Remark **3.5.5.** It is easy to see that uniform convergence implies convergence in measure and that a sequence that is convergent in measure is also Cauchy in measure.. However the sequence $\{\chi_{[n,n+1]}\}$ converges pointwise on $\mathbb{R}$ to 0, but it does not converge in measure.

Now suppose that $f_n \to f_0$ in L_p. Let $\alpha > 0$. We have that if $A_\alpha = \{x \in X \mid |f_n(x) - f_0(x)| \geq \alpha\}$, then

$$\int_X |f_n(x) - f_0(x)|^p \, d\mu \geq \int_{A_\alpha} |f_n(x) - f_0(x)|^p \, d\mu \geq \alpha^p \mu(A_\alpha) \to 0.$$

From this it follows that $\mu(A_\alpha) \to 0$. That is, $f_n \to f_0$ in measure.

Theorem **3.5.6.** *[F. Riesz]*

Let $\{f_n\}$ be a sequence of measurable real-valued functions which is Cauchy in measure. Then there is a subsequence which converges almost everywhere and in measure to a real-valued function f_0.

Proof. Choose a subsequence $g_k = f_{n_k}$ such that for each $k \in \mathbb{N}$ if

$$E_k = \{x \in X \mid |g_{k+1}(x) - g_k(x_0)| \geq 2^{-k}, \}$$

then $\mu(E_k) < 2^{-k}$. We also let

$$F_k = \bigcup_{j=k}^{\infty} E_j.$$

In particular

$$\mu(F_k) \leq \sum_{j=k}^{\infty} \mu(E_j) < \sum_{j=k}^{\infty} 2^{-j} = 2^{-(k-1)}.$$

Let $i \geq j \geq k$ and $x \notin F_k$. Then

$$\begin{aligned} |g_i(x) - g_j(x)| &\leq |g_i(x) - g_{i-1}(x)| + |g_{i-1}(x) - g_{i-2}(x)| + \cdots + |g_{j+1}(x) - g_j(x)| \\ &\leq \frac{1}{2^{i-1}} + \frac{1}{2^{i-2}} + \cdots + \frac{1}{2^j} \\ &< \frac{1}{2^{j-1}} \end{aligned}$$

Let $F = \bigcap_{k=1}^{\infty} F_k$. Then $F \in \mathcal{A}$ and $\mu(F) = 0$. It is also clear from above that $\{g_k\}$ is pointwise Cauchy, and hence convergent on $X \setminus F$. Let

$$f(x) = \begin{cases} \lim_{k\to\infty} g_k(x) & \text{if } x \notin F, \\ 0 & \text{if } x \in F. \end{cases}$$

Next we see that if $j \geq k$ and $x \notin F$, then

$$|f(x) - g_j(x)| < \frac{1}{2^{j-1}} \leq \frac{1}{2^{k-1}}.$$

So now let $\alpha, \epsilon > 0$. Choose k large enough so that

$$\mu(F_k) < \frac{1}{2^{k-1}} < \min\{\alpha, \epsilon\}.$$

If $j \geq k$, then

$$\begin{aligned} \{x \in X \mid |f(x) - g_j(x)| \geq \alpha\} &\subseteq \{x \in X \mid |f(x) - g_j(x)| \geq \frac{1}{2^{k-1}}\} \\ &\subseteq F_k \end{aligned}$$

Consequently,

$$\mu(\{x \in X \mid |f(x) - g_j(x)| \geq \alpha\}) \leq \mu(F_k) < \epsilon$$

for all $j \geq k$.

∎

The next corollary is reminiscent of the fact that a Cauchy sequence with a convergent subseqeunce converges.

Corollary **3.5.7.** *Let $\{f_n\}$ be a sequence of measurable real-valued functions which is Cauchy in measure. Then there is a measurable real-valued function to which this sequence converges in measure. Moreover, this function is uniquely determined almost every where.*

Proof. We know that there is a subsequence $\{f_{n_k}\}$ that converges in measure to a real-valued function f. Now

$$|f_n - f(x)| \leq |f_n - f_{n_k}| + |f_{n_k}(x) - f(x)|.$$

As such

$$\{x \in X \mid |f_n(x) - f(x)| \geq \alpha\} \subseteq \{x \in X \mid |f_n(x) - f_{n_k}(x)| \geq \frac{\alpha}{2}\} \cup \{x \in X \mid |f_{n_k}(x) - f(x)| \geq \frac{\alpha}{2}\}$$

However, if $\epsilon > 0$, we can find an $N_0 \in \mathbb{N}$ so that if $n, m > N_0$ then

$$\mu(\{x \in X \mid |f_n(x) - f_m(x)| \geq \frac{\alpha}{2}\}) < \frac{\epsilon}{2}.$$

So let $n \geq N_0$. Then by choosing n_k large enough we get

$$\mu(\{x \in X \mid |f_n(x) - f_{n_k}(x)| \geq \frac{\alpha}{2}\}) < \frac{\epsilon}{2}$$

and

$$\mu(\{x \in X \mid |f_{n_k}(x) - f(x)| \geq \frac{\alpha}{2}\}) < \frac{\epsilon}{2}.$$

Hence if $n \geq N_0$, then

$$\mu(\{x \in X \mid |f_n(x) - f(x)| \geq \alpha\}) < \epsilon.$$

To establish the uniqueness assume that $f_n \to f$ and $f_n \to g$ in measure. Suppose that $\alpha > 0$ and that

$$\mu(\{x \in X \mid |f(x) - g(x)| \geq 2\alpha\}) = 2\epsilon > 0.$$

Then for every $n \in \mathbb{N}$ we have either

$$\mu(\{x \in X \mid |f(x) - f_n(x)| \geq \alpha\}) \geq \epsilon$$

or

$$\mu(\{x \in X \mid |f_n(x) - g(x)| \geq \alpha\}) \geq \epsilon$$

which is impossible if $f_n \to f$ and $f_n \to g$ in measure. Thus

$$\mu(\{x \in X \mid |f(x) - g(x)| > 0\}) = 0.$$

∎

We have see that convergence in measure does not imply convergence in L_p. But we now see that it does if the convergence is dominated.

THEOREM **3.5.8.** *Let $\{f_n\}$ be a sequence of measurable real-valued functions which converges to f in measure. Let $1 \leq p < \infty$ and let $g \in L_p(X, \mathcal{A}, \mu)$ such that*

$$|f_n(x)| \leq g(x) \quad a.e.$$

Then $f \in L_p(X, \mathcal{A}, \mu)$ and $f_n \to f$ in L_p.

Proof. If $\{f_n\}$ does not converge to f in L_p, then there exists an $\epsilon > 0$ and a subsequence $\{f_{n_k}\}$ such that

$$\|f_{n_k} - f\|_p \geq \epsilon$$

for every $k \in \mathbb{N}$. But $f_{n_k} \to f$ in measure as well. So by passing to a new subsequence we can assume that $f_{n_k} \to f$ almost everywhere. But then since $\{f_{n_k}\}$ is dominated, we also have $f_{n_k} \to f$ in L_p which is a contradiction.

∎

REMARK **3.5.9.** In case $f_n \to f$ in measure, a close glance at the proof that $\{f_n\}$ has a subsequence converging almost everywhere to f shows that we were actually able to construct this subsequence with the property that for any $\epsilon > 0$ we have a set $F \in \mathcal{A}$ such that $\mu(X \setminus F) < \epsilon$ and $f_n \to f$ uniformly on F.

DEFINITION **3.5.10.** A sequence $\{f_n\}$ of measurable functions is said to converge almost uniformly to a measurable fucuntion f if for every $\delta > 0$ there exists a set $F \in \mathcal{A}$ so that $\mu(F) < \delta$ and $\{f_n\}$ converges uniformly to f on $X \setminus F$.

We say that a sequence $\{f_n\}$ of measurable functions is said to almost uniformly Cauchy if for every $\delta > 0$ there exists a set $F \in \mathcal{A}$ so that $\mu(X \setminus F) < \delta$ and $\{f_n\}$ is uniformly Cauchy on F.

REMARK **3.5.11.** Note that in the definition of almost uniform convergence from our typical use of the qualifier "almost" one may have expected the definition to be that the sequence converges uniformly except on a set of measure zero. In fact, this does not follow from the definition above. For example, if

$$f_n(x) = n \cdot \chi|_{[\frac{1}{n}, \frac{2}{n}]}$$

then the sequence converges uniformly off of the set $[0, \delta]$ for any $\delta > 0$. However, there is no set A with $m(A) = 0$ for which $\{f_n\}$ converges uniformly on $\mathbb{R} \setminus A$.

Proposition **3.5.12.** *Let $\{f_n\}$ be a sequence of measurable functions that is almost uniformly Cauchy. Then there exists a measurable function f so that $\{f_n\}$ converges almost everywhere and almost uniformly to f.*

Proof. For each $k \in \mathbb{N}$, let $E_j \in \mathcal{A}$ be such that $\mu(E_j) < \frac{1}{2^j}$ with $\{f_n\}$ being uniformly Cauchy on $X \setminus E_j$. Next let

$$F_k = \bigcup_{j=1}^{k} E_j.$$

Then $\mu(F_k) < \frac{1}{2^{k-1}}$ and $\{f_n\}$ converges uniformly on $X \setminus F_k$. In particular, because $\{F_k\}$ is decreasing,

$$\lim_{n\to\infty} f_n(x) = f(x)$$

is well defined on $\bigcup\limits_{k=1}^{\infty} (X \setminus F_k) = X \setminus \bigcap\limits_{k=1}^{\infty} F_k$.

We can of course extend f to all of X by letting it be identically equal to 0 on $\bigcap\limits_{k=1}^{\infty} F_k$. Since $\mu(\bigcap\limits_{k=1}^{\infty} F_k) = 0$ we have established almost everywhere convergence. To see that it is also the almost uniform limit, let $\delta > 0$ and choose k large enough so that $\frac{1}{2^{k-1}} < \delta$. Then $\mu(F_k) < \frac{1}{2^{k-1}} < \delta$ and $\{f_n\}$ converges uniformly to f on F_k.

∎

We are now able to make a strong link between almost uniform convergence on convergence in measure.

Theorem **3.5.13.** *Let $\{f_n\}$ be a sequence of measurable functions which converges almost uniformly to a measurable function f, then $\{f_n\}$ converges in measure to f.*

Conversely, if $\{f_n\}$ converges in measure to f, then there is a subsequence $\{f_{n_k}\}$ which converges almost uniformly to f.

Proof. Assume that $\{f_n\}$ converges almost uniformly to f. Let $\alpha, \epsilon > 0$. Then there exists a set $F \in \mathcal{A}$ such that $\mu(F) < \epsilon$ and $f_n \to f$ uniformly on $X \setminus F$. As such, we can also find an $N_0 \in \mathbb{N}$ so that if $n \geq N_0$, then

$$|f_n(x) - f(x)| < \alpha$$

for all $x \in X \setminus F$. In particular, if $n \geq N_0$, then

$$\mu(\{x \in X \mid |f_n(x) - f(x)| \geq \alpha\}) \leq \mu(F) < \epsilon.$$

This shows that $f_n \to f$ in measure.

The converse has already been established.

∎

The following corollary is immediate from what we have already established.

Corollary **3.5.14.** *Let $\{f_n\} \subseteq L_p(X, \mathcal{A}, \mu)$ where $1 \leq p < \infty$. If $f_n \to f$ in L_p, then $\{f_n\}$ has a subsequence f_{n_k} such that $f_{n_k} \to f$ almost uniformly.*

Conversely, if $\{f_n\} \subseteq L_p(X, \mathcal{A}, \mu)$, and if there is a $g \in L_p(X, \mathcal{A}, \mu)$ such that

$$|f_n(x)| \leq g(x)$$

for all $x \in X$, then $f_n \to f$ almost uniformly, implies that $f_n \to f$ in L_p.

REMARK **3.5.15.** It is clear that almost uniform convergence implies almost every where convergence. But the converse fails. To see this consider the sequence $\{\chi|_{[n,\infty)}\}$. This converges everywhere to 0 but it does not converge almost uniformly. However we can show that in a finite measure space, almost everywhere convergence does indeed imply almost uniform convergence. This result, usually attributed to Egoroff but first proved by Severini, is rather surprising and powerful, yet it has a very simple proof.

THEOREM **3.5.16.** *[Egoroff's Theorem]*

Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $\{f_n\}$ be a sequence of measurable real-valued functions which converge almost everywhere to a real-valued measurable function f. Then $f_n \to f$ almost uniformly.

Proof. By redefining our functions on a set of measure zero if necessary, we may assume that $f_n(x) \to f(x)$ for all $x \in X$. Now for each $n, m \in \mathbb{N}$ let

$$F_{n,m} = \bigcup_{k=n}^{\infty} \{x \in X \mid |f_k(x) - f(x)| \geq \frac{1}{m}\}.$$

It is clear that $F_{n+1,m} \subseteq F_{n,m}$. Moreover, since $f_n(x) \to f(x)$ everywhere, for each $m \in \mathbb{N}$ we have

$$\bigcap_{n1}^{\infty} F_{n,m} = \emptyset.$$

Since $\mu(X) < \infty$, continuity from above shows that

$$\lim_{n\to\infty} \mu(F_{n,m}) = 0.$$

Fix $\delta > 0$. For each $m \in \mathbb{N}$ choose k_m large enough for that $\mu(F_{k_m,m}) < \frac{\delta}{2^m}$. Then let

$$F = \bigcup_{m=1}^{\infty} F_{k_m,m}.$$

We get that $\mu(F) < \delta$, and if $x \notin F$, then $x \notin F_{k_m,m}$ for any $m \in \mathbb{N}$. In particular,

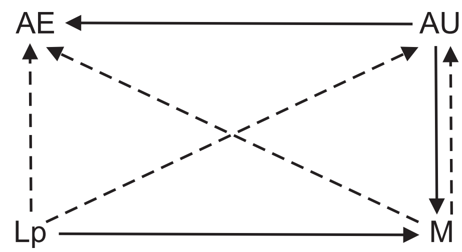
$$|f_k(x) - f(x)| < \frac{1}{m}$$

whenever $k \geq k_m$. This is sufficient to show that $f_n \to f$ uniformly on $X \setminus F$.

∎

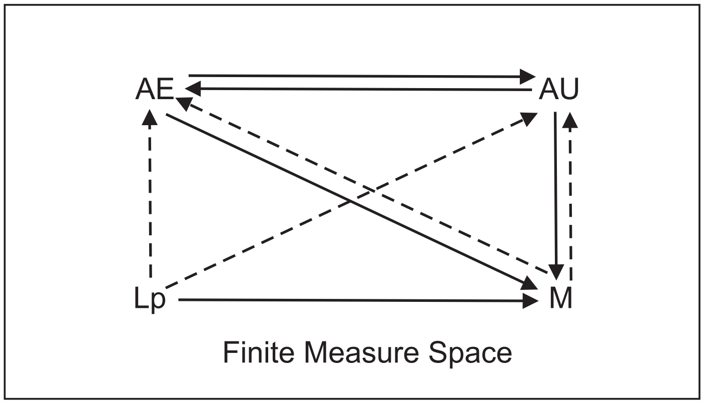
In this section we have considered 4 fundamental modes of convergence, almost every where convergence, almost uniform convergence, convergence in measure, and convergence in L_p. We are now in a position to summarize the relationship between these modes. To do so we will present three diagrams representing the relationships *a*) for a general measure space, *b*) for a finite measure space and *c*) when the convergence is dominated in L_p for some $1 \leq p < \infty$. In each diagram a solid line from **mode A** to **mode B** indicates that convergent in **mode A** implies convergence in **mode B**. A dashed line indicates that convergence in **mode A** implies convergence of a subsequence in **mode B**. The absence of a line means that there is a counter example.

Case 1: General Measure Space



General Case

Case 2: Finite Measure Spaces



Case 3: Dominated Convergence