Chapter 2

Measures and Measure Spaces

In summarizing the flaws of the Riemann integral we can focus on two main points:

1) Many *nice* functions are not Riemann integrable.
2) The Riemann integral does not behave well with respect to limits of sequence of functions. That is, we cannot interchange the limit and the integral.

In the core part of this course we will see how it is possible to develop a theory of integration that goes a long way to repair these flaws. To do so we will start by looking at how we go about defining an abstract notion of *length* for sets.

2.1 Algebras and σ-algebras

Definition **2.1.1.** Let X be a non-empty set. We denote by

$$\mathcal{P}(X) = \{A \mid A \subseteq X\}.$$

$\mathcal{P}(X)$ is called the *power set* of X

An *algebra* of subsets of X is a collection $\mathcal{A} \subseteq \mathcal{P}(X)$ such that

1. $\emptyset \in \mathcal{A}$
2. $E_1, E_2 \in \mathcal{A}$ implies $E_1 \cup E_2 \in \mathcal{A}$
3. $E \in \mathcal{A}$ implies $E^c = X \setminus E \in \mathcal{A}$

$\mathcal{A}$ is said to be a σ-algebra if

1) $\emptyset \in \mathcal{A}$
2) $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ implies $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$
3) $E \in \mathcal{A}$ implies $E^c = X \setminus E \in \mathcal{A}$

Remark **2.1.2.** 1) Every σ-algebra is an algebra.

2) $E_1 \cap E_2 = (E_1^c \cup E_2^c)^c$, so algebras are closed under intersection as well. A similar use of DeMorgan's Laws shows that a σ-algebra is closed under countable intersections.
3) Let $\mathcal{A}$ be an algebra. Let $\{F_n\} \subseteq \mathcal{A}$. Let $E_1 = F_1$, $E_2 = F_2 \setminus F_1$, $E_3 = F_3 \setminus (F_1 \cup F_2)$ and then proceed recursively to define $E_{n+1} = F_{n+1} \setminus (F_1 \cup F_2 \cup \cdots \cup F_n)$. Then $\{E_n\} \subseteq \mathcal{A}$, $\{E_n\}$ is pairwise disjoint and

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n.$$

4) Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra (σ-algebra) and let $E \in \mathcal{A}$. Let

$$\mathcal{A}_{|_E} = \{A \cap E | A \in \mathcal{A}\}.$$

Then $\mathcal{A}_{|_E}$ is an algebra (σ-algebra) in $\mathcal{P}(E)$.

Example **2.1.3.** $\mathcal{P}(X)$ is a σ-algebra.

Proposition **2.1.4.** *If $\{\mathcal{A}_\alpha\}_{\alpha \in I}$ is a collection of algebras (resp. σ-algebras) then $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is an algebra (resp. σ-algebra).*

Proof. This proof is left as an exercise. ∎

The previous proposition leads us immediately to the following important observation.

Observation**.** Given any set $S \subseteq \mathcal{P}(X)$ then there exists a smallest algebra (resp. σ-algebra) that contains S, namely

$$\bigcap \{\mathcal{A} \mid S \subseteq \mathcal{A}\}.$$

Notation **2.1.6.** *Given $S \subseteq \mathcal{P}$, let $\mathcal{A}(S)$, the algebra generated by S, be the smallest algebra containing S, and $\sigma(S)$, the σ-algebra generated by S, be the smallest σ-algebra containing S.*

Example **2.1.7.** The collection of all finite unions of sets of the form $\{\mathbb{R}, (-\infty, b], (a, b], (a, \infty)\}$ is an algebra, but it is not a σ-algebra.

Definition **2.1.8.** Let $S = \{U \subseteq \mathbb{R} \mid U \text{ is open}\}$. The σ-algebra generated by S is called the *Borel* σ-algebra of $\mathbb{R}$ and is denoted $\mathcal{B}(\mathbb{R})$.

More generally, we may take the Borel σ-algebra of any topological space $(\mathbb{X}, \tau)$ which we denote by $\mathcal{B}(\mathbb{X})$.

Of course open and closed sets are certainly the most important elements elements of $\mathcal{B}(\mathbb{X})$. However, there are other classes of sets in $\mathcal{B}(\mathbb{X})$ which also play an important role. In particular, a set $A \subseteq \mathbb{X}$ is called G_δ if $A = \bigcap_{n=1}^\infty U_n$ where each U_n is open. A set $A \subseteq \mathbb{X}$ is called F_σ if $A = \bigcup_{n=1}^\infty F_n$ where each F_n is closed. A set is $A \subseteq \mathbb{X}$ called $G_{\delta\sigma}$ if $A = \bigcup_{n=1}^\infty A_n$ where each A_n is G_δ and it is $F_{\sigma\delta}$ if $A = \bigcap_{n=1}^\infty A_n$ where each A_n is F_σ.

Additional subscripts may be appended in the most obvious way.

Remark **2.1.9.** The G_δ sets are exactly the complements of the F_σ sets, and *vice versa.*

Notice that in a metric space (X, d) closed sets are always G_δ. To see why suppose that F is closed. For each $n \in \mathbb{N}$ let

$$U_n = \bigcup_{x \in F} B(x, \frac{1}{n}).$$

where $B(x, r) = \{y \in X \mid d(x, y) < r\}$. Then U_n is open and

$$F = \bigcap_{n=1}^\infty U_n.$$

Problem **2.1.10.** *Are there sets in $\mathbb{R}$ which are neither F_σ or G_δ?*

The next fact, which is proved using transfinite induction, shows that the answer to the previous problem is clearly yes.

FACT **2.1.11.** *$|\mathcal{B}(\mathbb{R})| = c$, the cardinality of $\mathbb{R}$.*

PROBLEM **2.1.12.** *Since singletons are closed in $\mathbb{R}$ it is clear that every countable set is an F_σ set. But is $\mathbb{Q}$ a G_δ set in $\mathbb{R}$?*

To answer this problem we first recall the notion of *category* for a topological space.

DEFINITION **2.1.13.** A set A in a topological space (X, τ) is said to be nowhere dense if $\overline{A}$ has empty interior. It is of first category in (X, τ) if

$$A = \bigcup_{n=1}^{\infty} A_n$$

where A_n is nowhere dense for each $n \in \mathbb{N}$.

A is of second category in (X, τ) if it is not of first category.

A is residual if A^c is of first category.

THEOREM **2.1.14.** ***[Baire's Category Theorem].*** *Let X be a complete metric space. If $\{U_n\}_{n=1}^{\infty}$ is a collection of dense open sets then $\bigcap_{n=1}^{\infty} U_n$ is dense.*

Proof. Let W be open and non-empty. Then there exists an $x_1 \in X$ and $0 < r_1 \leq 1$ such that

$$B(x_1, r_1) \subseteq B[x_1, r_1] \subseteq W \cap U_1.$$

Next we can find $x_2 \in X$ and $0 < r_2 < \frac{1}{2}$ such that

$$B(x_2, r_2) \subseteq B[x_2, r_2] \subseteq B(x_1, r_1) \cap U_2.$$

We can then proceed recursively to find sequences $\{x_n\} \subseteq X$ and $\{r_n\} \subset \mathbb{R}$ with $0 < r_n \leq \frac{1}{n}$, and

$$B(x_{n+1}, r_{n+1}) \subseteq B[x_{n+1}, r_{n+1}] \subseteq B(x_n, r_n) \cap U_{n+1}.$$

Since $r_n \to 0$ and $B[x_{n+1}, r_{n+1}] \subseteq B[x_n, r_n]$, Cantor's Intersection Theorem implies that there exists an

$$x_0 \in \bigcap_{n=1}^{\infty} B[x_n, r_n.]$$

But then $x_0 \in B[x_1, r_1] \subseteq W$ and $x_0 \in B[x_n, r_n] \subseteq U_n$ for each $n \in \mathbb{N}$. This shows that

$$x_0 \in W \cap (\bigcap_{n=1}^{\infty} U_n).$$

∎

REMARK **2.1.15.** The Baire Category Theorem shows that if $\{U_n\}$ is a sequence of open dense sets, then $\bigcap\limits_{n=1}^{\infty} U_n$ is dense in X. We also know that $\bigcap\limits_{n=1}^{\infty} U_n$ is a G_δ. These dense G_δ subsets of a complete metric space are always residual, as we will see below, and as such are topologically fat.

Our next corollary shows the connection between the Baire Category Theorem and our notion of *category*.

COROLLARY **2.1.16.** *[Baire Category Theorem II]*

Every complete metric space (X, d) is of second category in itself.

Proof. Assume that X is of first category. Then there exists a sequence A_n of nowhere dense sets so that

$$X = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \overline{A_n}.$$

Now let $U_n = (\overline{A_n})^c$. Then U_n is open and dense. But

$$\bigcap_{n=1}^{\infty} U_n = \emptyset$$

which is impossible. ∎

COROLLARY **2.1.17.** $\mathbb{Q}$ *is not* G_δ *in* $\mathbb{R}$.

Proof. Assume that $\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n$, where U_n is open for each n. Then $\mathbb{R}/\mathbb{Q} = \bigcup_{n=1}^{\infty} F_n$, where $F_n = U_n^c$ is closed and nowhere dense. Let $\mathbb{Q} = \{r_1, r_2, \ldots\}$. Then $F_n' = F_n \cup \{r_n\}$ is closed and nowhere dense and $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n'$. This is a contradiction since $\mathbb{R}$ is not of first category. ∎

The next example is a set in $\mathbb{R}$ which is not Borel. Observe that the construction of this set requires the Axiom of Choice. While this is not necessary to construct non-Borel sets it turns out that this set is even more pathological than simply being non-Borel. We will see later that it is an example of a set that is also not *Lebesgue measurable*.

EXAMPLE **2.1.18.** Consider $[0,1]$. Define an equivalence relation on $[0,1]$ by $x \sim y$ if $x - y \in \mathbb{Q}$. Use the Axiom of Choice to choose one element from each equivalence class and denote this set by S. We will show later that S is not Borel.

REMARK **2.1.19.** There is nothing special about using the open sets of $\mathbb{R}$ to define $\mathcal{B}(\mathbb{R})$.

$$\mathcal{B}(\mathbb{R}) = \sigma\{(a,b) \mid a,b \in \mathbb{R}\} = \sigma\{(a,b] \mid a,b \in \mathbb{R}\} = \sigma\{[a,b) \mid a,b \in \mathbb{R}\} = \sigma\{[a,b] \mid a,b \in \mathbb{R}\}$$

2.2 Measures and Measure Spaces

In trying to find a notion of *length or measure* for an arbitray subset $A \subseteq \mathbb{R}$ we are looking for a function $m : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ with the following properties:

1. $m(I) =$ the usual length of I for all intervals I
2. If $\{A_n\}$ is a sequence of pairwise disjoint subsets of $\mathbb{R}$, then
$$m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n).$$
3. $m(x + A) = m(A)$ for all $A \in \mathcal{P}(\mathbb{R})$ and $x \in \mathbb{R}$.

The problem is that no such measure exists. If we weaken our conditions and only look for a measure on some sufficiently large sub-σ-algebra $\mathcal{A}$ of $\mathcal{P}(\mathbb{R})$ then we may succeed. More specifically, we would like $\mathcal{A}$ to contain $\mathcal{B}(\mathbb{R})$. The measure function that we get is called the *Lebesgue measure* on $\mathbb{R}$. It will serve as our motivation for defining and constructing abstract measures on arbitrary sets.

DEFINITION **2.2.1.** A pair $(X, \mathcal{A})$ consisting of a set X together with a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ is called a *measurable space*. A (*countably additive*) *measure* on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ such that

1) $\mu(\emptyset) = 0$

2) $\mu(E) \geq 0$ for all $E \in \mathcal{A}$

3) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ is a sequence of pairwise disjoint sets, then $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$.

A triple $(X, \mathcal{A}, \mu)$ is called a *measure space* where $\mathcal{A}$ is a σ-algebra in $\mathcal{P}(X)$, and μ is a measure on $\mathcal{A}$. If $E \in \mathcal{A}$, then E is called *$\mathcal{A}$-measurable*, or more commonly *measurable* .

$(X, \mathcal{A}, \mu)$ is *complete* if $\mu(E) = 0$ and $S \subseteq E$ implies that $S \in \mathcal{A}$.

If $\mu(X) = 1$ then μ is called a *probability measure* and $(\mathbb{X}, \mathcal{A}, \mu)$ is called a *probability space*. In this case, the elements of $\mathcal{A}$ are called *events*.

Remark **2.2.2.** Condition (3) is known as *countable additivity*. If we replace (3) by

$3'$. If $\{E_n\}_{n=1}^{N} \subseteq \mathcal{A}$ is a finite sequence of pairwise disjoint sets, then $\mu(\bigcup_{n=1}^{N} E_n) = \sum_{n=1}^{N} \mu(E_n)$, then μ is called a *finitely additive* measure on $\mathcal{A}$.

We will see that the assumption of countable additivity for a measure leads to powerful convergence results for integration. Finitely additive measures are less useful but still play an important role in analysis. We will generally not consider finitely additive measures in this course.

Example **2.2.3.** 1) The simplest example of a measure is the *counting measure* μ on the set X. More specifically, for any $A \subseteq X$ we define

$$\mu(A) = \begin{cases} \mid A \mid & \text{if } A \text{ is finite.} \\ \infty & \text{if } A \text{ is infinite.} \end{cases}$$

where $\mid A \mid$ denotes the cardinality of A.

If X is finite, then $\mu_*(A) = \frac{|A|}{|X|}$ is caled the *normalized counting measure* on X.

2) Let $(X, \mathcal{A})$ be a measurable space. Let $x_0 \in X$. The point mass at x_0 is a measure μ_{x_0} defined by

$$\mu_{x_0}(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{if } x_0 \notin A. \end{cases}$$

Definition **2.2.4.** We call a measure μ *finite* if $\mu(X) < \infty$. We call μ *σ-finite* if there exists there exists a sequence $\{E_n\} \subseteq \mathcal{A}$ such that $X = \bigcup\limits_{n=1}^{\infty} E_n$ with $\mu(E_n) < \infty$ for each $n \in \mathbb{N}$.

Example **2.2.5.** Let $X = \mathbb{N}$ and $\mathcal{A} = \mathcal{P}(\mathbb{N})$. Let $f : \mathbb{N} \to \mathbb{R}_+^*$ be any function. Define

$$\mu_f(E) := \sum_{n \in E} f(n) = \sum_{n=1}^{\infty} f(n)\chi_E(n).$$

It is easy to see that properties 1) and 2) for a measure are satisfied by μ_f. The fact that μ_f is countably additive is simply a restatement of the fact that for positive series the sum is independent of order. That is if $a_{i,j} \geq 0$, then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{i,j}.$$

Hence μ_f is a countably additive measure on $\mathcal{P}(\mathbb{N})$.

The measure μ_f is finite if and only if $\sum\limits_{n=1}^{\infty} f(n) < \infty$. That is to say that μ_f is finite if and only if

$$f \in l^1(\mathbb{N})^+ = \{g \in l^1(\mathbb{N}) \mid g(n) \geq 0 \ \text{ for all } n \in \mathbb{N}\}.$$

In addition, μ_f is σ-finite if $f(n) < \infty$ for each $n \in \mathbb{N}$.

Moreover, if μ is a countably additive measure on $\mathcal{P}(\mathbb{N})$, then if we define $f_\mu : \mathbb{N} \to \mathbb{R}_+^*$ by

$$f_\mu(n) = \mu(\{n\}),$$

then $\mu = \mu_{f_\mu}$. That is, there is a one to one correspondence between measures on $\mathbb{R}$ and functions $f : \mathbb{N} \to \mathbb{R}_+^*$.

More generally, suppose that X is any set. Given a function $f : X \to \mathbb{R}_+^*$, we can define

$$\mu_f(E) := \sup\{f(x_1) + f(x_2) + \cdots + f(x_n) \mid x_1, x_2, \ldots, x_n \in E, \text{ with } x_i \neq x_j \text{ if } i \neq j\}.$$

Then μ_f is a countably additive measure on $\mathcal{P}(\mathcal{X})$. If $f \in l^1(X)^+$, then μ_f is finite. It is σ-finite if $f(x) = 0$ for all but countably many $x \in X$ and $f(x) < \infty$ for all $x \in X$.

PROBLEM **2.2.6.** *Given a measure μ on $(X, \mathcal{P}(X))$ can we always find a function $f : X \to \mathbb{R}_+^*$ such that $\mu = \mu_f$?*

Consider the measure μ on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ given by

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is countable.} \\ \infty & \text{if } E \text{ is uncountable.} \end{cases}$$

Then it is easy to see that there is no function $f : X \to \mathbb{R}_+^*$ such that $\mu = \mu_f$.

In this case the measure μ is not σ-finite. If μ was assumed to be σ-finite could such a function f be found?. What would happen if μ is assumed to be finite?

The proof of the following useful proposition is left as an exercise:

PROPOSITION **2.2.7.** *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $E \in \mathcal{A}$.*

1) Define $\gamma_E : \mathcal{A} \to \mathbb{R}^*$ by

$$\gamma_E(A) = \mu(A \cap E).$$

Then γ_E is a measure on $\mathcal{A}$.

2) $\mu_{|\mathcal{A}_{|E}}$ is a measure on $\mathcal{A}_{|E}$.

3) Given measures $\{\mu_1, \mu_2, \ldots, \mu_n\}$ on $\mathcal{A}$ and positive scalars $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$, then

$$\gamma = \sum_{i=1}^{n} \alpha_i \mu_i$$

is also a measure on $\mathcal{A}$.

The next proposition is almost trivial but it is none the less very important:

PROPOSITION **2.2.8.** *(Monotonicity). Let $(X, \mathcal{A}, \mu)$ be a measure space. If $E, F \in \mathcal{A}$ with $E \subseteq F$, then $\mu(E) \leq \mu(F)$. If $\mu(E) < \infty$, then $\mu(F \setminus E) = \mu(F) - \mu(E)$.*

Proof. This follows since

$$\mu(F) = \mu(E) + \mu(F \setminus E)$$

and hence if $\mu(E) < \infty$, then we get

$$\mu(F) - \mu(E) = \mu(F \setminus E).$$

∎

The next lemma which is also straight forward, will be used many times throughout the course.

LEMMA **2.2.9.** *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra. Assume that $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$. Then there exists $\{F_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ such that $\bigcup\limits_{n=1}^{k} E_n = \bigcup\limits_{n=1}^{k} F_n$ for all $k \in \mathbb{N}$, $F_n \subseteq E_n$ for all $n \in \mathbb{N}$, and $\{F_n\}_{n=1}^{\infty}$ is pairwise disjoint. Moreover, $\bigcup\limits_{n=1}^{\infty} E_n = \bigcup\limits_{n=1}^{\infty} F_n$.*

Proof. Let $F_1 := E_1 \in \mathcal{A}$. For each $n \geq 2$, let $F_n := E_n \setminus (\bigcup\limits_{i=1}^{n-1} F_i)$.

∎

PROPOSITION **2.2.10.** *(Countable Subadditivity) Let $(X, \mathcal{A}, \mu)$ be a measure space with $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$. Then*

$$\mu(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

Proof. Let $\{F_n\}_{n=1}^{\infty}$ be as in Lemma 2.2.9. Then

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \mu(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} \mu(F_n) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

∎

THEOREM **2.2.11.** *(Continuity from Below) Let $(X, \mathcal{A}, \mu)$ be a measure space with $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$. If $E_i \subseteq E_{i+i}$ for each $i \in \mathbb{N}$, then*

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n).$$

Proof. First observe that if $\mu(E_n) = \infty$ for some $n \in \mathbb{N}$, then monotonicity shows that

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \infty = \lim_{n\to\infty} \mu(E_n).$$

As such we may assume that $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$.

Let $F_1 := E_1$ and for $n \geq 2$ let $F_n = E_n \setminus E_{n-1}$. Then $\{F_n\}_{n=1}^{\infty}$ is a pairwsie disjoint sequence with $E_n = \bigcup\limits_{i=1}^{n} F_i$. Moreover,

$$\mu(F_n) = \mu(E_n) - \mu(E_{n-1})$$

for $n > 1$. It follows that

$$\sum_{n=1}^{m} \mu(F_n) = \mu(E_m).$$

Finally, we have

$$\begin{aligned}
\mu(\bigcup_{n=1}^{\infty} E_n) &= \mu(\bigcup_{n=1}^{\infty} F_n) \\
&= \sum_{n=1}^{\infty} \mu(F_n) \\
&= \lim_{m\to\infty} \sum_{n=1}^{m} \mu(F_n) \\
&= \lim_{m\to\infty} \mu(E_m)
\end{aligned}$$

$\blacksquare$

Remark **2.2.12.** Theorem 2.2.11 is also called the *Monotone Convergence Theorem for Measures.*

Theorem **2.2.13.** *(Continuity from Above) Let $(X, \mathcal{A}, \mu)$ be a measure space with $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$. If $\mu(E_1) < \infty$ and if $E_{i+1} \subseteq E_i$ for each $i \in \mathbb{N}$, then*

$$\mu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n).$$

Proof. Let $F_n = E_1 \setminus E_n$. Then $\{F_n\}$ is an increasing sequence of $\mathcal{A}$-measurable sets with

$$\bigcup_{n=1}^{\infty} F_n = E_1 \setminus \bigcap_{n=1}^{\infty} E_n.$$

and as such by Continuity from Below we have

$$\begin{aligned} \mu(E_1) - \mu(\bigcap_{n=1}^{\infty} E_n) &= \mu(\bigcup_{n=1}^{\infty} F_n) \\ &= \lim_{n\to\infty} \mu(F_n) \\ &= \lim_{n\to\infty} \mu(E_1) - \mu(E_n) \\ &= \mu(E_1) - \lim_{n\to\infty} \mu(E_n). \end{aligned}$$

It follows that

$$\mu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n).$$

$\blacksquare$

Remark **2.2.14.** In the previous Theorem the assumption that $\mu(E_1), \infty$ was necessary in the sense that if $\mu(E_n) = \infty$ for each $n \in \mathbb{N}$, then the result may fail. For example, Let $X = \mathbb{N}$ and let μ be the counting measure on $\mathcal{P}(\mathbb{N})$. If $E_n = \{n, n+1, n+2, \ldots\}$, then $\mu(E_n) = \infty$ for each $n \in \mathbb{N}$, but

$$\mu(\bigcap_{n=1}^{\infty} E_n) = \mu(\emptyset) = 0 \neq \lim_{n\to\infty} \mu(E_n).$$

2.3 Constructing Measures: The Carathéodory Method

Suppose that one is looking for a generalized notion of length of subsets of $\mathbb{R}$. It would be reasonable that we ask for a measure m on $\mathcal{P}(\mathbb{R})$ with the following properties:

1) $m(I) =$ length for I for all intervals I

2) m is countably additive

3 $m(x + A) = m(A)$ for all $A \in \mathcal{P}(\mathbb{R})$ and $x \in \mathbb{R}$

The problem is that no such measure exists. If we weaken our conditions and only look for a measure on some sufficiently large sub-σ-algebra $\mathcal{A}$ of $\mathcal{P}(\mathbb{R})$ then we may suceed. More specifically, we would like $\mathcal{A}$ to contain $\mathcal{B}(\mathbb{R})$ and be complete with respect to the measure. To see how this measure, which is known as Lebesgues measure, is contructed we will introduce a general process known as the Carathéodory Method for constructing a measure.

Definition **2.3.1.** [Outer Measure and Measurable sets]

Let X be any non-empty set. A function $\mu^* : \mathcal{P}(X) \to \mathbb{R}^*$ is called an *outer measure* if

1. $\mu^*(\emptyset) = 0$
2. $\mu^*(A) \le \mu^*(B)$ if $A \subseteq B$ (monotonicity)
3. If $\{E_n\}_{n=1}^{\infty} \subseteq A$ then $\mu^*(\bigcup_{n=1}^{\infty} E_n) \le \sum_{n=1}^{\infty} \mu^*(E_n)$. (countable subadditivity)

We say that μ^* is *finite* if $\mu^*(X) < \infty$. We say that μ^* is *σ-finite* if there exists a sequence $\{E_n\} \subseteq \mathcal{P}(X)$ such that $X = \bigcup_{n=1}^{\infty} E_n$ and $\mu^*(E_n) < \infty$ for all n.

A set $E \in \mathcal{P}(\mathbb{R})$ is said to be μ^**-measurable* or *measurable* if for every $A \in \mathcal{P}(\mathbb{R})$

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Otherwise we say that E is non-measurable.

Remark **2.3.2.** Notice that $\mu^*(A) \le \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for every $A \in \mathcal{P}(X)$. Thus we need only show that $\mu^*(A) \ge \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for every $A \in \mathcal{P}(X)$ to show that E is measurable. Furthermore, we need only assume that $\mu^*(A) < \infty$.

In the case that $\mu^*(A) < \mu^*(A \cap E) + \mu^*(A \cap E^c)$ then $A = (A \cap E) \cup (A \cap E^c)$ is called a *paradoxical decomposition* of the non-measurable set E.

Example **2.3.3.** Let I be an interval in $\mathbb{R}$ and let $\ell(I)$ denote its length. For any $E \subseteq \mathbb{R}$ define

$$m^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) \mid E \subseteq \bigcup_{n=1}^{\infty} I_n,\ I_n\text{'s are open intervals} \right\}$$

m^* is called the Lebesgue outer measure on $\mathbb{R}$.

To see that m^* is indeed an outer measure on $\mathbb{R}$ observe that from the definition, it is clear that:

1. $m^*(\emptyset) = 0$
2. $m^*(E) \ge 0$
3. If $F \subseteq E$, then $m^*(F) \le m^*(E)$.

As such to confirm that m^* is an outer measure we need only show that m^* is countably subadditive.

Theorem **2.3.4.** *If m^* is the Lebesgue outer measure on $\mathbb{R}$, then m^* is countably subadditive.*

Proof. Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(\mathbb{R})$. We may assume without lost of generality that $m^*(E_n) < \infty$ for all n. Let $\epsilon > 0$. For each $n \in \mathbb{N}$ choose a countable collection $\{I_{i,n}\}$ of open intervals with $E_n \subseteq \bigcup_{i=1}^{\infty} I_{i,n}$ such that $\sum_{i=1}^{\infty} \ell(I_{i,n}) \le m^*(E_n) + \frac{\epsilon}{2^n}$. Note that

$$\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{i,n=1}^{\infty} I_{i,n},$$

so

$$\begin{aligned} m^*\left(\bigcup_{n=1}^{\infty} E_n\right) &\leq \sum_{i,n=1}^{\infty} \ell(I_{i,n}) \\ &= \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \ell(I_{i,n}) \\ &\leq \sum_{n=1}^{\infty} \left(m^*(E_n) + \frac{\epsilon}{2^n}\right) \\ &\leq \left(\sum_{n=1}^{\infty} m^*(E_n)\right) + \epsilon \end{aligned}$$

Since this is true for all $\epsilon > 0$ we get $m^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} m^*(E_n)$. ∎

It is reasonable to ask if m^*, or indeed if any given outer measure μ^* is in fact a measure. Clearly, if there is a set $E \in \mathcal{P}(X)$ which is non-measurable with respect to μ^*, then μ^* fails to be even finitely additive.

The next theorem illustrates that if we restrict an outer measure μ^* to the set of all measurable sets, then we do indeed get a true measure.

Theorem **2.3.5.** *[Carathéodory's Theorem] Let μ^* be an outer measure on X. The set $\mathcal{B}$ of μ^*-measurable sets in $\mathcal{P}(X)$ is a σ-algebra and if $\mu = \mu^*_{|\mathcal{B}}$, then μ is a complete measure on $\mathcal{B}$.*

Proof. : It is clear that $\emptyset \in \mathcal{B}$ since for any $A \in \mathcal{P}(X)$

$$\mu^*(A) = \mu^*(A \cap \mathbb{R}) = \mu^*(A \cap \emptyset) + \mu^*(A \cap \emptyset^c)$$

It is also clear that if $E \in \mathcal{B}$, then $E^c \in \mathcal{B}$ by symmetry of the definition of μ^*-measurable.

Let E_1, $E_2 \in \mathcal{B}$ and let $A \in \mathcal{P}(X)$. Since E_2 is measurable,

$$\mu^*(A) = \mu^*(A \cap E_2) + \mu^*(A \cap E_2^c)$$

Since E_1 is measurable,

$$\mu^*(A \cap E_2^c) = \mu^*(A \cap E_2^c \cap E_1) + \mu^*(A \cap E_2^c \cap E_1^c)$$

These together imply that

$$\mu^*(A) = \mu^*(A \cap E_2) + \mu^*(A \cap E_2^c \cap E_1) + \mu^*(A \cap E_2^c \cap E_1^c)$$

Notice that $A \cap (E_1 \cup E_2) = (A \cap E_2) \cup (A \cap E_1 \cap E_2^c)$. Thus

$$\mu^*(A \cap (E_1 \cup E_2)) \leq \mu^*(A \cap E_2) + \mu^*(A \cap E_1 \cap E_2^c)$$

by subadditivity, so

$$\mu^*(A) \geq \mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap E_2^c \cap E_1^c) = \mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_2 \cup E_1)^c)$$

which implies $E_1 \cup E_2 \in \mathcal{B}$. Therefore $\mathcal{B}$ is an algebra.

Now let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{B}$ be a sequence of pairwise disjoint μ^*-measurable sets and let $E = \bigcup_{i=1}^{\infty} E_i$. Let $G_n = \bigcup_{i=1}^{n} E_i$. Then $G_n \in \mathcal{B}$ and for any $A \in \mathcal{P}(X)$

$$\mu^*(A) = \mu^*(A \cap G_n) + \mu^*(A \cap G_n^c) \geq \mu^*(A \cap G_n) + \mu^*(A \cap E^c)$$

since $E^c \subseteq G_n^c$. Notice $G_n \cap E_n = E_n$ and $G_n \cap E_n^c = G_{n-1}$. Since E_n is measurable

$$\mu^*(A \cap G_n) = \mu^*(A \cap G_n \cap E_n) + \mu^*(A \cap G_n \cap E_n^c) = \mu^*(A \cap E_n) + \mu^*(A \cap G_{n-1})$$

An inductive argument shows that $\mu^*(A \cap G_n) = \sum_{i=1}^{n} \mu^*(A \cap E_i)$. Therefore

$$\mu^*(A) \geq \mu^*(A \cap E^c) + \sum_{i=1}^{n} \mu^*(A \cap E_i)$$

for all n. Hence

$$\mu^*(A) \geq \mu^*(A \cap E^c) + \sum_{n=1}^{\infty} \mu^*(A \cap E_i) \geq \mu^*(A \cap E^c) + \mu^*\left(\bigcup_{n=1}^{\infty} (A \cap E_i)\right) = \mu^*(A \cap E^c) + \mu^*(A \cap E)$$

Thus E is measurable. Given any sequence $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{B}$ with $E = \bigcup_{i=1}^{\infty} E_i$, we can find a pairwise disjoint sequence $\{F_i\}_{i=1}^{\infty} \subseteq \mathcal{B}$ with $E = \bigcup_{i=1}^{\infty} F_i$. Hence $E \in \mathcal{B}$ and $\mathcal{B}$ is a σ-algebra.

Let E_1, $E_2 \in \mathcal{B}$ be disjoint. Then since E_2 is measurable,

$$\mu(E_1 \cup E_2) = \mu^*(E_1 \cup E_2) = \mu^*((E_1 \cup E_2) \cap E_2) + \mu^*((E_1 \cup E_2) \cap E_2^c) = \mu^*(E_2) + \mu^*(E_1) = \mu(E_2) + \mu(E_1)$$

Hence μ is finitely additive. Let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{B}$ be a sequence of pairwise disjoint μ^*-measurable sets and let $E = \bigcup_{i=1}^{\infty} E_i$. Then

$$\mu(E) = \mu^*(E) \geq \mu^*\left(\bigcup_{i=1}^{n} E_i\right) = \mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i)$$

for every n. Taking the limit, we have $\mu(E) \geq \sum_{i=1}^{\infty} \mu(E_i)$. On the other hand,

$$\mu(E) = \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i) = \sum_{i=1}^{\infty} \mu(E_i)$$

Thus $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$ and μ is countable additive. Clearly $\mu(\emptyset) = 0$ and $\mu(E) \geq 0$ for all $E \in \mathcal{B}$, so μ is a measure on $\mathcal{B}$.

Let $\mu(E) = 0$ and $F \subseteq E$. Then

$$\mu(A) \geq \mu^*(A \cap F^c) = \mu^*(A \cap F^c) + \mu^*(A \cap E) \geq \mu^*(A \cap F^c) + \mu^*(A \cap F)$$

so $F \in \mathcal{B}$. Therefore $(X, \mathcal{B}, \mu^*_{|\mathcal{B}})$ is complete. ∎

2.4 Lebesgue Measure

Remark **2.4.1.** Recall that the Lebesgue outer measure m^* was defined on $\mathcal{P}(\mathbb{R})$ by letting

$$m^*(E) = \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) \mid E \subseteq \bigcup_{n=1}^{\infty} I_n,\ I_n\text{'s are open intervals}\right\}$$

for any $E \subseteq \mathbb{R}$.

We denote the σ-algebra of m^*-measurable sets by $\boldsymbol{M}(\mathbb{R})$. Elements of $\boldsymbol{M}(\mathbb{R})$ are said to be *Lebesgue measurable.* $m = m^*|_{\boldsymbol{M}(\mathbb{R})}$ is called the *Lebesgue measure* on $\mathbb{R}$.

In this section we will take a brief look at some of the properties of the Lebesgue measure.

Proposition **2.4.2.** *If I is an interval, then $m^*(I) = \ell(I)$.*

Proof. Assume that $I = [a, b]$ and let $\epsilon > 0$. Then if $I_1 = (a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$, we have $I \subseteq I_1$, so

$$m^*(I) \leq \ell(I_1) = b - a + \epsilon = \ell(I) + \epsilon$$

Since ϵ was arbitary, $m^*(I) \leq \ell(I)$.

Assume that $I \subseteq \bigcup_{i=1}^{\infty} I_i$, where each I_i is an open interval. Since $[a,b]$ is compact, there is a finite subcover. Out of these finitely many elements, choose the longest one that contains a. Without lose of generality, we may suppose it is $I_1 = (a_1, b_1)$. If $b \in I_1$ we are done. Otherwise, choose the longest such interval that contains b_1 and call it I_2. Continue this process to get $I_1, \ldots, I_k$ such that $I \subseteq \bigcup_{i=1}^{k} I_i$. Furthermore, if $I_j = (a_j, b_j)$, then $a_1 < a < b < b_k$ and $a_1 < a_2 < b_1 < a_3 < b_2 < \ldots < a_k < b_{k-1} < b_k$ and $b_i - a_i \geq b_i - b_{i-1}$. Hence

$$\sum_{i=1}^{k} b_i - a_i \geq b_1 - a_1 + \sum_{i=2}^{k} b_i - b_{i-1} = b_k - a_1 > b - a$$

Therefore

$$\sum_{i=1}^{\infty} \ell(I_i) \geq \sum_{i=1}^{k} \ell(I_i) = \sum_{i=1}^{k} b_i - a_i > b - a = \ell(I)$$

Hence $m^*(I) \geq \ell(I)$, so $m^*(I) = \ell(I)$.

Assume that I is a finite interval. For any $\epsilon > 0$, we can find a closed interval $J \subseteq I$ such that $\ell(I) < \ell(J) + \epsilon$. Hence

$$\ell(I) - \epsilon < \ell(J) = m^*(J) \leq m^*(I) \leq m^*(\bar{I}) = \ell(\bar{I}) = \ell(I)$$

so $m^*(I) = \ell(I)$.

Finally, if $\ell(I) = \infty$, then for any $M > 0$ we can find a finite interval $J \subseteq I$ with $\ell(J) > M$. Then $m^*(I) \geq m^*(J) = \ell(J) > M$, so $m^*(I) = \infty = \ell(I)$.

∎

Lemma **2.4.3.** *The intervals (a, ∞) and $(-\infty, a]$ are m^*-measurable.*

Proof. Let $A \subseteq \mathbb{R}$. We need to show that

$$m^*(A) \geq m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a])$$

and moreover, we may assume that $m^*(A) < \infty$. Let $\epsilon > 0$. We can find a collection of open intervals with $A \subseteq \bigcup_{i=1}^{\infty} I_i$ and $\sum_{i=1}^{\infty} \ell(I_i) < m^*(A) + \epsilon$. Let $I_n' = I_n \cap (a, \infty)$ and $I_n'' = I_n \cap (-\infty, a]$. Then I_n' and I_n'' are intervals and $\ell(I_n) = \ell(I_n') + \ell(I_n'') = m^*(I_n') + m^*(I_n'')$. We have

$$m^*(A \cap (a, \infty)) \leq m^*\left(\bigcup_{i=1}^{\infty} I_i'\right) \leq \sum_{i=1}^{\infty} m^*(I_i')$$

and

$$m^*(A \cap (-\infty, a]) \leq m^*\left(\bigcup_{i=1}^{\infty} I_i''\right) \leq \sum_{i=1}^{\infty} m^*(I_i'')$$

Hence

$$\begin{aligned} m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a]) &\leq \sum_{i=1}^{\infty} m^*(I_i') + \sum_{i=1}^{\infty} m^*(I_i'') \\ &= \sum_{i=1}^{\infty} m^*(I_i') + m^*(I_i'') \\ &= \sum_{i=1}^{\infty} \ell(I_i') + \ell(I_i'') \\ &= \sum_{i=1}^{\infty} \ell(I_i) \\ &< \quad m^*(A) + \epsilon \end{aligned}$$

Since ϵ was arbitrary, $m^*(A) \geq m^*(A \cap (a,\infty)) + m^*(A \cap (-\infty, a])$. ∎

Given that intervals of the type (a,∞) and $(-\infty, a]$ generate the Borel σ-algebra, the next theorem is immediate.

THEOREM **2.4.4.** $\mathcal{B}(\mathbb{R}) \subseteq \boldsymbol{M}(\mathbb{R})$

REMARK **2.4.5.** 1) Clearly every countable or finite set has Lebesgue measure 0.

2) Since the length of an interval is translation invariant, it follows that m^* is also translation invariant. Now let $E \in \boldsymbol{M}(\mathbb{R})$, $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Then

$$\begin{aligned} m^*(x+A) &= m^*(A) \\ &= m^*(A \cap E) + m^*(A \cap E^c) \\ &= m^*((x+A) \cap (x+E)) + m^*((x+A) \cap (x+E^c)) \\ &= m^*((x+A) \cap (x+E)) + m^*((x+A) \cap (x+E)^c) \end{aligned}$$

If A is arbitrary then so is $x+A$, so $x+E \in \boldsymbol{M}(\mathbb{R})$. That is $\boldsymbol{M}(\mathbb{R})$ is translation invariant and as such so is m.

The following is a useful characterization of Lebesgue measurable sets. It's proof will be left as an exercise.

PROPOSITION **2.4.6.** *Let $E \subseteq \mathbb{R}$. The following are equivalent:*

1. $E \in \boldsymbol{M}(\mathbb{R})$
2. *Given $\epsilon > 0$ there is an open set $U \subseteq \mathbb{R}$ with $E \subseteq U$ and $m^*(U \setminus E) < \epsilon$.*
3. *Given $\epsilon > 0$ there is a closed set $F \subseteq \mathbb{R}$ with $F \subseteq E$ and $m^*(E \setminus F) < \epsilon$.*

DEFINITION **2.4.7.** Let (X, τ) be a topological space. Let $\mathcal{A}$ be a σ-algebra in $\mathcal{P}(X)$ which contains the Borel sets $\mathcal{B}(X)$. Let μ be a measure on $\mathcal{A}$. Then we say that:

1) μ is inner regular if $m(E) = \sup\{m(K) \mid K \subseteq E \text{ is compact}\}$ for all $E \in \mathcal{A}$.

2) μ is outer regular if $m(E) = \inf\{m(U) \mid U \supseteq E \text{ is open}\}$ for all $E \in \mathcal{A}$.

We are now in a position to obtain some important properties of the Lebesgue measure including the fact that m is both inner and outer regular. The results are either immediate or the follow easily from the previous proposition.

THEOREM **2.4.8.** *[Regularity of the Lebesgue measure] Let $E \in \mathcal{B}(\mathbb{R})$. Then*

1) $m(E) = \sup\{m(K) \mid K \subseteq E \textit{ is compact}\}$

2) $m(E) = \inf\{m(U) \mid U \supseteq E \textit{ is open}\}$

3) If U is open, then $m(U) = \sum_{n=1}^{\infty}(b_n - a_n)$ where $U = \bigcup_{n=1}^{\infty}(a_n, b_n)$ is a decomposition of U into disjoint open intervals.

4) $m(\{x\}) = 0$

5) If $K \subset \mathbb{R}$ is compact, then $m(K) < \infty$.

PROBLEM **2.4.9.** *It is reasonable to ask if every $A \subseteq \mathbb{R}$ is Lebesgue measurable?*

Example **2.4.10.** Let x, $y \in [0,1)$ and let $x \oplus y := x + y \pmod 1$. If $E \subseteq [0,1)$ is measurable, then $x \oplus E$ is measurable and $m(E) = m(x \oplus E)$. Indeed, let $E_1 = \{y \in E \mid x + y < 1\} = E \cap (-\infty, 1-x)$ and $E_2 = \{y \in E \mid x + y \geq 1\} = E \cap [1-x, \infty)$. Then $E = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$, and E_1, E_2 are both measurable. It follows that

$$x \oplus E = (x \oplus E_1) \cup (x \oplus E_2) = (x \oplus E_1) \cup ((x-1) \oplus E_2)$$

is measurable and $m(x \oplus E) = m(E)$.

Define an equivalence relation on $[0,1)$ by $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$. Using the Axiom of Choice, construct a set $E \subseteq [0,1)$ consisting of one element from each equivalence class. Let $\mathbb{Q} \cap [0,1) = \{r_1, r_2, \ldots\}$ be an enumeration of the rationals in $[0,1)$. Let $E_n := r_n \oplus E$. Is E measurable?

If E is measurable, then E_n is measurable and $m(E) = m(E_n)$ for each n. But $[0,1) = \bigcup_{n=1}^{\infty} E_n$, and the E_n's are pariwise disjoint, so

$$1 = m([0,1)) = \left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} m(E_n) = \sum_{n=1}^{\infty} m(E)$$

which is impossible. The set E in this example is not Borel, since the Borel sets are measurable. Unfortunately, E is not a particularily nice non-Borel set, as its existence depends on the Axiom of Choice.

We have seen that using the Aziom of Choice we can construct a non-measurable set. (As an exercise, show that, assuming the Axiom of Choice, if $m(E) > 0$, then E contains a non-measurable set.) In contrast, without the Axiom of Choice we can find a model for the real line in which every subset is in fact Lebesgue measurable.

Problem **2.4.11.** *Given a Borel set Λ in $\mathbb{R}^2$, is it necessarily true that the projection of Λ onto the real line is Borel? (The answer is no, but this is not obvious why this would be so).*

Example **2.4.12.** We have already noted that since singletons are measurable with measure 0, it follows that every countable subset of $\mathbb{R}$ is measurable with measure 0.

The Cantor set C is compact, nowhere dense, and has cardinality c. It turns out that it also has measure zero. Since m is complete, it follows from this that the cardinality of $\boldsymbol{M}(\mathbb{R})$ is 2^c. As such most Lebesgue measurable sets are non-Borel.

2.5 Extending Measures

The Carathéodory Method allowed us to construct a measure from an outer measure. In this section we will see how to extend measures from an algebra to a full σ-algebra.

Definition **2.5.1.** Let $\mathcal{A} \subseteq \mathcal{P}(x)$ be an algebra. A measure on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to \mathbb{R}^*$ such that

1) $\mu(\emptyset) = 0$
2) $\mu(E) \geq 0$ for all $E \in \mathcal{A}$
3) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ is a sequence of pairwise disjoint sets with $\bigcup\limits_{n=1}^{\infty} E_n \in \mathcal{A}$, then $\mu(\bigcup\limits_{n=1}^{\infty} E_n) = \sum\limits_{n=1}^{\infty} \mu(E_n)$.

Proposition **2.5.2.** *Given such a measure μ on an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ define $\mu^* : \mathcal{P}(X) \to \mathbb{R}^*$ by*

$$\mu^*(A) = \inf \Big\{ \sum_{n=1}^{\infty} \mu(E_n) | \{E_n\} \subseteq \mathcal{A}, A \subseteq \bigcup_{n=1}^{\infty} E_n \Big\}.$$

Then

1) $\mu^*(\emptyset) = 0$.

2) $\mu^*(B) \geq 0$ *for each* $B \in \mathcal{P}(X)$.

3) If $A \subseteq B$, *then* $\mu^*(A) \leq \mu^*(B)$.

4) If $B \in \mathcal{A}$, *then* $\mu^*(B) = \mu(B)$.

5) If $\{B_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$, *then* $\mu^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum\limits_{n=1}^{\infty} \mu^*(B_n)$.

Proof. The proof of 1), 2), 3) are immediate conseqeunces of the definition.

To prove 4) we first note that $\mu^*(B) \leq \mu(B)$ for all $B \in \mathcal{A}$.

Let $\{F_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be such that $B \subseteq \bigcup\limits_{n=1}^{\infty} F_n$.

Let $E_1 = F_1,\ E_2 = F_2 \setminus F_1,\ E_3 = F_3 \setminus (F_1 \cup F_2)$ and then proceed recursively to define $E_{n+1} = F_{n+1} \setminus (F_1 \cup F_2 \cup \cdots \cup F_n)$. Then $\{E_n\} \subseteq \mathcal{A}$, $\{E_n\}$ is pairwise disjoint and

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n.$$

Observe also that

$$B = \bigcup_{n=1}^{\infty} (B \cap E_n)$$

It follows that

$$\begin{aligned} \mu(B) &= \mu(\bigcup_{n=1}^{\infty} (B \cap E_n)) \\ &= \sum_{n=1}^{\infty} \mu(B \cap E_n) \\ &\leq \sum_{n=1}^{\infty} \mu(E_n) \\ &\leq \sum_{n=1}^{\infty} \mu(F_n). \end{aligned}$$

Hence $\mu(B) \leq \mu^*(B)$.

To prove 5), let $\epsilon > 0$ and choose sequences $\{E_{n,k}\} \subset \mathcal{A}$ such that

$$B_n \subseteq \bigcup_{k=1}^{\infty} E_{n,k} \quad \text{and} \quad \sum_{k=1}^{\infty} \mu(E_{n,k}) \leq \mu^*(B_n) + \frac{\epsilon}{2^n}.$$

Then $\{E_{n,k}\}$ is a countable collection of sets in $\mathcal{A}$ whose union contains $\bigcup\limits_{n=1}^{\infty} B_n$. Hence

$$\begin{aligned} \mu^*(\bigcup_{n=1}^{\infty} B_n) &\leq \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{\infty} \mu(E_{n,k}) \Big) \\ &\leq \sum_{n=1}^{\infty} (\mu^*(B_n) + \frac{\epsilon}{2^n}) \\ &= (\sum_{n=1}^{\infty} \mu^*(B_n)) + \epsilon \end{aligned}$$

∎

Remark **2.5.3.** It is clear from the previous proposition that μ^* is an outer measure on $\mathcal{P}(X)$. We call μ^* the outer measure generated by μ. The next theorem shows us that μ can be extended to a measure on the full σ-algebra generated by the algebra $\mathcal{A}$.

Theorem **2.5.4.** *[Carathéodory Extension Theorem]*

Let μ be a measure on an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$. Let μ^* be the outer measure generated by μ. Let $\mathcal{A}^*$ be the σ-algebra of μ^* measurable sets. Then $\mathcal{A} \subseteq \mathcal{A}^*$ and μ extends to a measure $\bar{\mu}$ on $\mathcal{A}^*$.

Proof. We need only show that $\mathcal{A} \subseteq \mathcal{A}^*$. Let $E \in \mathcal{A}$ and let $A \subseteq X$. As before, we can assume that $\mu^*(A) < \infty$ and that we need only show that

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Let $\epsilon > 0$ and choose$\{F_n\} \subset \mathcal{A}$ such that

$$A \subseteq \bigcup_{n=1}^{\infty} F_n \quad \text{and} \quad \sum_{n=1}^{\infty} \mu(F_n) \leq \mu^*(A) + \epsilon.$$

Note that

$$A \cap E \subseteq \bigcup_{n=1}^{\infty} E \cap F_n \quad \text{and} \quad A \cap E^c \subseteq \bigcup_{n=1}^{\infty} E^c \cap F_n.$$

It follows that

$$\mu^*(A \cap E) \leq \sum_{n=1}^{\infty} \mu(E \cap F_n) \quad \text{and} \quad \mu^*(A \cap E^c) \leq \sum_{n=1}^{\infty} \mu(E^c \cap F_n).$$

Hence

$$\begin{aligned} \mu^*(A \cap E) + \mu^*(A \cap E^c) &\leq \sum_{n=1}^{\infty} \mu(E \cap F_n) + \sum_{n=1}^{\infty} \mu(E^c \cap F_n) \\ &= \sum_{n=1}^{\infty} \mu(F_n) \\ &\leq \mu^*(A) + \epsilon. \end{aligned}$$

Since ϵ was arbitray, we have $\mu^*(A \cap E) + \mu^*(A \cap E) \leq \mu^*(A)$.

The remainder of the Proposition follows from Carathéodory's Theorem. ∎

Remark **2.5.5.** The measure $\bar{\mu}$ constructed in the previous theorem is called the Carathéodory extension of the measure μ.

We can ask: given a measure μ on an algebra $\mathcal{A}$, is the extension of μ to $\mathcal{A}^*$ unique?

Example **2.5.6.** Let $X = (0, 1] \cap \mathbb{Q}$. Let $\mathcal{A}$ be the algebra of all finite union of sets of the form $(a, b] \cap \mathbb{Q}$ where $a, b \in X$. It is easy to see that the smallest σ-algebra containing $\mathcal{A}$ is all of $\mathcal{P}(X)$. Let μ be the counting measure on $\mathcal{A}$. Then

$$\mu(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ \infty & \text{otherwise.} \end{cases}$$

This implies that for each $A \in \mathcal{P}(X)$,

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \infty & \text{otherwise.} \end{cases}$$

From the Carathéodory Extension Theorem we can conclude that $\mathcal{A}^* = \mathcal{P}(X)$ and that $\bar{\mu} = \mu^*$. On the other hand, the counting measure on $\mathcal{P}(X) = \mathcal{A}^*$ also extends μ but it is not equal to $\bar{\mu}$.

Observe that in this example the original measure μ fails to be σ-finite.

Theorem **2.5.7.** *[Hahn Extension Theorem]*

Suppose that μ is a σ-finite measure on an algebra $\mathcal{A}$. Then there is a unique extension $\bar{\mu}$ to a measure on $\mathcal{A}^*$, the σ-algebra of all μ^*-measurable sets.

Let γ be a measure on $\mathcal{A}^*$ that agrees with μ on $\mathcal{A}$. Let $\bar{\mu}$ be the Carathéodory extension of μ.

Case 1: Assume that $\mu(X) < \infty$, and hence that both γ and $\bar{\mu}$ are also finite.

Let $E \in \mathcal{A}^*$ and let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ with $E \subseteq \bigcup\limits_{n=1}^{\infty} E_n$. Then

$$\gamma(E) \leq \gamma(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \gamma(E_n) = \sum_{n=1}^{\infty} \mu(E_n).$$

From this it follows that

$$\gamma(E) \leq \mu^*(E) = \bar{\mu}(E).$$

A similar argument shows that

$$\gamma(E^c) \leq \mu^*(E^c) = \bar{\mu}(E^c).$$

However

$$\gamma(E) + \gamma(E^c) = \gamma(X) = \mu(X) = \bar{\mu}(X) = \bar{\mu}(E) + \bar{\mu}(E^c).$$

and since $\gamma(E) \leq \mu^*(E)$ and $\gamma(E^c) \leq \mu^*(E^c)$, we must have $\gamma(E) = \mu^*(E)$.

Case 2: Assume that μ is a σ-finite. Let $\{F_n\} \subset \mathcal{A}$ be an increasing sequence with $\mu(F_n) < \infty$ and $X = \bigcup\limits_{n=1}^{\infty} F_n$. Then $\bar{\mu}(E \cap F_n) = \gamma(E \cap F_n)$ for each $n \in \mathbb{N}$ and all $E \in \mathcal{A}^*$. It follows that

$$\bar{\mu}(E) = \lim_{n\to\infty} \bar{\mu}(E \cap F_n) = \lim_{n\to\infty} \gamma(E \cap F_n) = \gamma(E)$$

and we are done.

2.6 Lebesgue-Stieltjes Measures

In this section we will use the Carathéodory Extension Theorem to identify a fundamental class of measures which generalize the Lebesgue measure on $\mathbb{R}$.

Let $\mathcal{A}$ be the collection of all finite unions of sets of the form $(-\infty, b], (a, \infty), (a, b]$. Then it is easy to show that $\mathcal{A}$ is an algebra. Let $F(x)$ be a non-decreasing function on $\mathbb{R}$ with

$$F(c) = \lim_{x\to c^+} F(x)$$

for each $c \in \mathbb{R}$. Note also that $\lim\limits_{x\to-\infty} F(x)$ and $\lim\limits_{x\to\infty} F(x)$ both exist as extended real numbers. Define

1) $\mu_F((a, b]) = F(b) - F(a)$
2) $\mu_F((a, \infty)) = \lim\limits_{x\to\infty} F(x) - F(a)$
3) $\mu_F((-\infty, b]) = F(b) - \lim\limits_{x\to-\infty} F(x)$
4) $\mu_F((-\infty, \infty)) = \lim\limits_{x\to\infty} F(x) - \lim\limits_{x\to-\infty} F(x)$
5) $\mu_F(\emptyset) = 0$

We can then extend μ_F to all of $\mathcal{A}$ in the obvious way.

THEOREM **2.6.1.** *Let $F(x)$ be a non-decreasing function on $\mathbb{R}$ with*

$$F(c) = \lim_{x \to c^+} F(x)$$

for each $c \in \mathbb{R}$. Let μ_F be defined as above. Then μ_F is a measure on the algebra $\mathcal{A}$ of all finite unions of sets of the form $(-\infty, b], (a, \infty), (a, b]$.

Proof. We will prove that if $\{(a_n, b_n]\}$ is a pairwise disjoint sequence of intervals with

$$(a, b] = \bigcup_{n=1}^{\infty} (a_n, b_n],$$

then

$$\mu_F((a, b]) = \sum_{n=1}^{\infty} \mu_F((a_n, b_n]).$$

Let $\{(a_{n_1}, b_{n_1}], (a_{n_2}, b_{n_2}], \ldots, (a_{n_k}, b_{n_k}]\}$ be any finite collection of theses intervals. We may assume that

$$a \leq a_{n_1} < b_{n_1} \leq a_{n_2} < b_{n_2} \leq \cdots \leq a_{n_k} < b_{n_k} \leq b$$

Then

$$\begin{aligned}
\sum_{i=1}^{k} \mu_F((a_{n_i}, b_{n_i}]) &= \sum_{i=1}^{k} F(b_{n_i}) - F(a_{n_i}) \\
&= F(b_{n_1}) - F(a_{n_1}) + F(b_{n_2}) - F(a_{n_2}) + \cdots + F(b_{n_k}) - F(a_{n_k}) \\
&= F(b_{n_k}) - (F(a_{n_k}) - F(b_{n_{k-1}})) - \cdots (F(a_{n_2} - F(b_{n_1})) - F(a_{n_1}) \\
&\leq F(b_{n_k}) - F(a_{n_1}) \\
&\leq F(b) - F(a) \\
&= \mu_F((a, b]
\end{aligned}$$

Since the number of subintervals in the collection was arbitrary, this shows that

$$\mu_F((a, b]) \geq \sum_{n=1}^{\infty} \mu_F((a_n, b_n]).$$

From here on we may assume that $a_1 = a$.
Now let $\epsilon > 0$. Then choose a sequence $\{\epsilon_n\}$ of positive numbers so that

$$\sum_{n=1}^{\infty} \epsilon_n < \frac{\epsilon}{2}.$$

Next since F is continuous from the right, for each $n = 1, 2, 3, \ldots$ we can pick a $\delta_n > 0$ such that

$$F(b_n + \delta_n) - F(b_n) < \epsilon_n.$$

And finally, let δ_0 be chosen so that $0 < \delta_0 < b - a$ and

$$F(a + \delta_0) - F(a) < \frac{\epsilon}{2}$$

Since the interval $[a + \delta_0, b]$ is compact and since $\{(a_n, b_n + \delta_n)\}$ is a cover we can find finitely many of these intervals

$$\{(a_{n_1}, b_{n_1} + \delta_{n_1}), (a_{n_2}, b_{n_2} + \delta_{n_2}), \ldots, (a_{n_k}, b_{n_k} + \delta_{n_k})\}$$

which also cover $[a+\delta_0, b]$. By reordering if necessary and omitting unnecessary intervals we can assmue that

$$a_{n_1} < a + \delta_0 < b_{n_1} + \delta_{n_1}$$

and

$$a_{n_1} < a_{n_2} < b_{n_1} + \delta_{n_1} < a_{n_3} < b_{n_2} + \delta_{n_2} < a_{n_4} < \cdots a_{n_k} < b_{n_{k-1}} + \delta_{n_{k-1}} \leq b < b_{n_k} + \delta_{n_k}$$

Then

$$\begin{aligned} F(b) - F(a+\delta_0) &\leq F(b_{n_k} + \delta_k) - F(a_{n_1}) \\ &\leq \sum_{i=1}^{k} F(b_{n_i} + \delta_i) - F(a_{n_i}) \\ &\leq \sum_{i=1}^{k} (F(b_{n_i}) - F(a_{n_i}) + \epsilon_i) \\ &< (\sum_{i=1}^{k} F(b_{n_i}) - F(a_{n_i})) + \frac{\epsilon}{2} \end{aligned}$$

From this it follows that

$$F(b) - F(a) < (\sum_{i=1}^{k} F(b_{n_i}) - F(a_{n_i})) + \epsilon$$

and since ϵ was arbitrary we get

$$\mu_F((a,b]) = F(b) - F(a) \leq \sum_{n=1}^{\infty} F(b_n) - F(a_n) = \sum_{n=1}^{\infty} \mu_F((a_n, b_n]).$$

Finally this shows that

$$\mu_F((a,b]) = \sum_{n=1}^{\infty} \mu_F((a_n, b_n]).$$

The remaining cases are similar. ∎

Definition **2.6.2.** Let $F(x)$ be a non-decreasing function on $\mathbb{R}$ with

$$F(c) = \lim_{x \to c^+} F(x)$$

for each $c \in \mathbb{R}$. The measure μ_F obtained in the previous proposition is called the Lebesgue-Stieltjes measure associated with F.

Theorem **2.6.3.** *Let $F(x)$ be a non-decreasing function on $\mathbb{R}$. Then μ_F is regular.*

Proof. Let $E \subseteq \mathbb{R}$ be μ_F-measurable.

Step 1: First assume that $E \subseteq [a,b]$. Let $\epsilon > 0$. Then there exists a sequence $\{(a_n, b_n]\}$ such that $(a_n, b_n] \subseteq [a-1, b+1]$,

$$E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n]$$

and

$$\sum_{n=1}^{\infty} \mu_F((a_n, b_n]) < \mu_F(E) + \frac{\epsilon}{2}.$$

For each $n \in \mathbb{N}$ choose $0 < \delta_n < 1$ so that

$$\mu_F((a_n, b_n + \delta_n)) < \mu_F((a_n, b_n]) + \frac{\epsilon}{2^n}.$$

Then

$$E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n]$$

and

$$\sum_{n=1}^{\infty} \mu_F((a_n, b_n + \delta_n)) < \sum_{n=1}^{\infty} [\mu_F((a_n, b_n]) + \frac{\epsilon}{2^n}] < \mu_F(E) + \epsilon.$$

Hence we have shown that there is an open set $U = \bigcup\limits_{n=1}^{\infty} (a_n, b_n + \delta_n)$ with $E \subseteq U$ and $\mu_F(U) - \mu_F(E) < \epsilon$. Now let $E' = [a, b] \setminus E$. Then there exists an open set U' such that $E' \subseteq U'$ and $\mu_F(U') - \mu_F(E') < \epsilon$. Let $K = [a, b] \setminus U'$. Then K is compact, $K \subseteq E$ and

$$\mu_F(E \setminus K) < \epsilon.$$

Step 2: Next assume that $\mu_F(E) < \infty$. Let $\{E_j\}_{j\in\mathbb{Z}} = \{(j, j+1]\}_{n\in\mathbb{Z}}$. Then

$$E = \bigcup_{j\in\mathbb{Z}} E_j.$$

For each $j \in \mathbb{Z}$, let $\epsilon_j > 0$ be chosen so that $\sum\limits_{j\in\mathbb{Z}} \epsilon_j < \frac{\epsilon}{2}$. Then we can find compact sets $\{K_j\}_{j\in\mathbb{Z}}$ and open sets $\{U_j\}_{j\in\mathbb{Z}}$ such that $K_j \subseteq E_j \subseteq U_j$ with

$$\mu_F(U_j) - \mu_F(E_j) < \epsilon_j$$

and

$$\mu_F(E_j) - \mu_F(K_j) < \epsilon_j$$

Then if $U = \bigcup\limits_{j\in\mathbb{Z}} U_j$, then $E \subseteq U$ and $\mu_F(U \setminus E) < \frac{\epsilon}{2} < \epsilon$.

$$\mu_F(E) = \inf\{\mu_F(U) | E \subseteq U, U \text{ is open}\}.$$

We can find an $M \in \mathbb{N}$ so that $\mu_F(E \setminus [-M, M]) < \frac{\epsilon}{2}$. Let

$$K = \bigcup_{j=-M-1}^{M} K_j.$$

Then K is compact, $K \subseteq E$ and $\mu_F(E \setminus K) < \epsilon$. Hence

$$\mu_F(E) = \sup\{\mu_F(K) | K \subseteq E, K \text{ is compact}\}.$$

Step 3: If $\mu_F(E) = \infty$, then clearly $\mu_F(U) = \infty$ for any open set U with $E \subset U$. Hence

$$\mu_F(E) = \inf\{\mu_F(U) | E \subseteq U, U \text{ is open}\}.$$

For any $M > 0$ there is an $n \in \mathbb{N}$ such that

$$\mu_F(E \cap [-n, n]) > M + 1.$$

But then there exists a compact set K such that $K \subseteq E \cap [-n, n]$ with $\mu_F(K) > M$. It follows that

$$\mu_F(E) = \sup\{\mu_F(K) | K \subseteq E, K \text{ is compact}\}.$$

∎

At this point we have seen that every right continuous non-decreasing function induces a unique regular measure on $\mathcal{B}(\mathbb{R})$. We shall see that if we add the additional requirement that the measure be finite, then all such measures arise this way.

THEOREM **2.6.4.** *Let μ be a finite regular measure on $\mathcal{B}(\mathbb{R})$. Let*

$$F(x) = \mu((-\infty, x]).$$

Then $F(x)$ is nondecreasing, right continuous and

$$\mu_F(E) = \mu(E)$$

for every $E \in \mathcal{B}(\mathbb{R})$.

Proof. It is clear that $F(x)$ is nondecreasing. Since μ is finite and since

$$(-\infty, x] = \bigcap_{n=1}^{\infty} (-\infty, x + \frac{1}{n}]$$

we have that

$$F(x) = \mu((-\infty, x]) = \lim_{n\to\infty} (-\infty, x + \frac{1}{n}] = \lim_{n\to\infty} F(x + \frac{1}{n})$$

Since $F(x)$ is nondecreasing, this shows that

$$F(x) = \lim_{t\to x^+} F(t).$$

That is $F(x)$ is right continuous.

Finally, we have that for each interval I of the type $(-\infty, b], (a, \infty), (a, b]$, it is immediate that

$$\mu(I) = \mu_F(I).$$

It follows from the Hahn Extension Theorem that $\mu = \mu_F$ on $\mathcal{B}(\mathbb{R})$. ∎