Chapter 1

Review of the Riemann Integral

This chapter provides a quick review of the basic properties of the Rieman integral.

1.0 Integrals and Riemann Sums

DEFINITION **1.0.1.** Let $[a, b]$ be a finite, closed interval. A *partition* $\mathcal{P}$ of $[a, b]$ is a finite subset of $[a, b]$ of the form

$$\mathcal{P} = \{x_i :\ a = x_0 < x_1 < \cdots < x_n = b\}.$$

DEFINITION **1.0.2.** For a partition $\mathcal{P}$ of $[a, b]$ having n elements,

1. define $\Delta x_i := x_i - x_{i-1}$, and
2. define the *norm* of $\mathcal{P}$, denoted as $\|\mathcal{P}\|$, by

$$\|\mathcal{P}\| := \max\{\Delta x_i \ :\ \ i = 1, 2, \ldots, n\}.$$

EXAMPLE **1.0.3.** For any $n \in N$ we can construct the *n-regular partition* $\mathcal{P}_n$ of $[a, b]$ by subdividing $[a, b]$ into n identical parts with the length of each subinterval being $\frac{b-a}{n}$. *I.e.*, $\Delta x_i = \frac{b-a}{n}$ for each i, and $\|\mathcal{P}\| = \frac{b-a}{n}$. Note that

$$\begin{aligned} x_1 &= a + \frac{b-a}{n}, \\ x_2 &= a + 2 \cdot \frac{b-a}{n}, \\ &\vdots \\ x_i &= a + i \cdot \frac{b-a}{n}, \\ &\vdots \\ x_n &= a + n \cdot \frac{b-a}{n} = b. \end{aligned}$$

□

Assume that $f(x)$ is bounded on $[a, b]$. Let $\mathcal{P}$ be a partition on $[a, b]$. Let

$$M_i = \sup\{f(x) :\ x \in [x_{i-1}, x_i]\},$$

and let

$$m_i = \inf\{f(x) :\ x \in [x_{i-1}, x_i]\}.$$

Then $m_i \leq f(x) \leq M_i$ for all $x \in [x_{i-1}, x_i]$.

Definition **1.0.4.** Given $f(x)$, bounded on $[a, b]$, and a partition $\mathcal{P}$ of $[a, b]$, we define the *upper Riemann sum of $f(x)$ with respect to $\mathcal{P}$* by

$$\mathrm{U}(f, \mathcal{P}) := \mathrm{U}_a^b(f, \mathcal{P}) = \sum_{i=1}^{n} M_i \cdot \Delta x_i,$$

and the *lower Riemann sum of $f(x)$ with respect to $\mathcal{P}$* by

$$\mathrm{L}(f, \mathcal{P}) := \mathrm{L}_a^b(f, \mathcal{P}) = \sum_{i=1}^{n} m_i \cdot \Delta x_i,$$

where M_i and m_i are defined as above. Finally, for each i, choose $c_i \in [x_{i-1}, x_i]$; a *Riemann sum for $f(x)$ on $[a, b]$ with respect to $\mathcal{P}$* is defined by

$$\mathrm{S}_a^b(f, \mathcal{P}) = \sum_{i=1}^{n} f(c_i) \Delta x_i.$$

The bounds of summation a and b from definition 1.0.4 are usually omitted.

Remark **1.0.5.** Since $m_i \leq f(c_i) \leq M_i$ and $\Delta x_i > 0$, we can conclude that

$$\mathrm{L}_a^b(f, \mathcal{P}) \leq \mathrm{S}_a^b(f, \mathcal{P}) \leq \mathrm{U}_a^b(f, \mathcal{P}).$$

□

Definition **1.0.6.** Given a partition $\mathcal{P}$, we say that a partition $\mathcal{Q}$ is a *refinement of* $\mathcal{P}$ if and only if $\mathcal{P} \subseteq \mathcal{Q}$. We also say that $\mathcal{Q}$ *refines* $\mathcal{P}$ or $\mathcal{P}$ *is refined by* $\mathcal{Q}$.

Theorem **1.0.7.** *Let $f(x)$ be bounded on $[a, b]$. Let $\mathcal{P}$, $\mathcal{Q}$ be partitions on $[a, b]$ with $\mathcal{Q}$ a refinement of $\mathcal{P}$. Then*

$$\mathrm{L}(f, \mathcal{P}) \leq \mathrm{L}(f, \mathcal{Q}) \leq \mathrm{U}(f, \mathcal{Q}) \leq \mathrm{U}(f, \mathcal{P}).$$

Proof. Assume that $\mathcal{Q} = \mathcal{P} \cup \{y_0\}$ for some $y_0 \in [a, b] \setminus \mathcal{P}$. Hence $\mathcal{Q} = \{x_i, y_0 : a = x_0 < x_1 < \cdots < x_{i-1} < y_0 < x_i < \cdots < x_n = b\}$ and $\mathcal{P} = \{x_i : a = x_0 < x_1 < \cdots < x_n = b\}$. Let

$$M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\},$$
$$m_i = \inf\{f(x) : x \in [x_{j-1}, x_j]\}.$$

Also let

$$M_{i,1} = \sup\{f(x) : x \in [x_{i-1}, y_0]\},$$
$$m_{i,1} = \inf\{f(x) : x \in [x_{i-1}, y_0]\},$$
$$M_{i,2} = \sup\{f(x) : x \in [y_0, x_i]\},$$
$$m_{i,2} = \inf\{f(x) : x \in [y_0, x_i]\}.$$

Note that since $f([x_{i-1}, x_i]) \supseteq f([x_{i-1}, y_0])$ and $f([x_{i-1}, x_i]) \supseteq f([y_0, x_i])$, we have that $M_{i,1} \leq M_i$ and

$M_{i,2} \leq M_i$. Similarly, $m_{i,1} \geq m_i$ and $m_{i,2} \geq m_i$. Now

$$\begin{aligned}
\mathrm{U}(f, \mathcal{P}) &= \sum_{j=1}^{n} M_j \cdot \Delta x_j \\
&= \sum_{\substack{j=1 \\ j \neq i}}^{n} + M_j \cdot \Delta x_j + M_i \cdot \Delta x_i \\
&= \sum_{\substack{j=1 \\ j \neq i}}^{n} + M_j \cdot \Delta x_j + M_i(y_0 - x_{i-1}) + M_i(x_i - y_0) \\
&\geq \sum_{\substack{j=1 \\ j \neq i}}^{n} + M_j \cdot \Delta x_j + M_{i,1}(y_0 - x_{i-1}) + M_{i,2}(x_i - y_0) \\
&= \mathrm{U}(f, \mathcal{Q}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{L}(f, \mathcal{P}) &= \sum_{j=1}^{n} m_j \cdot \Delta x_j \\
&= \sum_{\substack{j=1 \\ j \neq i}}^{n} + m_j \cdot \Delta x_j + m_i \cdot \Delta x_i \\
&= \sum_{\substack{j=1 \\ j \neq i}}^{n} + m_j \cdot \Delta x_j + m_i(y_0 - x_{i-1}) + m_i(x_i - y_0) \\
&\leq \sum_{\substack{j=1 \\ j \neq i}}^{n} + m_j \cdot \Delta x_j + m_{i,1}(y_0 - x_{i-1}) + m_{i,2}(x_i - y_0) \\
&= \mathrm{L}(f, \mathcal{Q}).
\end{aligned}$$

Due to remark 1.0.5, we conclude that

$$\mathrm{L}(f, \mathcal{P}) \leq \mathrm{L}(f, \mathcal{Q}) \leq \mathrm{U}(f, \mathcal{Q}) \leq \mathrm{U}(f, \mathcal{P}).$$

We can use induction on the number of points in $\mathcal{Q} \setminus \mathcal{P}$ to establish the theorem. ∎

Corollary **1.0.8.** *If $\mathcal{P}$ and $\mathcal{Q}$ are partitions of $f(x)$ on $[a, b]$, then $\mathrm{L}(f, \mathcal{P}) \leq \mathrm{U}(f, \mathcal{Q})$.*

Proof. Let $\mathcal{T} = \mathcal{P} \cup \mathcal{Q}$. Then $\mathcal{T}$ refines both $\mathcal{P}$ and $\mathcal{Q}$. Hence $\mathrm{L}(f, \mathcal{P}) \leq \mathrm{L}(f, \mathcal{T}) \leq \mathrm{U}(f, \mathcal{T}) \leq \mathrm{U}(f, \mathcal{Q})$, as desired. ∎

Definition **1.0.9.** Let $f(x)$ be bounded on $[a, b]$. The *upper Riemann integral for $f(x)$ over $[a, b]$* is

$$\overline{\int_a^b} f(x)\, dx = \inf\{\mathrm{U}(f, \mathcal{P}) : \mathcal{P} \text{ a partition of } [a, b]\},$$

and the *lower Riemann integral for $f(x)$ over $[a, b]$* is

$$\underline{\int_a^b} f(x)\, dx = \sup\{\mathrm{L}(f, \mathcal{P}) : \mathcal{P} \text{ a partition of } [a, b]\},$$

Observation.
$$\overline{\int_a^b} f(x)\,dx \geq \underline{\int_a^b} f(x)\,dx.$$

□

1.1 Riemann Integrable Functions

Definition **1.1.1.** We say that $f(x)$ is *Riemann integrable on* $[a,b]$ if
$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx,$$
in which case we denote this common value by
$$\int_a^b f(x)\,dx.$$

In definitions 1.0.9 and 1.1.1, we encountered new notations: the $\int$ is the *integral sign*, a and b are *endpoints of integration*, $f(x)$ is the *integrand*, and dx refers to the *variable of integration* (also called the *dummy variable*)—in this case it is x.

Example **1.1.2.** Let
$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ -1 & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$
If $\mathcal{P} = \{x_i : \ 0 = x_0 < x_1 < \cdots < x_n = 1\}$, then let
$$M_i = \sup\{f(x) : \ x \in [x_{i-1}, x_i]\},$$
$$m_i = \inf\{f(x) : \ x \in [x_{i-1}, x_i]\}.$$
We have that
$$\mathrm{U}(f,\mathcal{P}) = \sum_{i=1}^n M_i \cdot \Delta x_i = \sum_{i=1}^n \Delta x_i = 1, \text{ and}$$
$$\mathrm{L}(f,\mathcal{P}) = \sum_{i=1}^n m_i \cdot \Delta x_i = \sum_{i=1}^n -\Delta x_i = -1.$$
Hence
$$\overline{\int_0^1} f(x)\,dx = 1 \neq -1 = \underline{\int_0^1} f(x)\,dx,$$
and hence this function is not Riemann integrable on $[0,1]$. □

Observation. $f(x)$ in the above example is discontinuous everywhere on $[a,b]$.

□

Theorem **1.1.4.** *Let $f(x)$ be bounded on $[a,b]$. Then $f(x)$ is Riemann integrable on $[a,b]$ if and only if for every $\epsilon > 0$ there exists a partition $\mathcal{P}$ of $[a,b]$ such that* $\mathrm{U}(f,\mathcal{P}) - \mathrm{L}(f,\mathcal{P}) < \epsilon$.

Proof. Assume that $f(x)$ is Riemann integrable on $[a,b]$. *I.e.*,
$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

Let $\epsilon > 0$. Since $\overline{\int_a^b} f(x)\,dx = \inf\{\mathrm{U}(f,\mathcal{P}) :\ \mathcal{P} \text{ is a partition}\}$, there exists a partition $\mathcal{P}_1$ such that

$$\int_a^b f(x)\,dx \le \mathrm{U}(f,\mathcal{P}_1) < \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{2}.$$

Similarly, since $\underline{\int_a^b} f(x)\,dx = \sup\{\mathrm{L}(f,\mathcal{P}) :\ \mathcal{P} \text{ is a partition}\}$, there exists a partition $\mathcal{P}_2$ such that

$$\underline{\int_a^b} f(x)\,dx - \frac{\epsilon}{2} < \mathrm{L}(f,\mathcal{P}_2) \le \int_a^b f(x)\,dx.$$

Let $\mathcal{Q} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then

$$\begin{aligned}
\int_a^b f(x)\,dx - \frac{\epsilon}{2} &= \underline{\int_a^b} f(x)\,dx - \frac{\epsilon}{2} \\
&< \mathrm{L}(f,\mathcal{P}_2) \\
&\le \mathrm{L}(f,\mathcal{Q}) \quad \text{(by theorem 1.0.7)} \\
&\le \mathrm{U}(f,\mathcal{Q}) \\
&\le \mathrm{U}(f,\mathcal{P}_1) \quad \text{(by theorem 1.0.7)} \\
&< \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{2} = \int_a^b f(x)\,dx + \frac{\epsilon}{2}.
\end{aligned}$$

This implies that

$$\mathrm{U}(f,\mathcal{Q}) - \mathrm{L}(f,\mathcal{Q}) < \epsilon.$$

To prove the converse, assume that for each $\epsilon > 0$ there exists a partition $\mathcal{P}$ of $[a,b]$ such that

$$\mathrm{U}(f,\mathcal{P}) - \mathrm{L}(f,\mathcal{P}) < \epsilon.$$

However, by definition 1.0.9, we have that

$$\mathrm{L}(f,\mathcal{P}) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le \mathrm{U}(f,\mathcal{P}),$$

and hence

$$0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \epsilon.$$

Since ϵ is arbitrary, we obtain

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx;$$

therefore $f(x)$ is Riemann integrable on $[a,b]$, as required. ∎

Example **1.1.5.** Let $f(x) = x^2$. Let $\mathcal{P}_n$ be the n-regular partition of $[0,1]$. Then

$$\mathrm{U}_0^1(f,\mathcal{P}_n) = \sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} \frac{i^2}{n^2} \cdot \frac{1}{n}$$

and

$$\mathrm{L}_0^1(f,\mathcal{P}_n) = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i = \sum_{i=1}^{n} \frac{(i-1)^2}{n^2} \cdot \frac{1}{n}.$$

Then

$$\begin{aligned} \mathrm{U}(f,\mathcal{P}_n) - \mathrm{L}(f,\mathcal{P}_n) &= \left(\frac{n^2}{n^2}\cdot\frac{1}{n}\right) - \left(\frac{0^2}{n^2}\cdot\frac{1}{n}\right) \\ &= \frac{1}{n}. \end{aligned}$$

This shows that for all $\epsilon > 0$, we can find a partition $\mathcal{P}$ such that $\mathrm{U}(f,\mathcal{P}) - \mathrm{L}(f,\mathcal{P}) < \epsilon$ (simply select a large enough n by the Archimedean Principle and use the n-regular partition $\mathcal{P}_n$). Hence by theorem 1.1.4, $f(x)$ is Riemann integrable on $[0,1]$. Later, we will be able to show that

$$\int_0^1 x^2\,dx = \frac{1}{3}.$$

□

Recall the following definition.

Definition **1.1.6.** We say that $f(x)$ is *uniformly continuous on an interval* I if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all x, $y \in I$,

$$|x-y| < \delta \qquad \text{implies} \qquad |f(x)-f(y)| < \epsilon.$$

Also recall the following theorem.

Theorem **1.1.7** [Sequential Characterization of Uniform Continuity]**.** *A function $f(x)$ is uniformly continuous on an interval I if and only if whenever $\{x_n\}$, $\{y_n\}$ are sequences in I,*

$$\lim_{n\to\infty}(x_n - y_n) = 0 \qquad \text{implies} \qquad \lim_{n\to\infty}(f(x_n) - f(y_n)) = 0.$$

Example **1.1.8.** Let $I = \mathbb{R}$ and $f(x) = x^2$. Take $\{x_n\} = \left\{n + \frac{1}{n}\right\}$, $\{y_n\} = \{n\}$. Then $\lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty}\frac{1}{n} = 0$, but

$$\begin{aligned} \lim_{n\to\infty}(f(x_n) - f(y_n)) &= \lim_{n\to\infty}\left[\left(n+\frac{1}{n}\right)^2 - n^2\right] \\ &= \lim_{n\to\infty}\left(2 + \frac{1}{n^2}\right) \\ &= 2 \neq 0. \end{aligned}$$

Hence $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. □

One should already be familiar with the following theorem.

Theorem **1.1.9.** *If $f(x)$ is continuous on $[a,b]$, then $f(x)$ is uniformly continuous on $[a,b]$.*

Proof. Assume that $f(x)$ is continuous on $[a,b]$ but not uniformly continuous on $[a,b]$. Then by theorem 1.1.7, there exists sequences $\{x_n\}$ and $\{y_n\}$ with $\lim_{n\to\infty}(x_n - y_n) = 0$ but $\lim_{n\to\infty}(f(x_n) - f(y_n)) \neq 0$. (Note that this limit may not even exist.) By choosing a subsequence if necessary, we can assume without loss of generality that there exists an $\epsilon_0 > 0$ such that

$$|(f(x_n) - f(y_n)) - 0| = |f(x_n) - f(y_n)| \geq \epsilon_0 \tag{1.1}$$

for all $n \in \mathbb{N}$.

Since $\{x_n\} \subset [a,b]$, by the Bolzano-Weierstrass Theorem $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ with $\lim_{k\to\infty} x_{n_k} = x_0 \in [a,b]$. Note that $\lim_{k\to\infty}(x_{n_k} - y_{n_k}) = 0$, hence $\lim_{k\to\infty} y_{n_k} = x_0$ also. By the sequential characterization of continuity, we have

$$\lim_{k\to\infty} f(x_{n_k}) = f(x_0),$$
$$\lim_{k\to\infty} f(y_{n_k}) = f(x_0).$$

Therefore, we can select a $K \in \mathbb{N}$ with

$$|f(x_{n_k}) - f(x_0)| < \frac{\epsilon_0}{2} \quad \text{and}$$
$$|f(y_{n_k}) - f(x_0)| < \frac{\epsilon_0}{2}$$

for all $k \geq K$. Hence we have, for all $k \geq K$,

$$\begin{aligned} 0 \leq |f(x_{n_k}) - f(y_{n_k})| &\leq |f(x_{n_k}) - f(x_0)| + |f(y_{n_k}) - f(x_0)| \\ &< \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} \\ &= \epsilon_0, \end{aligned}$$

directly contradicting equation (1.1). Hence $f(x)$ is uniformly continuous on $[a,b]$. ∎

Theorem **1.1.10** [Integrability Theorem for Continuous Functions]. *If $f(x)$ is continuous on $[a,b]$, then $f(x)$ is Riemann integrable on $[a,b]$.*

Proof. Let $\epsilon > 0$. Since $f(x)$ is continuous on $[a,b]$, $f(x)$ is also uniformly continous on $[a,b]$ by theorem 1.1.9 above. We can find a $\delta > 0$ such that if $|x-y| < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{b-a}$. Let $\mathcal{P}$ be a partition of $[a,b]$ with $\|\mathcal{P}\| = \max\{\Delta x_i\} < \delta$. For example, we can choose the n-regular partition of $[a,b]$ with n large enough (chosen by the Archimedean Principle) so that $\frac{b-a}{n} = \|\mathcal{P}_n\| < \delta$. After we have chosen $\mathcal{P}$, let

$$M_i = \sup\{f(x) : \ x \in [x_{i-1}, x_i]\},$$
$$m_i = \inf\{f(x) : \ x \in [x_{i-1}, x_i]\}.$$

By the extreme value theorem, there exists $c_i, d_i \in [x_{i-1}, x_i]$ such that $f(c_i) = m_i$ and $f(d_i) = M_i$; note that $M_i - m_i = |M_i - m_i| = |f(d_i) - f(c_i)|$. Since $|c_i - d_i| < \Delta x_i < \delta$, we have that $|f(d_i) - f(c_i)| < \frac{\epsilon}{b-a}$. This shows that

$$\begin{aligned} \mathrm{U}(f,\mathcal{P}) - \mathrm{L}(f,\mathcal{P}) &= \sum_{i=1}^{n} M_i \cdot \Delta x_i - \sum_{i=1}^{n} m_i \cdot \Delta x_i \\ &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\ &< \sum_{i=1}^{n} \frac{\epsilon}{b-a} \cdot \Delta x_i \\ &= \frac{\epsilon}{b-a} \sum_{i=1}^{n} \Delta x_i \\ &= \frac{\epsilon}{b-a} \cdot (b-a) \\ &= \epsilon. \end{aligned}$$

Hence $f(x)$ is Riemann integrable by theorem 1.1.4. ∎

REMARK **1.1.11.** Assume that $f(x)$ is continuous on $[a, b]$. For each n, let $\mathcal{P}_n$ be the n-regular partition. Then if $\mathrm{S}(f, \mathcal{P}_n)$ is any Riemann sum associated with $\mathcal{P}_n$, we have

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \mathrm{S}(f, \mathcal{P}_n).$$

Proof. Given $\epsilon > 0$, we can find an N large enough so that if $n \geq N$, then $\frac{b-a}{n} < \delta$ as defined in theorem 1.1.10 above. Hence

$$\mathrm{U}(f, \mathcal{P}_n) - \mathrm{L}(f, \mathcal{P}_n) < \epsilon,$$

via a similar argument made in theorem 1.1.10. But $\mathrm{U}(f, \mathcal{P}_n) \geq \mathrm{S}(f, \mathcal{P}_n) \geq \mathrm{L}(f, \mathcal{P}_n)$ and $\mathrm{U}(f, \mathcal{P}_n) \geq \int_a^b f(x)\,dx \geq \mathrm{L}(f, \mathcal{P}_n)$. This implies that

$$\left| \int_a^b f(x)\,dx - \mathrm{S}(f, \mathcal{P}_n) \right| < \epsilon$$

and so $\lim_{n\to\infty} \mathrm{S}(f, \mathcal{P}_n) = \int_a^b f(x)\,dx$, as remarked. ∎

Note that the above remark shows that if $f(x)$ is continuous, we have

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[f\left(a + i \cdot \frac{b-a}{n} \right) \frac{b-a}{n} \right].$$

DEFINITION **1.1.12.** Given a partition $\mathcal{P} = \{x_i :\ a = x_0 < x_1 < \cdots < x_n = b\}$, define the *right-hand Riemann sum*, $\mathrm{S_R}$, by

$$\mathrm{S_R} := \sum_{i=1}^{n} f(x_i) \cdot \Delta x_i;$$

define the *left-hand Riemann sum*, $\mathrm{S_L}$, by

$$\mathrm{S_L} := \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x_i;$$

define the *midpoint Riemann sum*, $\mathrm{S_M}$, by

$$\mathrm{S_M} := \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2} \right) \cdot \Delta x_i.$$

THEOREM **1.1.13.** *Assume that $f(x)$ is Riemann integrable on $[a, b]$. Given $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a, b]$ with $\|\mathcal{P}\| < \delta$, then*

$$\left| \mathrm{S}(f, \mathcal{P}) - \int_a^b f(x)\,dx \right| < \epsilon$$

for any Riemann sum $\mathrm{S}(f, \mathcal{P})$.

Proof. Since $f(x)$ is Riemann integrable, we can find a partition $\mathcal{P}_1$ of $[a, b]$ with $\mathrm{U}(f, \mathcal{P}_1) - \mathrm{L}(f, \mathcal{P}_1) < \frac{\epsilon}{2}$. In particular, for any Riemann sum $\mathrm{S}(f, \mathcal{P}_1)$, we have $\mathrm{U}(f, \mathcal{P}_1) \geq \mathrm{S}(f, \mathcal{P}_1) \geq \mathrm{L}(f, \mathcal{P}_1)$ and $\mathrm{U}(f, \mathcal{P}_1) \geq \int_a^b f(x)\,dx \geq \mathrm{L}(f, \mathcal{P}_1)$. Suppose $\mathcal{P}_1 = \{x_i :\ a = x_0 < x_1 < \cdots < x_n = b\}$. Then let

$$M = \sup\{f(x) :\ x \in [a, b]\},$$
$$m = \inf\{f(x) :\ x \in [a, b]\}.$$

If $M - m = 0$, then $f(x)$ is constant on $[a, b]$, so say $f(x) = c$ for all $x \in [a, b]$. In this case $\mathrm{U}(f, \mathcal{P}) = c(b - a) = \mathrm{L}(f, \mathcal{P})$ for any partition $\mathcal{P}$. Hence any $\delta > 0$ would satisfy the theorem and we are done. So assume $M > m$.

Let $\delta < \frac{\epsilon}{2n(M-m)}$. Let $\mathcal{P} = \{y_i : a = y_0 < y_1 < \cdots < y_j < \cdots < y_k = b\}$ be any partition of $[a, b]$ with $\|\mathcal{P}\| < \delta$. Let

$$\mathbf{T} = \{j : j \in \{1, 2, \ldots, k\}, \text{ and } [y_{j-1}, y_j] \subseteq [x_{i-1}, x_i] \text{ for some } i\}.$$

Also, let

$$\begin{aligned} M_j &= \sup\{f(x) : x \in [y_{j-1}, y_j]\}, \\ m_j &= \inf\{f(x) : x \in [y_{j-1}, y_j]\}. \end{aligned}$$

Then we have that

$$\begin{aligned} \mathrm{U}(f, \mathcal{P}) - \mathrm{L}(f, \mathcal{P}) &= \sum_{j=1}^{k} M_j \cdot \Delta y_j - \sum_{j=1}^{k} m_j \cdot \Delta y_j \\ &= \sum_{j=1}^{k} (M_j - m_j)\Delta y_j \\ &= \sum_{j \in \mathbf{T}} (M_j - m_j)\Delta y_j + \sum_{j \in \{1,2,\ldots,k\} \setminus \mathbf{T}} (M_j - m_j)\Delta y_j. \end{aligned}$$

Note that

$$\begin{aligned} \sum_{j \in \mathbf{T}} (M_j - m_j)\Delta y_j &\leq \mathrm{U}(f, \mathcal{P}_1 \cup \mathcal{P}) - \mathrm{L}(f, \mathcal{P}_1 \cup \mathcal{P}) \\ &\leq \mathrm{U}(f, \mathcal{P}_1) - \mathrm{L}(f, \mathcal{P}_1) \\ &< \frac{\epsilon}{2}. \end{aligned}$$

Observe that $\{1, 2, \ldots, k\} \setminus \mathbf{T}$ has at most n elements (since for each $j \notin \mathbf{T}$, we have a unique i such that $y_{j-1} < x_{i-1} < y_j < x_i$; conversely for each such i there exists a unique $j \notin \mathbf{T}$—but there are only n points in $\mathcal{P}_1$). Hence for each $j \in \{0, 1, \ldots, k\} \setminus \mathbf{T}$, we have

$$\begin{aligned} (M_j - m_j)\Delta y_j &\leq (M - m) \cdot \|\mathcal{P}\| \\ &< (M - m) \cdot \frac{\epsilon}{2n(M - m)} \\ &= \frac{\epsilon}{2n}. \end{aligned}$$

This shows that

$$\begin{aligned} \sum_{j \in \{1,2,\ldots,k\} \setminus \mathbf{T}} (M_j - m_j)\Delta y_j &< \sum_{j \in \{1,2,\ldots,k\} \setminus \mathbf{T}} \frac{\epsilon}{2n} \\ &\leq \frac{\epsilon}{2n} \cdot n \\ &= \frac{\epsilon}{2}. \end{aligned}$$

Hence, if $\|\mathcal{P}\| < \delta$, $\mathrm{U}(f, \mathcal{P}) - \mathrm{L}(f, \mathcal{P}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$; therefore, if $\mathrm{S}(f, \mathcal{P})$ is any Riemann sum, then

$$\left| \mathrm{S}(f, \mathcal{P}) - \int_a^b f(x)\, dx \right| < \epsilon,$$

as required. ∎

This thoerem is a generalization of remark 1.1.11. In particular, this theorem provides an easy, alternate proof of remark 1.1.11. Here we state the remark again as a corollary.

COROLLARY **1.1.14.** *If $f(x)$ is Riemann integrable on $[a,b]$, then*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \mathrm{S}(f,\mathcal{P}_n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\cdot \frac{b-a}{n},$$

where $\mathcal{P}_n$ is the n-regular partition of $[a,b]$ and $c_i \in [x_{i-1}, x_i]$. ($\mathrm{S}(f,\mathcal{P}_n)$ is any Riemann sum.)

Proof. Let $\epsilon > 0$. By the above theorem, we can find $\delta > 0$ so that if $\|\mathcal{P}_n\| = \frac{1}{n} < \delta$,

$$\left|\mathrm{S}(f,\mathcal{P}_n) - \int_a^b f(x)\,dx\right| < \epsilon$$

for any Riemann sum $\mathrm{S}(f,\mathcal{P}_n)$. The result follows by choosing an N (using the Archimedean Principle) so that $\frac{1}{n} \le \frac{1}{N} < \delta$ for all $n \ge N$. ∎

In general, we write

$$\int_a^b f(x)\,dx = \lim_{\|\mathcal{P}\|\to 0} \mathrm{S}(f,\mathcal{P}).$$

THEOREM **1.1.15.** *Let $f(x)$ be monotonic on $[a,b]$. Then $f(x)$ is Riemann integrable on $[a,b]$.*

Proof. Let $\mathcal{P}_n$ be the n-regular partition. Assume, without loss of generality, that $f(x)$ is nondecreasing on $[a,b]$. Then

$$\mathrm{U}(f,\mathcal{P}_n) = \mathrm{S_R}(f,\mathcal{P}_n) = \sum_{i=1}^{n} f(x_i)\cdot\frac{b-a}{n}, \text{ and}$$
$$\mathrm{L}(f,\mathcal{P}_n) = \mathrm{S_L}(f,\mathcal{P}_n) = \sum_{i=1}^{n} f(x_{i-1})\cdot\frac{b-a}{n}.$$

So

$$\begin{aligned}
\mathrm{U}(f,\mathcal{P}_n) - \mathrm{L}(f,\mathcal{P}_n) &= \sum_{i=1}^{n} f(x_i)\cdot\frac{b-a}{n} - \sum_{i=0}^{n-1} f(x_i)\cdot\frac{b-a}{n} \\
&= \frac{b-a}{n}[(f(x_1)+\cdots+f(x_n)) \\
&\qquad -(f(x_0)+\cdots+f(x_{n-1}))] \\
&= \frac{b-a}{n}(f(x_n)-f(x_0)) \\
&= \frac{b-a}{n}(f(b)-f(a)).
\end{aligned}$$

Since $\lim_{n\to\infty}\frac{b-a}{n}(f(b)-f(a)) = 0$, $f(x)$ is Riemann integrable on $[a,b]$ by theorem 1.1.4. ∎

Question: Assume that $f(x)$ is continuous on $[a,b]$ except at c. (Note that $f(x)$ is bounded on $[a,b]$.) Is $f(x)$ still Riemann integrable on $[a,b]$?

THEOREM **1.1.17.** *Assume that $f(x)$ is continuous on $[a,b]$ except at $c \in [a,b]$. Then $f(x)$ is Riemann integrable on $[a,b]$.*

Proof. Let $\mathcal{P} = \{x_i : a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$ so that $c \in (x_{i_0 - 1}, x_{i_0})$ for some i_0. Let $\epsilon > 0$. Let

$$\begin{aligned} M &= \sup\{f(x) : x \in [a, b]\}, \\ m &= \inf\{f(x) : x \in [a, b]\}, \\ M_i &= \sup\{f(x) : x \in [x_{i-1}, x_i]\}, \\ m_i &= \inf\{f(x) : x \in [x_{i-1}, x_i]\}. \end{aligned}$$

We can choose $\mathcal{P}$ so that $\Delta x_{i_0} < \frac{\epsilon}{3(M-m)}$. Then $(M_{i_0} - m_{i_0})\Delta x_{i_0} \leq (M-m)\Delta x_{i_0} < (M-m)\frac{\epsilon}{3(M-m)} = \frac{\epsilon}{3}$. Since $f(x)$ is uniformly continuous on $[a, x_{i_0-1}]$, by refining as necessary, we can assume that if $0 \leq i \leq i_0$, then $M_i - m_i < \frac{\epsilon}{3(x_{i_0} - a)}$. We now have

$$\sum_{i=1}^{i_0-1}(M_i - m_i)\Delta x_i < \sum_{i=1}^{i_0-1} \frac{\epsilon}{3(x_{i_0-1} - a)}\Delta x_i = \frac{\epsilon}{3}\sum_{i=1}^{i_0-1}\frac{\Delta x_i}{x_{i_0-1} - a} = \frac{\epsilon}{3}.$$

A similar argument shows, by refining if necessary, that

$$\sum_{i=i_0+1}^{n}(M_i - m_i)\Delta x_i < \frac{\epsilon}{3}.$$

Then

$$\begin{aligned} \mathrm{U}(f, \mathcal{P}) - \mathrm{L}(f, \mathcal{P}) &= \sum_{i=1}^{n}(M_i - m_i)\Delta x_i \\ &= \sum_{i=1}^{i_0-1}(M_i - m_i)\Delta x_i + (M_{i_0} - m_{i_0})\Delta x_{i_0} \\ &\qquad + \sum_{i=i_0+1}^{n}(M_i - m_i)\Delta x_i \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

This shows that $f(x)$ is Riemann integrable on $[a, b]$ by theorem 1.1.4. ∎

Observation. Observe the following facts:

1. If $f(x)$ is bounded on $[a, b]$ and continuous except at possibly at finitely many points, then $f(x)$ is Riemann integrable on $[a, b]$.
2. If $f(x)$ is Rieman integrable on $[a, b]$ and $g(x) = f(x)$ except at finitely many points, then $g(x)$ is Riemann integrable and
$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

□

1.2 Flaws in the Riemann Integral

The following problem will be a key to our motivation for the construction of countably additive measures.

Problem **1.2.1.** *Assume that* $\{f_n\}$ *is a sequence of Riemann integrable functions with* $\{f_n\}$ *converging pointwise on* $[a,b]$ *to a function* f_0*. Is* f_0 *Riemann integrable, and if so is*

$$\int_a^b f_0(x)\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx?$$

Unfortunately, as we will see below, the answer to both questions above is negative.

Example **1.2.2.** 1) Let $[0,1]\bigcap\mathbb{Q} = \{r_1, r_2, \cdots, r_n, \cdots\}$ and

$$f_n(x) = \begin{cases} 0 & \text{if } x \in [0,1]\setminus\{r_1, r_2, \cdots, r_{n-1}\} \\ 1 & \text{if } x \in \{r_1, r_2, \cdots, r_{n-1}\}. \end{cases}$$

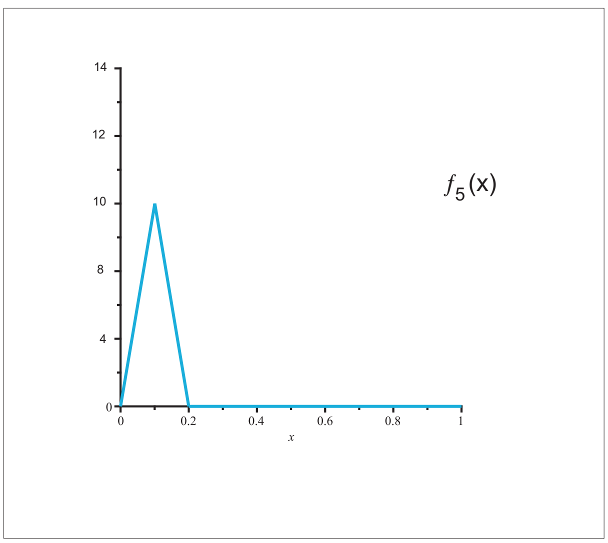
Then each $f_n(x)$ is Riemann integrable. Moreover, $f_n(x) \to f_0(x)$ pointwise, where

$$f_0(x) = \begin{cases} 0 & \text{if } x \in [0,1]\setminus\mathbb{Q} \\ 1 & \text{if } x \in [0,1]\bigcap\mathbb{Q}. \end{cases}$$

and $f_0(x)$ is not Riemann integrable.

2) Let

$$f_n(x) = \begin{cases} 4n^2x & \text{if } x \in [0, \frac{1}{2n}] \\ 2n - 4n^2(x - \frac{1}{2n}) & \text{if } x \in (\frac{1}{2n}, \frac{1}{n}] \\ 0 & \text{if } x \in (\frac{1}{n}, 1] \end{cases}$$



The diagram above represents the graph of $f_5(x)$. It is easy to see geometrically that $\int_0^1 f_5(t)\,dt = 1$ since the integral represents the area of a triangle with base 0.2 and height 10. Generically, $\int_0^1 f_n(t)\,dt = 1$ since the integral represents the area of a triangle with base $\frac{1}{n}$ and height $2n$. From this it follows that $f_n(x) \to f_0(x) = 0$ for each $x \in [0,1]$, and

$$1 = \lim_{n\to\infty}\int_0^1 f_n(t)\,dt \neq \int_0^1 \lim_{n\to\infty} f_n(t)\,dt = 0.$$

1.3 A Vector-Valued Riemann Integral

In this section we will see how we may define a vector-valued version of the Riemann integral. In doing so we will assume that B is a Banach space and that $F : [a, b] \to B$ is a continuous function.

Definition **1.3.1.** Let B be a Banach space. Let $F : [a, b] \to B$ be continuous. Given a partition $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[0, b]$, by a *Riemann sum* we will mean a sum $S(F, P)$ of the form

$$S(F,\mathcal{P}) = \sum_{i=1}^{n} F(c_i)(t_i - t_{i-1})$$

where $c_i \in [t_{i-1}, t_i]$.

Lemma **1.3.2.** *Assume that $F : [a, b] \to B$ is continuous. Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that if $\mathcal{P}$ is any partion of $[a, b]$ with $\|\mathcal{P}\| < \delta$ and if $\mathcal{P}_1$ is a refinement of $\mathcal{P}$, then for any Riemann sums $\mathrm{S}(F, \mathcal{P})$ and $\mathrm{S}(F, \mathcal{P}_1)$ we have*

$$\| \mathrm{S}(F,\mathcal{P}) - \mathrm{S}(F,\mathcal{P}_1) \| < \epsilon.$$

Proof. Given $\epsilon > 0$, since F is uniformly continuous, we can find a δ such that if $| x - y | < \delta$, then

$$\| F(x) - F(y) \| < \frac{\epsilon}{(b-a)}.$$

Assume that $\|\mathcal{P}\| = \max\{\triangle t_i\} < \delta$. Let $\mathcal{P}_1$ be a refinement of $\mathcal{P}$ with each interval $[t_{i-1}, t_i]$ being partitioned into $\{t_{i-1} = s_{i,0} < s_{i,1} < \cdots < s_{i,k_i} = t_i\}$. Let $\mathrm{S}(F, \mathcal{P}_1)$ be any Riemann sum associated with $\mathcal{P}_1$.

Since we know that

$$F(c_i)(t_i - t_{i-1}) = \sum_{j=1}^{k_i} F(c_i)(s_{i,j} - s_{i,j-1}),$$

we have

$$\begin{aligned}
\| \mathrm{S}(F,P) - \mathrm{S}(F,\mathcal{P}_1) \| &= \| \sum_{i=1}^{n} F(c_i)(t_i - t_{i-1}) - \sum_{i=1}^{n}\sum_{j=1}^{k_i} F(c_{i,j})(s_{i,j} - s_{i,j-1}) \| \\
&= \| \sum_{i=1}^{n}\sum_{j=1}^{k_i} F(c_i)(s_{i,j} - s_{i,j-1}) - \sum_{i=1}^{n}\sum_{j=1}^{k_i} F(c_{i,j})(s_{i,j} - s_{i,j-1}) \| \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{k_i} \| F(c_i) - F(c_{i,j}) \| (s_{i,j} - s_{i,j-1}) \\
&\leq \frac{\epsilon}{b-a} \cdot \sum_{i=1}^{n}\sum_{j=1}^{k_i} (s_{i,j} - s_{i,j-1}) \\
&= \epsilon.
\end{aligned}$$

∎

Theorem **1.3.3.** *Assume that $F : [a, b] \to B$ is continuous. Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that if $\mathcal{P}$ and $\mathcal{Q}$ are any two partitions of $[a, b]$ with $\|\mathcal{P}\| < \delta$ and $\|\mathcal{Q}\| < \delta$, then for any Riemann sums $s(F, \mathcal{P})$ and $s(F, \mathcal{Q})$ we have*

$$\| \mathrm{S}(F,\mathcal{P}) - \mathrm{S}(F,\mathcal{Q}) \| < \epsilon.$$

Proof. From the previous lemma, we know that we can choose a $\delta > 0$ such that if $\mathcal{P}_1$ is any partion of $[a, b]$ with $\| \mathcal{P}_1 \| < \delta$ and if $\mathcal{R}$ is a refinement of $\mathcal{P}_1$, then for any Riemann sums $\mathrm{S}(F, \mathcal{P}_1)$ and $\mathrm{S}(F, \mathcal{R})$ we have

$$\| \mathrm{S}(F,\mathcal{P}_1) - \mathrm{S}(F,\mathcal{R}) \| < \frac{\epsilon}{2}.$$

Next assume that $\mathcal{P}$ and $\mathcal{Q}$ are any two partitions of $[a, b]$ with $\|\mathcal{P}\| < \delta$ and $\|\mathcal{P}\| < \delta$. Let $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$ be a refinement of both $\mathcal{P}$ and $\mathcal{Q}$, and let $\mathrm{S}(F, \mathcal{P})$ and $\mathrm{S}(F, \mathcal{Q})$ be Riemann sums for F associated with $\mathcal{P}$ and $\mathcal{Q}$ respectively. Next let $\mathrm{S}(F, \mathcal{R})$ be any Riemann sums for F associated with $\mathcal{R}$ respectively. Then

$$\begin{aligned} \| \mathrm{S}(F, \mathcal{P}) - \mathrm{S}(F, \mathcal{Q}) \| &\leq \| \mathrm{S}(F, \mathcal{P}) - \mathrm{S}(F, \mathcal{R}) \| + \| \mathrm{S}(F, \mathcal{R}) - \mathrm{S}(F, \mathcal{Q}) \| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

∎

Corollary **1.3.4.** *Let $F : [a, b] \to B$ be continuous. If $\{\mathcal{P}_k\}$ is a sequence of partitions of $[a, b]$ such that $\lim_{k\to\infty} \| \mathcal{P}_k \| = 0$, then for any choice of Riemann sums $\{\mathrm{S}(F, \mathcal{P}_k)\}$ is Cauchy in B, and hence converges. Moreover, the limit is independent of both the choice of partitions and of the choice of Riemann sums.*

Proof. Since $\lim_{k\to\infty} \| \mathcal{P}_k \| = 0$, the previous theorem allows us to immediately deduce that $\{\mathrm{S}(F, \mathcal{P}_k)\}$ is Cauchy in B.

Let $\{\mathcal{P}_k\}$ and $\{\mathcal{Q}_k\}$ be two sequence of partitions of $[a, b]$ such that

$$\lim_{k\to\infty} \| \mathcal{P}_k \| = \lim_{k\to\infty} \| \mathcal{Q}_k \| = 0.$$

Let $\{\mathrm{S}(F, \mathcal{P}_k)\}$ and $\{\mathrm{S}(F, \mathcal{Q}_k)\}$ be such that

$$\lim_{k\to\infty} \{\mathrm{S}(F, \mathcal{P}_k)\} = x_0 \in B$$

and

$$\lim_{k\to\infty} \{\mathrm{S}(F, \mathcal{Q}_k)\} = y_0 \in B.$$

Then we create the sequence $\{\mathcal{P}_1, \mathcal{Q}_1, \mathcal{P}_2, \mathcal{Q}_2, \mathcal{P}_3, \mathcal{Q}_3, \cdots\}$ of partitions. It follows that the sequence

$$\{\mathrm{S}(F, \mathcal{P}_1), \mathrm{S}(F, \mathcal{Q}_1), \mathrm{S}(F, \mathcal{P}_2), \mathrm{S}(F, \mathcal{Q}_2), \cdots\}$$

is also Cauchy in B. From this we can conclude that $x_0 = y_0$ and hence that the limit is in fact independent of both the choice of partitions and of the choice of Riemann sums. ∎

Definition **1.3.5. [Vector-valued Riemann Integral]** Let $F : [a, b] \to B$ be continuous. We define the *B-valued Riemann integral of $F(X)$ on* $[a, b]$ by

$$\int_a^b F(t)\, dt = \lim_{\|P\|\to 0} \mathrm{S}(F, \mathcal{P}).$$

where $\lim_{\|P\|\to 0} \mathrm{S}(F, \mathcal{P})$ is the unique limit obtained as in the Corollary above.