Weak amenability of commutative Beurling algebras

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Yong Zhang

Department of Mathematics
University of Manitoba, Winnipeg, Canada

Outline

1. Preliminaries
2. Weak amenability of Beurling algebras
3. 2-weak amenability of Beurling algebras

Beurling algebras

Let G be a locally compact group. A weight function on G is a positive valued continuous function ω on G that satisfies

$$\omega(st) \leq \omega(s)\omega(t) \quad (s, t \in G).$$

If ω is a weight on G, we consider

$$L^1(G, \omega) = \{f : \ f\omega \in L^1(G)\}.$$

With the convolution product and the norm

$$\|f\|_\omega = \int_G |f(t)|\omega(t)dt \quad (f \in L^1(G, \omega))$$

$L^1(G, \omega)$ is a Banach algebra, called a Beurling algebra on G. When $\omega \equiv 1$ this is just the usual group algebra $L^1(G)$.

Derivation

Let $\mathcal{A}$ be a Banach algebra and X a Banach $\mathcal{A}$-bimodule.
A linear map $D: \mathcal{A} \to X$ is a derivation if it satisfies

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in \mathcal{A}).$$

Given an $x \in X$, the map $\mathrm{ad}_x: a \mapsto a \cdot x - x \cdot a$ ($a \in \mathcal{A}$) is a continuous derivation, called an inner derivation.

If X is a Banach $\mathcal{A}$-bimodule, then X^*, the dual space of X is naturally a Banach $\mathcal{A}$-bimodule with the module actions given by

$$\langle x, a \cdot f \rangle = \langle x \cdot a, f \rangle, \ \langle x, f \cdot a \rangle = \langle a \cdot x, f \rangle \quad (a \in \mathcal{A}, f \in X^*, x \in X).$$

The Banach algebra $\mathcal{A}$ itself is a Banach $\mathcal{A}$-bimodule with the module actions given by the product. So the nth dual $\mathcal{A}^{(n)}$ of $\mathcal{A}$ is naturally a Banach $\mathcal{A}$-bimodule for each $n \in \mathbb{N}$.

Weak and n-weak amenability

- The Banach algebra $\mathcal{A}$ is called weakly amenable if every continuous derivation $D: \mathcal{A} \to \mathcal{A}^*$ is inner.
- Given $n \in \mathbb{N}$, if every continuous derivation $D: \mathcal{A} \to \mathcal{A}^{(n)}$ is inner, then $\mathcal{A}$ is called n-weakly amenable.
- If $\mathcal{A}$ is n-weakly amenable for all $n \in \mathbb{N}$, then $\mathcal{A}$ is called permanently weakly amenable.

For example, all C*-algebras are permanently weakly amenable. So are all group algebras $L^1(G)$.

A commutative Banach algebra is permanently weakly amenable if and only if it is weakly amenable (**Dales-Ghahramani-Grønbæk, 1998**).

We investigate the conditions for a commutative Beurling algebra to be weakly or 2-weakly amenable.

Some related spaces

- The dual space of $L^1(G,\omega)$ is
$$L^\infty(G,1/\omega) =: \{f : f/\omega \in L^\infty(G)\}$$
with the norm given by
$$\|f\|_{\sup,1/\omega} = \operatorname*{ess\,sup}_{t\in G}\left|\frac{f(t)}{\omega(t)}\right| \quad (f \in L^\infty(G,1/\omega)).$$
- The multiplier algebra of $L^1(G,\omega)$ is $M(G,\omega)$ consisting of all Radon measures μ such that
$$\|\mu\|_\omega = \int_G \omega(t)d|\mu|(t) < \infty,$$
where $|\mu|$ denotes the total variation measure of μ.
- $M(G,\omega)$ is also the dual space of
$$C_0(G,1/\omega) = \{f \in L^\infty(G,1/\omega) : f/\omega \in C_0(G)\}$$
which is a closed $L^1(G,\omega)$-submodule of $L^\infty(G,1/\omega)$.

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Weak amenability of $L^1(G, \omega)$

Weak amenability of $L^1(G, \omega)$ was first studied by W.G. Bade, P.C. Curtis ans H.G. Dales in 1987.

1. (**Bade-Curtis-Dales, 1987**) For the additive group $\mathbb{Z}$ of all integers and the weight $\omega_\alpha(x) = (1 + |x|)^\alpha$ on $\mathbb{Z}$, the Beurling algebra $L^1(\mathbb{Z}, \omega_\alpha)$ is weakly amenable if and only if $0 \leq \alpha < \frac{1}{2}$.
2. (**Grønbæk, 1989**) The Beurling algebra $L^1(\mathbb{Z}, \omega)$ is weakly amenable if and only if

$$\liminf_{n \to \infty} \frac{\omega(n)\omega(-n)}{n} = 0.$$

3. (**Samei, 2008**) Let G be a LCAG and ω a weight on G. If

$$\liminf \frac{\omega(t^n)\omega(t^{-n})}{n} = 0$$

for all $t \in G$ then $L^1(G, \omega)$ is weakly amenable.

Weak amenability of $L^1(G,\omega)$

Theorem 1

Let G be a locally compact Abelian group and ω a weight on G. Then $L^1(G,\omega)$ is weakly amenable if and only if every nontrivial continuous group homomorphism ϕ: $G \to (\mathbb{C},+)$ satisfies

$$\sup_{t\in G}\frac{|\phi(t)|}{\omega(t)\omega(t^{-1})}=\infty. \tag{1}$$

Proof.

"$\Leftarrow$": If $L^1(G,\omega)$ is not weakly amenable, then there is a nonzero continuous derivation D: $L^1(G,\omega)\to L^\infty(G,1/\omega)$. Extend D to $M(G,\omega)$ and define $\Delta(t)=\delta_{t^{-1}}\cdot D(\delta_t)$ $(t\in G)$. Then $\exists h\in L^1(G,\omega)$ such that $\phi(t):=\langle h,\Delta(t)\rangle$ is a nontrivial continuous group homomorphism from G into $(\mathbb{C},+)$ and

$$|\phi(t)|\leq(\|D\|\|h\|_\omega)\omega(t)\omega(t^{-1})\quad(t\in G).$$

"$\Rightarrow$": Let ϕ: $G \to (\mathbb{C}, +)$ be a continuous nontrivial group homomorphism such that for all $t \in G$

$$\sup_{t \in G} \frac{|\phi(t)|}{\omega(t)\omega(t^{-1})} \leq m < \infty.$$

Fix a compact neibourghood B of e in G. $\forall h \in L^1(G, \omega)$ define

$$D(h)(t) = \int_B \phi(t^{-1}\xi)h(t^{-1}\xi)d\xi \quad (t \in G).$$

Then D: $L^1(G, \omega) \to L^\infty(G, 1/\omega)$ is indeed a nontrivial continuous derivation. Therefore $L^1(G, \omega)$ is not weakly amenable. □

Special consequences

We first note that if $G = (\mathbb{Z}, +)$ or $(\mathbb{R}, +)$ then $\phi \colon G \to (\mathbb{C}, +)$ is a continuous group homomorphism if and only if it takes the form $\phi(x) = xc_0$ for some $c_0 \in \mathbb{C}$. So from Theorem 1 we immediately derive

Corollary 1

Let $G = (\mathbb{N}, +)$ or $(\mathbb{R}, +)$ and let ω be a weight on G. Then the following statements are equivalent.

1. *The Beurling algebra $L^1(G, \omega)$ is weakly amenable.*
2. $\liminf_{t\to\infty} \frac{\omega(t)\omega(-t)}{|t|} = 0$.
3. $\liminf_{n\to\infty} \frac{\omega(n)\omega(-n)}{n} = 0$.
4. *There is $t_0 \in G$ such that $t_0 \neq 0$ and $\liminf_{n\to\infty} \frac{\omega(nt_0)\omega(-nt_0)}{n} = 0$.*

□

Corollary 2 (Samei)

Let G be a locally compact Abelian group and ω a weight on G. If for each $t \in G$

$$\inf_{n\in\mathbb{N}} \frac{\omega(t^n)\omega(t^{-n})}{n} = 0, \tag{2}$$

then $L^1(G,\omega)$ is weakly amenable.

Proof.

If $\phi\colon G \to \mathbb{C}$ is a group homomorphism such that $\phi(t_0) \neq 0$, we have

$$\sup_{t\in G} \frac{|\phi(t)|}{\omega(t)\omega(t^{-1})} \geq \sup_{n\in\mathbb{N}} \frac{|\phi(t_0^n)|}{\omega(t_0^n)\omega(t_0^{-n})} = |\phi(t_0)| \sup_{n\in\mathbb{N}} \frac{n}{\omega(t_0^n)\omega(t_0^{-n})} = \infty.$$

By Theorem 1, $L^1(G,\omega)$ is weakly amenable. □

Let H and R be two locally compact Abelian groups. We consider the product group $H \times R = \{(s,t) : s \in H, t \in R\}$. Let ω be a weight on $H \times R$. Then $\omega_H(s) = \omega(s, e_R)$ and $\omega_R(t) = \omega(e_H, t)$ are weights on H and R respectively, where e_H (e_R) is the identity of H (resp. R). We denote $\Omega(s,t) := \omega(s,t)\omega(s^{-1}, t^{-1})$ ($s \in H$, $t \in R$).

Corollary 3

If both $L^1(H, \omega_H)$ and $L^1(R, \omega_R)$ are weakly amenable then so is $L^1(H \times R, \omega)$. Conversely, the algebra $L^1(H \times R, \omega)$ is not weakly amenable if any of the following conditions holds:

1. *$L^1(H, \omega_H)$ is not weakly amenable and* $\sup_{(s,t)\in H\times R} \frac{\Omega(s,e_R)}{\Omega(s,t)} < \infty$.
2. *$L^1(R, \omega_R)$ is not weakly amenable and* $\sup_{(s,t)\in H\times R} \frac{\Omega(e_H,t)}{\Omega(s,t)} < \infty$.

Proof.

Straightforward calculation. □

Corollary 4

Let G_1 and G_2 be two locally compact Abelian groups and let ω_1 and ω_2 be weights on them, respectively. Then the tensor product $L^1(G_1, \omega_1)\hat{\otimes}L^1(G_2, \omega_2)$ is weakly amenable if and only if both $L^1(G_1, \omega_1)$ and $L^1(G_2, \omega_2)$ are weakly amenable.

Proof.

$L^1(G_1, \omega_1)\hat{\otimes}L^1(G_2, \omega_2) \cong L^1(G_1 \times G_2, \omega_1 \times \omega_2)$. The result follows from Corollary 3. □

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2-weak amenability of $L^1(G, \omega)$

- ([**Choi-Ghahramani-Z 2009, Losert 2010**]) $L^1(G)$ is always 2-weakly amenable.
- ([**Dales-Lau 2005**]) For $G = \mathbb{Z}$ or $\mathbb{R}$, and the weight $\omega_\alpha(t) = (1 + |t|)^\alpha$, the Beurling algebra $L^1(G, \omega_\alpha)$ is 2-weakly amenable if $0 \le \alpha < 1$; and it is not 2-weakly amenable if $\alpha \ge 1$.
- (**ibid.**) If G is Abelian and ω satisfies

$$\lim_{t\to\infty} \sup_{s\in K} |\omega(st)/\omega(t) - 1| = 0 \tag{3}$$

  for each compact set $K \subset G$ and

$$\liminf_{n\to\infty} \omega(t^n)/n = 0 \text{ for all } t \in G \tag{4}$$

  then $L^1(G, \omega)$ is 2-weakly amenable.
- ([**Ghahramani-Zabandan 2004, Samei 2008**]) The above is still true if the condition (3) is replaced by

$$\widehat{\omega}(s) = \limsup_{t\to\infty} \omega(ts)/\omega(t)$$

  being bounded on G.

The precise definition of $\widehat{\omega}$ is as follows.

$$\widehat{\omega}(t) = \limsup_{s\to\infty} \frac{\omega(ts)}{\omega(s)} := \inf_K \sup_{s\in G\setminus K} \frac{\omega(ts)}{\omega(s)},$$

where the infimum is taken over all compact subsets K of G.

Theorem 2

Let G be an locally compact Abelian group and ω a weight on G. If there is a constant $m > 0$ such that

$$\liminf_{n\to\infty} \frac{\omega(t^n)\widehat{\omega}(t^{-n})}{n} \le m \quad (t \in G),$$

then $L^1(G,\omega)$ is 2-weakly amenable.

I will skip the proof of Theorem 2 but give an example to show that our result is indeed more general.

Example

Let $\beta \geq 0$. Consider the additive group $\mathbb{Z}^2$ and the weight

$$\omega(s, t) = (1 + |s + t|)^{\beta}.$$

Then $\widehat{\omega} = \omega$ which is unbounded if $\beta > 0$. However, it is readily seen that $\lim \frac{\omega(ns,nt)\widehat{\omega}(-ns,-nt)}{n} = 0$ when $2\beta < 1$. So $\ell^1(\mathbb{Z}^2, \omega)$ is 2-weakly amenable if $\beta < 1/2$ due to Theorem 2.

Thank You.