Cuntz-Pimsner algebras for subproduct systems

Ami Viselter

University of Alberta

Banach Algebras 2011, University of Waterloo
August 10
Let $\mathcal{M}$ denote a $C^*$-algebra throughout.

**Definition**

A (right) Hilbert $C^*$-module $E$ over $\mathcal{M}$ is a $C^*$-correspondence if it is also a left $\mathcal{M}$-module, with multiplication on the left given by adjointable operators.

That is: there exists a $\ast$-homomorphism $\varphi : \mathcal{M} \to \mathcal{L}(E)$ such that $a \cdot \zeta$ is defined to be $\varphi(a)\zeta$ for $a \in \mathcal{M}$ and $\zeta \in E$.

**Examples**

1. $\mathcal{M} = \mathbb{C}$, $E = \mathcal{H}$ and $\varphi(\alpha)\zeta := \alpha\zeta$.
2. $E = \alpha\mathcal{M}$ where $\alpha$ is an endomorphism of $\mathcal{M}$, that is: $E = \mathcal{M}$ as sets and $\varphi(a)\zeta := \alpha(a)\zeta$. 
The Toeplitz algebra

Definition

Let $E$ denote a $C^*$-correspondence over $\mathcal{M}$.

- The Fock space is the correspondence

$$
\mathcal{F}_E := \bigoplus_{n \in \mathbb{Z}_+} E^\otimes n = \mathcal{M} \oplus E \oplus E^\otimes 2 \oplus \ldots.
$$

- For $a \in \mathcal{M}$ and $\zeta \in E$, let $\varphi_\infty(a), S(\zeta) \in \mathcal{L}(\mathcal{F}_E)$ be given by

$$
\varphi_\infty(a) : \eta \mapsto a \cdot \eta \quad S(\zeta) : \eta \mapsto \zeta \otimes \eta
$$

($\eta \in X(m), m \in \mathbb{Z}_+$).

It is a simple calculation that $S(\xi)^* S(\zeta) = \varphi_\infty(\langle \xi, \zeta \rangle)$.

- The Toeplitz algebra $\mathcal{T}(E)$ is the $C^*$-subalgebra of $\mathcal{L}(\mathcal{F}_E)$ generated by the operators $\varphi_\infty(\cdot), S(\cdot)$. 

Ami Viselter (University of Alberta)
The Cuntz-Pimsner algebra

Definition (Pimsner, 1995)

Let $E$ denote a faithful $C^*$-correspondence over $M$.

- The ideal $\mathcal{I} \trianglelefteq M$ is defined by
  \[ \mathcal{I} := \varphi^{-1}(\mathcal{K}(E)). \]

- We have $\mathcal{K}(\mathcal{F}_E \mathcal{I}) \trianglelefteq \mathcal{T}(E)$. More precisely,
  \[ \mathcal{T}(E) \cap \mathcal{K}(\mathcal{F}_E) = \mathcal{K}(\mathcal{F}_E \mathcal{I}). \]

- The **Cuntz-Pimsner** algebra is
  \[ O(E) := \mathcal{T}(E)/\mathcal{K}(\mathcal{F}_E \mathcal{I}). \]
Embed $\mathcal{K}(E) \hookrightarrow \mathcal{T}(E)$ by $\psi : \zeta \otimes \eta^* \mapsto S(\zeta)S(\eta)^*$.

**Theorem (Pimsner, 1995)**

A $C^*$-representation $\pi$ of $\mathcal{T}(E)$ factors through $O(E) \iff$ for all $a \in \mathcal{I}$ we have $\pi(\psi(\varphi(a))) = \pi(\varphi_{\infty}(a))$.

**Examples**

- $\mathcal{M} = E = \mathbb{C} \sim \mathcal{T}(E)$ is the (classical) Toeplitz algebra, $O(E) = C(\mathbb{T})$.
- $G$ is a finite graph of $d$ vertices, $E$ is the graph correspondence of $G$ (with $\mathcal{M} = \mathbb{C}^d$) $\sim O(E)$ is the Cuntz-Krieger algebra of $G$.
- $\mathcal{M}$ is a unital $C^*$-algebra, $\alpha \in \text{Aut} \mathcal{M}$, $E := \alpha \mathcal{M} \sim O(E) \cong \mathcal{M} \rtimes_{\alpha} \mathbb{Z}$.
- This could be generalized further to crossed products of Hilbert bimodules.
The Toeplitz algebra $\mathcal{T}(E)$ has a gauge action: for $\lambda \in \mathbb{T}$ there is $\alpha_\lambda \in \text{Aut}(\mathcal{T}(E))$ with

$$\varphi_\infty(a) \mapsto \varphi_\infty(a) \quad S(\zeta) \mapsto \lambda S(\zeta).$$

An ideal $I \subseteq \mathcal{T}(E)$ is called gauge invariant if $\alpha_\lambda(I) = I$ for all $\lambda$.

Recall that $O(E) = \mathcal{T}(E)/\mathcal{K}(\mathcal{F}_E \mathcal{J})$.

**The gauge-invariant uniqueness theorem (Katsura, 2007)**

The ideal $\mathcal{K}(\mathcal{F}_E \mathcal{J})$ is the largest among ideals $I$ of $\mathcal{T}(E)$ s.t.:

1. $\varphi_\infty(\mathcal{M}) \cap I = \{0\}$.
2. $I$ is gauge invariant.
Subproduct systems

**Definition**

A subproduct system is a family $X = (X(n))_{n \in \mathbb{Z}_+}$ of $C^*$-correspondences over the $C^*$-algebra $\mathcal{M} := X(0)$, such that

$$X(n + m) \subseteq X(n) \otimes X(m),$$

and moreover, $X(n + m)$ is orthogonally complementable in $X(n) \otimes X(m)$, for all $n, m \in \mathbb{Z}_+$.

**Product systems – the “trivial” example**

$E$ is an (essential) $C^*$-correspondence over $\mathcal{M}$ and $X(n) = E^\otimes n$ for all $n \in \mathbb{Z}_+$. 
Examples

**SSP\(d\) (the symmetric subproduct system), \(d \in \mathbb{N}\)**

\[ X(n) = \left( \mathbb{C}^d \right)^{\otimes n} \text{ (the } n\text{-fold symmetric tensor product of } \mathbb{C}^d \text{) for all } n. \]

**SSP\(\infty\) (the infinite-dimensional symmetric subproduct system)**

\[ X(n) = \left( \ell_2 \right)^{\otimes n}. \text{ Here dim } X(n) \text{ is infinite for all } n \in \mathbb{N}. \]

\[ P \in M_d, \ P_{ij} \geq 0 \text{ for all } i, j, \no \text{ all-zero columns} \]

\[ X(n) \text{ is the “support” quiver of the matrix } P^n. \]

**cp-semigroups**

A subproduct system can be associated with any cp-semigroup over \( \mathbb{Z}_+\).
The Toeplitz algebra for subproduct systems

Let $X = (X(n))_{n \in \mathbb{Z}_+}$ be a subproduct system.

**Definition (X-shifts)**

- Define
  
  $$ \mathcal{F}_X := \bigoplus_{n \in \mathbb{Z}_+} X(n) = \mathcal{M} \oplus X(1) \oplus X(2) \oplus X(3) \oplus \ldots. $$

- Setting $E := X(1)$, we have $X(n) \subseteq E^\otimes n$.

- For $n \in \mathbb{Z}_+$ and $\zeta \in X(n)$, define $S_n^X(\zeta) \in \mathcal{L}(\mathcal{F}_X)$ by
  
  $$(\forall m \in \mathbb{Z}_+, \eta \in X(m)) \quad S_n^X(\zeta)\eta := \text{Proj}_{X(n+m)}(\zeta \otimes \eta).$$

**Definition**

The **Toeplitz algebra** $\mathcal{T}(X)$ of $X$ is the $C^*$-subalgebra of $\mathcal{L}(\mathcal{F}_X)$ generated by $\{S_n^X(\zeta) : n \in \mathbb{Z}_+, \zeta \in X(n)\}$. 

-Ami Viselter (University of Alberta)
What about the Cuntz-Pimsner algebra?

The definition of the Toeplitz algebra for subproduct systems is natural.

But how should one define the Cuntz-Pimsner algebra for subproduct systems?

We will present a possible candidate, and try to justify it by demonstrating some of its virtues.
The Cuntz-Pimsner algebra for subproduct systems

Assume henceforth that $X(n)$ is faithful for all $n$. Write $Q_n \in \mathcal{L}(\mathcal{F}_X)$ for the projection on the direct summand $X(n)$.

**Theorem (V.)**

The set

$$I := \left\{ S \in \mathcal{T}(X) : \lim_{n \to \infty} \|SQ_n\| = 0 \right\}$$

is a gauge-invariant ideal of $\mathcal{T}(X)$ (and in fact, $I = \langle I \cap \mathcal{T}_0(X) \rangle$, where $\mathcal{T}_0(X)$ is the 0th spectral subset of $\mathcal{T}(X)$ w.r.t. the gauge action).

Clearly $\mathcal{K}(\mathcal{F}_X I) \subseteq \mathcal{T}(X) \cap \mathcal{K}(\mathcal{F}_X) \subseteq I$.

**Definition (V.)**

The **Cuntz-Pimsner algebra** of $X$ is defined as $O(X) := \mathcal{T}(X)/I$.

**Proposition**

If $X$ is a product system, then $I = \mathcal{K}(\mathcal{F}_X I)$. Thus, $O(X) = O(E)$. 

Ami Viselter (University of Alberta)  
Cuntz-Pimsner algebras  
Banach Algebras 2011
Examples

1. $X = \text{SSP}_d \sim I = \mathbb{K}$ and $O(X) = C(\partial B_d)$ (Arveson’s construction)

2. More generally: all fibers $X(n)$ are finite-dimensional Hilbert spaces $\sim I = \mathbb{K}$

3. $X = \text{SSP}_\infty \sim O(X) = C(B)$, where $B$ is the closed unit ball of $\ell_2$ with the Tychonoff topology (by the way: $O_\infty =$?)
   Question: is $I$ simple in this case? (our guess: no)

4. If $Q_n \in T(X)$ for all $n$ then $I = \langle Q_n : n \in \mathbb{Z}_+ \rangle$
   Example: the subproduct system of $P \in M_d$ with $P_{ij} \geq 0$ for all $i, j$ and no all-zero columns
Example

The Toeplitz algebra of $SSP_2$ does not admit a largest ideal which does not contain the unit $I$, and which is gauge invariant.

Idea of proof.

Suppose that such ideal $\mathcal{P} \leq T(SSP_2)$ exists.

1. $\mathcal{P}$ is largest $\sim \mathbb{I} \subseteq \mathcal{P}$
2. $0 \to \mathbb{I} \to T(SSP_2) \to C(\partial B_2)$
3. $\mathcal{P}/\mathbb{I}$ has a clear structure as an ideal of $C(\partial B_2)$

Now it is easy to find a larger ideal with the desired properties.
An equivalent definition

- Consider the subspaces $L(\bigoplus_{k=0}^{n}X(k))$ of $L(F_X)$
- Let $B$ be the ∗-algebra $\bigcup_{n=0}^{\infty} L(\bigoplus_{k=0}^{n}X(k))$
- $T(X)$ is contained in the multiplier algebra $M(\overline{B})$
- Let $q : M(\overline{B}) \to M(\overline{B})/\overline{B}$ be the quotient map.

Easy proposition

$O(X) \cong q(T(X))$. That is, $\ker q|_{T(X)} = \mathcal{I}$.

The proposition generalizes a result of Pimsner (1995). In fact, this was the original definition of the Cuntz-Pimsner algebra.
Morita equivalence

Definition (Muhly and Solel (2000))

Let $E, F$ be $C^*$-correspondences over $\mathcal{A}, \mathcal{B}$. $E$ is strongly Morita equivalent to $F$ if $\mathcal{A}$ is ME to $\mathcal{B}$ via an equivalence bimodule $M$, and there exists an isomorphism $W : M \otimes F \to E \otimes M$. Notation: $E \overset{\text{SME}}{\sim}_M F$.

If $E \overset{\text{SME}}{\sim}_M F$, define isomorphisms $W_n : M \otimes F^\otimes n \to E^\otimes n \otimes M$ by $W_1 := W$ and $W_n := (I_E \otimes W_{n-1})(W \otimes I_{F^\otimes (n-1)})$.

Definition (V.)

Subproduct systems $X, Y$ are strongly Morita equivalent if $X(1) \overset{\text{SME}}{\sim}_M Y(1)$ and $W_n(M \otimes Y(n)) = X(n) \otimes M$ for all $n$. 
For a subproduct system $X$, the tensor algebra $\mathcal{T}_+(X)$ is the operator subalgebra of $\mathcal{T}(X)$ generated by all $X$-shifts.

The following generalizes a theorem of Muhly and Solel (2000) for product systems.

**Theorem (V.)**

If $X$ is strongly Morita equivalent to $Y$, then:

1. $\mathcal{T}_+(X)$ is strongly Morita equivalent\(^a\) to $\mathcal{T}_+(Y)$
2. $\mathcal{T}(X)$ is Morita equivalent to $\mathcal{T}(Y)$
3. The Rieffel correspondence of $\mathcal{T}(X) \sim \mathcal{T}(Y)$ carries $I(X)$ to $I(Y)$. Therefore $O(X)$ is Morita equivalent to $O(Y)$.

\(^a\)as operator algebras, *a la* Blecher, Muhly and Paulsen (2000)

This is another evidence that our definition of the Cuntz-Pimsner algebra for subproduct systems is “natural”.

---

Ami Viselter (University of Alberta)  
Cuntz-Pimsner algebras  
Banach Algebras 2011  
16 / 21
Definition

A $C^*$-representation $\pi$ of $T(X)$ on $\mathcal{H}$ is essential if for every $n$,

$$\overline{\text{span}} \bigcup_{\zeta \in X(n)} \text{Im} \, \pi (S_n(\zeta)) = \mathcal{H}.$$ 

Remark

If $X$ is a product system, then $\pi$ is essential $\iff$ it is fully coisometric.

Theorem (Hirshberg (2005), Skeide (2009))

Let $E$ be a faithful and essential $C^*$-correspondence. Then

$$\bigcap_{\pi \text{ is an essential representation of } T(E)} \ker \pi = \mathcal{K}(\mathcal{F}_E \mathcal{I}).$$
Open question

Is it true that under (mild) hypotheses we have

$$\bigcap \ker \pi = \mathcal{I} ?$$

$$\pi$$ is an essential representation of $$\mathcal{T}(X)$$

We conjecture that it is.
What if “essential” is replaced by “fully coisometric”?

Proposition

$$\bigcap \ker \pi \supseteq \mathcal{I}.$$ 

$$\pi$$ is an essential representation of $$\mathcal{T}(X)$$
The conjecture (in its strict version) is true in many interesting cases. For instance:

1. “Finite-dimensional” subproduct systems:
   - All fibers $X(n)$ are finite-dimensional Hilbert spaces (e.g.: $X = \text{SSP}_d$, $d \in \mathbb{N}$)
   - The subproduct system of $P \in M_d$ with $P_{ij} \geq 0$ for all $i,j$ and no all-zero columns

2. But also $\text{SSP}_\infty$. 
More open questions

1. Is there a “strong” universality characterization of $O(X)$?
2. What is the ideal structure of $O(X)$?
   (The general case seems hopeless; what about specific families?)
3. In the spirit of Cuntz (1977), Pimsner used an “extension of scalars” method to find a $C^*$-algebra that is naturally isomorphic to $O(E)$, and for which there is a semi-split exact sequence with the Toeplitz algebra$^1$.
   Could this be done in our context?
4. Is there a relation between $O(X)$ and $C^*_\text{env}(\mathcal{T}+(X))$?
   Different cases have very different answers:
   \[ C^*_\text{env}(\mathcal{T}+(E)) = O(E), \]
   but
   \[ C^*_\text{env}(\mathcal{T}+(\text{SSP}_d)) = \mathcal{T}(\text{SSP}_d) \quad (d \in \mathbb{N}). \]
   We do not know what $C^*_\text{env}(\mathcal{T}+(\text{SSP}_\infty))$ is.

$^1$Pimsner used this to obtain a $KK$-theoretical six-term exact sequence.
Thank you for listening!