Approximately spectrum-preserving maps

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Part I

Introducing the problem

Kaplansky's problem

Identify the multiplicative linear maps among all linear maps, between complex Banach algebras A and B, in terms of spectra.

I. Kaplansky (1970)

Let A and B be complex Banach algebras and let $\Phi\colon A \to B$ be a linear map with the property that

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Is it true that Φ is a Jordan homomorphism, i.e.

$$\Phi(a^2) = \Phi(a)^2 \quad (a \in A)\ ?$$

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B. Aupetit (2000)

Let A and B be semisimple complex Banach algebras and let $\Phi\colon A \to B$ be a surjective linear map with the property that

$$\operatorname{sp}\big(\Phi(a)\big) = \operatorname{sp}(a) \quad (a \in A).$$

Is it true that Φ is a Jordan homomorphism?

Operator algebras

Theorem (A. A. Jafarian and A. R. Sourour (1986))

Let X and Y be complex Banach spaces and let $\Phi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ be a surjective linear map with the property that

$$\operatorname{sp}\big(\Phi(T)\big) = \operatorname{sp}(T) \ \ (T \in \mathcal{B}(X)).$$

Then Φ has the form $\Phi(T) = STS^{-1}$ $(T \in \mathcal{B}(X))$ for some isomorphism $S\colon X \to Y$ or $\Phi(T) = RT^*R^{-1}$ $(T \in \mathcal{B}(Y))$ for some isomorphism $R\colon X^* \to Y$.

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Let $\Phi \colon A \to B$ be a linear map between complex Banach algebras A and B and suppose that

$$\operatorname{sp}\big(\Phi(a)\big) \text{ is near } \operatorname{sp}(a) \text{ for each } a \in A.$$

Is Φ near a (Jordan) multiplicative map?

Replacing the preserving condition

For a linear map $\Phi\colon A \to B$.

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$$\operatorname{sp}\big(\Phi(a)\big) = \operatorname{sp}(a) \quad (a \in A),$$

into the property

ε-approximate preserving condition

$$\operatorname{dist}_{\mathrm{H}}\big(\operatorname{sp}(\Phi(a)), \operatorname{sp}(a)\big) < \varepsilon \quad (a \in A,\ \|a\| = 1),$$

for a given $\varepsilon > 0$.

The problem

$$\sup_{\|a\|=1} \operatorname{dist}_{\mathrm{H}}\Big(\operatorname{sp}(\Phi(a)), \operatorname{sp}(a)\Big) \quad \text{small}$$

$$\Downarrow$$

$$\operatorname{dist}\Big(\Phi, \operatorname{Hom}(A,B) \cup \operatorname{AntiHom}(A,B)\Big) \quad \text{small ?}$$

Part II

Motivating results for Kaplansky's problem

Theorem (G. Frobenius (1897))

Let $n \in \mathbb{N}$. A linear map $\Phi\colon \mathbb{M}_n \to \mathbb{M}_n$ satisfies the property

$$\det\big(\Phi(M)\big) = \det(M) \quad (M \in \mathbb{M}_n)$$

if and only if $\Phi = W\Psi$ for some automorphism or anti-automorphism Ψ of the Banach algebra $\mathbb{M}_n$ and some invertible matrix $W \in \mathbb{M}_n$ with $\det W = 1$.

Theorem (J. Dieudonné (1949))

Let $n \in \mathbb{N}$. A bijective linear map $\Phi \colon \mathbb{M}_n \to \mathbb{M}_n$ satisfies the property

$$M \in \mathbb{M}_n, \ \det(M) = 0 \ \Rightarrow \ \det\big(\Phi(M)\big) = 0$$

if and only if $\Phi = W\Psi$ for some automorphism or anti-automorphism Ψ of the Banach algebra $\mathbb{M}_n$ and some invertible matrix $W \in \mathbb{M}_n$.

Theorem (M. Marcus and R. Purves (1959))

Let $n \in \mathbb{N}$. A linear map $\Phi\colon \mathbb{M}_n \to \mathbb{M}_n$ satisfies the property

$$\mathrm{sp}\big(\Phi(M)\big) = \mathrm{sp}(M) \quad (M \in \mathbb{M}_n)$$

if and only if Φ is either an automorphism or an anti-automorphism of the Banach algebra $\mathbb{M}_n$.

Beyond finite-dimensionality

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Theorem (A. M. Gleason (1967), J. P. Kahane and W. Żelazko (1968))

Let A be a complex Banach algebra and let $\varphi\colon A \to \mathbb{C}$ be a linear functional such that

$$\varphi(a) \in \operatorname{sp}(a) \quad (a \in A).$$

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Shrinking the spectrum

$$\mathrm{sp}\big(\Phi(a)\big) \subset \mathrm{sp}(a) \quad (a \in A).$$

Part III

Motivating results for the approximate Kaplansky's problem

Approximate Gleason-Kahane-Żelazko theorem

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B. E. Johnson (1986) replaced

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$$\mathrm{dist}\big(\varphi(a), \mathrm{sp}(a)\big) < \varepsilon \quad (a \in A, \ \|a\| = 1)$$

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Theorem (B. E. Johnson (1986))

Let A be a commutative Banach algebra. Then for each $\varepsilon > 0$ there is $\delta > 0$ such that if $\varphi\colon A \to \mathbb{C}$ is a linear functional with

$$\mathrm{dist}(\varphi(a), \mathrm{sp}(a)) < \delta \quad (a \in A,\ \|a\| = 1),$$

then

$$\sup\{|\varphi(ab) - \varphi(a)\varphi(b)| \colon\ a, b \in A,\ \|a\| = \|b\| = 1\} < \varepsilon.$$

AMNM algebras

A Banach algebra A is **AMNM** if for each linear functional $\varphi\colon A \to \mathbb{C}$

$$\sup\{|\varphi(ab) - \varphi(a)\varphi(b)| : \; a, b \in A, \; \|a\| = \|b\| = 1\} \quad \text{small}$$

$$\Downarrow$$

$$\operatorname{dist}\big(\varphi, \operatorname{Hom}(A, \mathbb{C})\big) \quad \text{small.}$$

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Examples

1. B. E. Johnson (1986): $C_0(\Omega)$, $L^1(G)$ $\ell^1(\mathbb{Z}^+)$, $L^1(]0, +\infty[)$, $A(\mathbb{D})$.
2. R. A. J. Howey (2003): $C^n([0, 1])$.

This leads to translate

Johnson ε-approximate condition

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for a linear map $\Phi\colon A \to B$, into

ε-approximate preserving condition

$$\operatorname{dist}_{\mathrm{H}}\big(\Phi(a), \operatorname{sp}(a)\big) < \varepsilon \quad (a \in A,\ \|a\| = 1)$$

What about approximate versions of the theorems by

- G. Frobenius,
- J. Dieudonné,
- M. Marcus and R. Purves ?

Theorem (G. Frobenius)

Let $n \in \mathbb{N}$. A linear map $\Phi\colon \mathbb{M}_n \to \mathbb{M}_n$ satisfies the property

$$\det\big(\Phi(M)\big) = \det(M) \quad (M \in \mathbb{M}_n)$$

if and only if $\Phi = W\Psi$ for some automorphism or anti-automorphism Ψ of the Banach algebra $\mathbb{M}_n$ and some invertible matrix $W \in \mathbb{M}_n$ with $\det W = 1$.

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Theorem

Let $n \in \mathbb{N}$. Then for each $\varepsilon > 0$ there is $\delta > 0$ such that if $\Phi \colon \mathbb{M}_n \to \mathbb{M}_n$ is a linear map with

$$\sup_{\|M\|=1} \Big|\det\big(\Phi(M)\big) - \det(M)\Big| < \delta,$$

then

$$\operatorname{dist}\Big(\Phi, \mathrm{SL}_n\mathrm{Aut}(\mathbb{M}_n) \cup \mathrm{SL}_n\mathrm{AntiAut}(\mathbb{M}_n)\Big) < \varepsilon.$$

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Let $n \in \mathbb{N}$. Then for each $K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathbb{M}_n \to \mathbb{M}_n$ is a linear map with

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and $\|\Phi\| < K$, then

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Theorem (J. Dieudonné)

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Let $n \in \mathbb{N}$. Then for each $K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi \colon \mathbb{M}_n \to \mathbb{M}_n$ is a bijective linear map with

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$$\sup_{\|M\|=1} \mathrm{dist}_{\mathrm{H}}\Big(\mathrm{sp}\big(\Phi(M)\big), \mathrm{sp}(M)\Big) < \delta,$$

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Proof (approximate Marcus-Purves)

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Given $K, \varepsilon > 0$ put

$$C = \left\{ \Phi \in \mathcal{B}(\mathbb{M}_n) : \ \|\Phi\| \leq K, \ \text{dist}\big(\Phi, \text{Aut}(\mathbb{M}_n) \cup \text{AntiAut}(\mathbb{M}_n)\big) \geq \varepsilon \right\}$$

compact

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and for $\delta > 0$,

$$G_\delta = \bigcup_{\|M\|=1} \left\{\Phi \in \mathcal{B}(\mathbb{M}_n) \colon \ \operatorname{dist}_{\mathrm{H}}\big(\operatorname{sp}(\Phi(M)), \operatorname{sp}(M)\big) > \delta\right\}$$

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open

$C \subset \bigcup_{\delta>0} G_\delta \implies C \subset G_\delta$ for some $\delta > 0$ and this proves the theorem.

The objective

Approximate version of Jafarian-Sourour Theorem

Part IV

Approximately spectrum-preserving maps on operator algebras

Theorem

Let X and Y be superreflexive Banach spaces. Then for each $\varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ is a surjective linear map with

$$\sup_{\|T\|=1} \operatorname{dist}_{\mathrm{H}}\Big(\operatorname{sp}(\Phi(T)), \operatorname{sp}(T)\Big) < \delta,$$

then

$$\underbrace{\sup\{\|\Phi(ST) - \Phi(S)\Phi(T)\|\colon\ S, T \in \mathcal{B}(X),\ \|S\| = \|T\| = 1\}}_{\operatorname{mult}(\Phi)} < \varepsilon$$

or

$$\underbrace{\sup\{\|\Phi(ST) - \Phi(T)\Phi(S)\|\colon\ S, T \in \mathcal{B}(X),\ \|S\| = \|T\| = 1\}}_{\operatorname{amult}(\Phi)} < \varepsilon,$$

Theorem

Let X and Y be superreflexive Banach spaces. Then for each $k, K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ is a surjective linear map with

$$\sup_{\|T\|=1} \mathrm{dist_H}\Big(\mathrm{sp}(\Phi(T)), \mathrm{sp}(T)\Big) < \delta,$$

$\kappa(\Phi) > k$, and $\|\Phi\| < K$, then

$$\underbrace{\sup\{\|\Phi(ST) - \Phi(S)\Phi(T)\|\colon\ S, T \in \mathcal{B}(X),\ \|S\| = \|T\| = 1\}}_{\mathrm{mult}(\Phi)} < \varepsilon$$

or

$$\underbrace{\sup\{\|\Phi(ST) - \Phi(T)\Phi(S)\|\colon\ S, T \in \mathcal{B}(X),\ \|S\| = \|T\| = 1\}}_{\mathrm{amult}(\Phi)} < \varepsilon,$$

where $\kappa(\Phi)$ is the **surjectivity modulus** *of Φ, which is defined by*

$$\kappa(\Phi) = \sup\left\{\varrho \geq 0\colon\ \varrho\mathbb{B}_{\mathcal{B}(Y)} \subset \Phi\big(\mathbb{B}_{\mathcal{B}(X)}\big)\right\}.$$

Question

$$\min\left\{\mathrm{mult}(\Phi),\mathrm{amult}(\Phi)\right\} \quad \text{small}$$

$$\Downarrow$$

$$\mathrm{dist}\Big(\Phi,\mathrm{Hom}\big(\mathcal{B}(X),\mathcal{B}(Y)\big)\cup\mathrm{AntiHom}\big(\mathcal{B}(X),\mathcal{B}(Y)\big)\Big) \quad \text{small ?}$$

Theorem

Let H be a separable Hilbert space. Then for each $k, K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathcal{B}(H) \to \mathcal{B}(H)$ is a surjective linear map with

$$\sup_{\|T\|=1} \mathrm{dist}_{\mathrm{H}}\Big(\mathrm{sp}(\Phi(T)), \mathrm{sp}(T)\Big) < \delta,$$

$\kappa(\Phi) > k$, and $\|\Phi\| < K$, then

$$\mathrm{dist}\Big(\Phi, \mathrm{Aut}(\mathcal{B}(H)) \cup \mathrm{AntiAut}(\mathcal{B}(H))\Big) < \varepsilon.$$

Part V

Outline of the proofs

First theorem

Assume towards a contradiction that the first theorem is false.

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Then there exist $\Phi_n \colon \mathcal{B}(X) \to \mathcal{B}(Y)$ $(n \in \mathbb{N})$ such that

1. $\sup_{\|T\|=1} \operatorname{dist}_{\mathrm{H}}\Big(\operatorname{sp}(\Phi_n(T)), \operatorname{sp}(T)\Big) \to 0$,
2. $\kappa(\Phi_n) > k$, $\|\Phi_n\| < K$,
3. $\inf_{n \in \mathbb{N}} \{\operatorname{mult}(\Phi_n), \operatorname{amult}(\Phi_n)\} > 0$.

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$.

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Let us consider the map

$$\Phi = (\Phi_n)^{\mathcal{U}}\colon \underbrace{\mathcal{B}(X)^{\mathcal{U}}}_{\subset \mathcal{B}\left(X^{\mathcal{U}}\right)} \longrightarrow \underbrace{\mathcal{B}(Y)^{\mathcal{U}}}_{\subset \mathcal{B}\left(Y^{\mathcal{U}}\right)}, \quad \Phi(\mathbf{T}) = \left(\Phi_n(T_n)\right)^{\mathcal{U}}.$$

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1. Φ is surjective.
2. $\mathrm{sp}\big(\Phi(\mathbf{T})\big) = \mathrm{sp}(\mathbf{T})$ $\big(\mathbf{T} = (T_n) \in \mathcal{B}(X)^{\mathcal{U}}\big)$.
3. Both $\mathcal{B}(X)^{\mathcal{U}}$ and $\mathcal{B}(Y)^{\mathcal{U}}$ contain the finite-rank operators on $X^{\mathcal{U}}$ and $Y^{\mathcal{U}}$, respectively.

Φ is either a homomorphism or an anti-homomorphism.

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$$0 < \lim_{\mathcal{U}} \min\{\text{mult}(\Phi_n), \text{amult}(\Phi_n)\} =$$

$$\min\left\{\lim_{\mathcal{U}} \text{mult}(\Phi_n), \lim_{\mathcal{U}} \text{amult}(\Phi_n)\right\} = \min\{\text{mult}(\Phi), \text{amult}(\Phi)\} = 0.$$

a contradiction !

Second theorem

The **AMNM** problem

$$\Phi \in \mathcal{B}(A, B),\ \min\left\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\right\}\ \text{ small}$$

$$\Downarrow$$

$$\mathrm{dist}\Big(\Phi, \mathrm{Hom}(A, B) \cup \mathrm{AntiHom}(A, B)\Big)\ \text{ small?.}$$

Second theorem

The **AMNM** problem

$$\Phi \in \mathcal{B}(A,B),\ \min\left\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\right\}\ \text{ small}$$

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$$\mathrm{dist}\Big(\Phi, \mathrm{Hom}(A,B) \cup \mathrm{AntiHom}(A,B)\Big)\ \text{ small?.}$$

Examples (B. E. Johnson, 1988)

1. The Banach algebras A and B are finite-dimensional.
2. The Banach algebra A is finite-dimensional and semisimple.
3. The Banach algebra A is amenable and B is a two-sided ideal of a dual Banach algebra C. This applies to the pairs:
   - $(L^1(G_1), M(G_2))$ and $(L^1(G_1), L^1(G_2))$ for each amenable group G_1 and each locally compact group G_2,
   - $(\mathcal{K}(H_1), \mathcal{B}(H_2))$ and $(\mathcal{K}(H_1), \mathcal{K}(H_2))$ for all Hilbert spaces H_1 and H_2.
4. $(\mathcal{B}(H), \mathcal{B}(H))$ for each separable Hilbert space H.

Part VI

A converse result

Replacing the spectra: the pseudospectra

ε–pseudospectrum of $T \in \mathcal{B}(X)$

$$\mathrm{sp}_\varepsilon(T) = \left\{ z \in \mathbb{C} \colon \left\| (T - z\mathbf{1})^{-1} \right\| > \varepsilon^{-1} \right\} \quad (\varepsilon > 0)$$

Theorem

Let X and Y be superreflexive Banach spaces. Then the following assertions hold.

1. *For each $K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ is a bijective linear map with*
$$\min\{\operatorname{mult}(\Phi), \operatorname{amult}(\Phi)\} < \delta$$
and $\|\Phi\|, \|\Phi^{-1}\| < K$, then
$$\operatorname{sp}(\Phi(T)) \subset \operatorname{sp}_\varepsilon(T) \ \ (T \in \mathcal{B}(X), \|T\| = 1).$$

2. *For each $K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ is a bijective linear map with*
$$\operatorname{sp}(\Phi(T)) \subset \operatorname{sp}_\delta(T) \ \ (T \in \mathcal{B}(X), \|T\| = 1)$$
and $\|\Phi\|, \|\Phi^{-1}\| < K$, then
$$\min\{\operatorname{mult}(\Phi), \operatorname{amult}(\Phi)\} < \varepsilon.$$

Theorem

Let H be a separable Hilbert space. Then the following assertions hold.

1. *For each* $K, \varepsilon > 0$ *there is* $\delta > 0$ *such that if* $\Phi \colon \mathcal{B}(H) \to \mathcal{B}(H)$ *is a continuous linear map with* $\|\Phi\| < K$ *and*

$$\operatorname{dist}\Big(\Phi, \operatorname{Aut}(\mathcal{B}(H)) \cup \operatorname{AntiAut}(\mathcal{B}(H))\Big) < \delta,$$

then

$$\operatorname{sp}(\Phi(T)) \subset \operatorname{sp}_\varepsilon(T) \ \ (T \in \mathcal{B}(H), \|T\| = 1).$$

2. *For each* $K, \varepsilon > 0$ *there is* $\delta > 0$ *such that if* $\Phi \colon \mathcal{B}(H) \to \mathcal{B}(H)$ *is a bijective linear map with*

$$\operatorname{sp}(\Phi(T)) \subset \operatorname{sp}_\delta(T) \ \ (T \in \mathcal{B}(H), \|T\| = 1)$$

and $\|\Phi\|, \|\Phi^{-1}\| < K$*, then*

$$\operatorname{dist}\Big(\Phi, \operatorname{Aut}(\mathcal{B}(H)) \cup \operatorname{AntiAut}(\mathcal{B}(H))\Big) < \varepsilon.$$