Closable multipliers

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(joint with Victor Shulman and Ivan Todorov)

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Outline of the Talk

- Measurable Schur multipliers
- Local Schur multipliers
- Closable and w^*-closable multipliers
- Toeplitz type multipliers and harmonic analysis.

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Measurable Schur multipliers

(X, μ) and (Y, ν) standard measure spaces
$H_1 = L^2(X, \mu)$, $H_2 = L^2(Y, \nu)$

$\mathcal{B}(H_1, H_2)$ and $\mathcal{K}(H_1, H_2)$ the space of all bounded and resp. compact linear operators from H_1 into H_2.

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Similarly, $\mathcal{C}_1(H_2, H_1)$, the space of nuclear operators, will be identified with $\Gamma(X, Y) = L_2(X) \hat{\otimes} L_2(Y)$, the space of all $F : X \times Y \to \mathbb{C}$ s.t.

$$F(x, y) = \sum_{i=1}^{\infty} f_i(x) g_i(y),$$

$f_i \in L^2(X, \mu)$, $g_i \in L^2(Y, \nu)$, $\sum_{i=1}^{\infty} \|f_i\|_2^2 < \infty$, $\sum_{i=1}^{\infty} \|g_i\|_2^2 < \infty$.

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Theorem (Peller)

Let $\varphi \in L^\infty(X \times Y)$. TFAE
(i) φ is a Schur multiplier;
(ii) there exist measurable functions $a : X \to l^2$ and $b : Y \to l^2$ such that

$$\begin{aligned} \varphi(x, y) &= (a(x), b(y))_{l^2}, \text{ a.e. on } X \times Y \\ & \quad \text{and } \sup_{x \in X} \|a(x)\|_2 \sup_{y \in Y} \|b(y)\|_2 < \infty. \end{aligned}$$

(iii) $\varphi(x, y)k(x, y) \in^{\mu \times \nu} \Gamma(X, Y)$ whenever $k(x, y) \in \Gamma(X, Y)$.

φ is a Schur multiplier $\Rightarrow S_\varphi$ is bounded on $\mathcal{K}(H_1, H_2)$ and L^∞-bimodule, i.e. $S_\varphi(M_b T M_a) = M_b S_\varphi(T) M_a$, $\forall a \in L^\infty(X)$, $b \in L^\infty(Y)$.
By the Smith theorem, S_φ is a cb map and $S_\varphi(T) = \sum_i M_{b_i} T M_{a_i}$ where $\sum a_i(x) b_i(y) = (a(x), b(y))_{l^2} \in L^\infty(X) \otimes_{eh} L^\infty(Y)$ and hence $\varphi(x, y) = (a(x), b(y))_{l^2}$.

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- Non-Schur multiplier: Transformer of triangular truncation $\varphi(x, y) = \chi_\Delta(x, y)$, where $\Delta = \{(x, y) \in [0, 1]^2 : x \leq y\}$ (Gohberg-Krein)

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Theorem (Bozejko)

Let G be a locally compact amenable group with Haar measure m and $\varphi(x, y) = f(xy^{-1})$, where $f : G \to \mathbb{C}$.
Then φ is a Schur multiplier (w.r.t. m) iff $f \in B(G)$ (the Fourier Stieltjes algebra of G).

Obs! We have $B(G) = \{f : G \to \mathbb{C} : fA(G) \subset A(G)\} = MA(G)$.

Local Schur multipliers

A function $\varphi \in \mathfrak{B}(X \times Y)$ is called a **local Schur multiplier** if there exists a covering family $\{\kappa_m\}_{m=1}^{\infty}$ of rectangles in $X \times Y$ (i.e. $X \times Y = \cup k_m$) s.t. $\varphi|_{\kappa_m}$ is a Schur multiplier for each $m \in \mathbb{N}$. $\mathfrak{S}_{\rm loc}(X, Y)$ the set of all local Schur multipliers.

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Theorem

TFAE

(i) φ *is a local Schur multiplier;*

(ii) there exist measurable functions $a : X \to l^2$ *and* $b : Y \to l^2$ *such that*

$$\begin{aligned} \varphi(x, y) &= (a(x), b(y))_{l^2}, \text{ a.e. on } X \times Y \text{ and} \\ & \|a(x)\|_2 < \infty \text{ a.e. on } X, \|b(y)\|_2 < \infty \text{ a.e. on } Y. \end{aligned}$$

w^*-closable and closable multipliers

For $\varphi \in \mathfrak{B}(X \times Y)$, consider S_φ as (densely defined linear) map on $\mathcal{K}(H_1, H_2)$ ($D(S_\varphi) \subset \mathcal{C}_2(H_1, H_2) \subset \mathcal{K}(H_1, H_2)$).

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φ is called a **closable multiplier** if the map S_φ is closable, i.e. if $(x_k)_{k\in\mathbb{N}} \subseteq D(S_\varphi)$, $\|x_k\| \to 0$ and $\|S_\varphi(x_k) - y\| \to 0$ imply that $y = 0$.

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Similarly, φ is called a **weak* closable multiplier** if S_φ is weak* closable as map on $\mathcal{B}(H_1, H_2)$, i.e. if $(x_i) \subseteq D(S_\varphi)$, $x_i \xrightarrow{w^*} 0$ in $\mathcal{B}(H_1, H_2)$ and $S_\varphi(x_i) \xrightarrow{w^*} y$ in $\mathcal{B}(H_1, H_2)$ imply $y = 0$.

Denote these classes of functions by $\mathfrak{S}_{\rm cl}(X, Y)$ and $\mathfrak{S}_{w^*}(X, Y)$, resp.

- S_φ^* is a map on $(\mathcal{K}(H_1, H_2))^* \simeq \Gamma(X, Y)$, $S_\varphi^* : k \mapsto \varphi k$ with domain $D(S_\varphi^*) = \{k \in \Gamma(X, Y) : \varphi k \in^{\mu \times \nu} \Gamma(X, Y)\}$.

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- **Fact:** S_φ is w^*-closable (closable) iff $\overline{D(S_\varphi^*)}^{\|\cdot\|} = \Gamma(X, Y)$ (resp. $\overline{D(S_\varphi^*)}^{w^*} = \Gamma(X, Y)$).

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Lemma

Let $\varphi \in \mathfrak{B}(X \times Y)$. TFAE:
(i) φ is a w-closable multiplier;*
(ii) $\exists$ rectangles $\{\kappa_m\}_{m\in\mathbb{N}}$, $\cup k_m = X \times Y$, and $\exists s_m, t_m \in \Gamma(X, Y)$ s.t. $s_m(x, y) \neq 0$ m.a.e. on κ_m and

$$\varphi(x, y) = \frac{t_m(x, y)}{s_m(x, y)}, \quad a.e.\ on\ \kappa_m,\ m \in \mathbb{N}.$$

- S_φ^* is a map on $(\mathcal{K}(H_1, H_2))^* \simeq \Gamma(X, Y)$, $S_\varphi^* : k \mapsto \varphi k$ with domain $D(S_\varphi^*) = \{k \in \Gamma(X, Y) : \varphi k \in^{\mu \times \nu} \Gamma(X, Y)\}$.
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Theorem

A function $\varphi \in \mathfrak{B}(X \times Y)$ is a w-closable multiplier iff $\exists t, s \in \mathfrak{S}_{loc}(X, Y)$ s.t. $s(x, y) \neq 0$ m.a.e. on $X \times Y$ and $\varphi(x, y) = \frac{t(x,y)}{s(x,y)}$, a.e. on $X \times Y$.*

- Hence $\mathfrak{S}(X,Y) \subset \mathfrak{S}_{loc}(X,Y) \subset \mathfrak{S}_{w^*}(X,Y) \subset \mathfrak{S}_{cl}(X,Y)$.
- Clearly, $\mathfrak{S}(X,Y) \subset \mathfrak{S}_{loc}(X,Y)$ is proper. We will see that $\mathfrak{S}_{w^*}(X,Y) \subset \mathfrak{S}_{cl}(X,Y)$ is proper.
- **OPEN**: Is the inclusion $\mathfrak{S}_{loc}(X,Y) \subset \mathfrak{S}_{w^*}(X,Y)$ proper?

Non-closable multiplier

Example

Example is based on result on non-validity of generalized Fuglede-Putnam theorem on the set of compact operators. Namely, $\exists T \in \mathcal{K}(L^2(\mathbb{T}))$, f_n, $g_n \in L^\infty(\mathbb{T})$ s.t. for $\psi(t,s) = \sum_{n\in\mathbb{Z}} f_n(t)g_n(s) \in \mathfrak{S}(\mathbb{T},\mathbb{T})$

$$S_\psi(T) = \sum_{n\in\mathbb{Z}} M_{f_n} T M_{g_n} = 0 \text{ and } S_{\overline{\psi}}(T) = \sum_{n\in\mathbb{Z}} M_{\overline{f}_n} T M_{\overline{g}_n} \neq 0,$$

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Let $\{T_n\}_{n=1}^\infty \subseteq \mathcal{C}_2(L^2(\mathbb{T}))$ s.t. $T_n \to T$ in the operator norm. Then $S_\psi(T_n) \to S_\psi(T) = \sum_{n\in\mathbb{Z}} M_{f_n} T M_{g_n} = 0$.

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Let $\{T_n\}_{n=1}^\infty \subseteq \mathcal{C}_2(L^2(\mathbb{T}))$ s.t. $T_n \to T$ in the operator norm. Then $S_\psi(T_n) \to S_\psi(T) = \sum_{n\in\mathbb{Z}} M_{f_n} T M_{g_n} = 0$. However,

$$S_\varphi(S_\psi(T_n)) = S_{\overline{\psi}}(T_n) \to S_{\overline{\psi}}(T) = \sum_{n\in\mathbb{Z}} M_{\overline{f}_n} T M_{\overline{g}_n} \neq 0.$$

Closable multipliers and M-sets.

- ω-**topology**: $E \subseteq X \times Y$ is marginally null if $E \subseteq (X_0 \times Y) \cup (X \times Y_0)$, $\mu(X_0) = \nu(Y_0) = 0$.
  E is ω-open if $E = \cup_{n=1}^{\infty} \alpha_n \times \beta_n$, $(\omega\text{-open})^c = \omega$-closed.

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  For any $\mathcal{M} \subseteq \mathcal{B}(H_1, H_2)$, $\exists$ a smallest (up to marginal equivalence) ω-closed set $\operatorname{supp} \mathcal{M}$ which supports $\forall T \in \mathcal{M}$.

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- For any ω-closed set κ $\exists$ a smallest (resp. largest) w*-closed $L_\infty(X)$-$L_\infty(Y)$-bimodule $\mathfrak{M}_{\min}(\kappa)$ (resp. $\mathfrak{M}_{\max}(\kappa)$) with support κ, i.e. if $\mathfrak{M} \subseteq \mathcal{B}(H_1, H_2)$ is a w*-closed bimodule with supp $\mathfrak{M} = \kappa$ then

$$\mathfrak{M}_{\min}(\kappa) \subseteq \mathfrak{M} \subseteq \mathfrak{M}_{\max}(\kappa).$$

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$$\mathfrak{M}_{\min}(\kappa) \subseteq \mathfrak{M} \subseteq \mathfrak{M}_{\max}(\kappa).$$

- If $\mathfrak{M}_{\min}(\kappa) = \mathfrak{M}_{\max}(\kappa)$, the set κ is called *synthetic*.

Fact:

S_φ is closable iff

$$\overline{D(S_\varphi^*)}^{w^*} = \Gamma(X, Y) \Leftrightarrow D(S_\varphi^*)^\perp \cap \mathcal{K}(H_1, H_2) = \{0\},$$

where $D(S_\varphi^*) = \{k \in \Gamma(X, Y) : \varphi k \in^{\mu\times\nu} \Gamma(X, Y)\}$.

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Let null $D(S_\varphi^*) = \kappa_\varphi$.
$D(S_\varphi^*)^\perp = \{T \in B(H_1,H_2) : \mathrm{Tr}(TS) = 0, \forall S \in D(S_\varphi^*)\}$ is a w^*-closed bimodule supported by κ_φ. Hence

$$\mathfrak{M}_{\min}(\kappa_\varphi) \subseteq D(S_\varphi^*)^\perp \subseteq \mathfrak{M}_{\max}(\kappa_\varphi).$$

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$D(S_\varphi^*)^\perp = \{T \in B(H_1,H_2) : \mathrm{Tr}(TS) = 0, \forall S \in D(S_\varphi^*)\}$ is a w^*-closed bimodule supported by κ_φ. Hence

$$\mathfrak{M}_{\min}(\kappa_\varphi) \subseteq D(S_\varphi^*)^\perp \subseteq \mathfrak{M}_{\max}(\kappa_\varphi).$$

Following the analogy with harmonic analysis initiated by Arveson and later pursued by Froelich, we call an ω-closed set $E \subseteq X \times Y$ an **operator** M**-set** (respectively, **operator** M_1**-set**) if E supports a non-zero compact operator (resp. $\mathfrak{M}_{\min}(E)$ contains a non-zero compact operator).

Theorem

Let $\varphi \in \mathfrak{B}(X \times Y)$.

(i) If κ_φ *is not an operator M-set then* φ *is a closable multiplier.*

(ii) If κ_φ *is an operator* M_1*-set then* φ *is not a closable multiplier.*

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Claim

χ_Δ *is closable but not w**-closable.*

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Claim

χ_Δ *is closable but not w*-closable.*

Idea: χ_Δ is closable: In fact, as χ_Δ is constant on each rectangle disjoint from $\Lambda = \{(x, x) : x \in [0, 1]\}$, null $D(S^*_{\chi_\Delta}) \subset \Lambda$.

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Theorem (Shulman)

Let $E = \{(x, y) : f_j(x) \leq g_j(y),\ \textit{for } j = 1, .., n.\}$, $f_j : X \to Z$, $g_j : Y \to Z$, *Z odered metric space. E is not an M-set iff* $\mu \times \nu(E) = 0$;

Multipliers of Toeplitz type

- G is a locally compact second countable group, $\mu = ds$ is the left invariant Haar measure on G.
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- G is a locally compact second countable group, $\mu = ds$ is the left invariant Haar measure on G.
- $A(G)$ (resp. $B(G)$) is the Fourier (resp. the Fourier-Stieltjes) algebra of G.
- We say that $f : G \to \mathbb{C}$ belongs (resp. almost belongs) to $A(G)$ at $t \in G$ if $\exists$nbhd$U$ of t and $g \in A(G)$ s.t. $f(s) = g(s)$ everywhere (resp. a.e.) on U.
- If f belongs to $A(G)$ at each $t \in G$ then we say that f **locally belongs** to $A(G)$ ($f \in A(G)^{\rm loc}$).

  $A(G)^{\rm loc} \subseteq C(G)$ and if G is compact then $A(G)^{\rm loc} = A(G)$. In general,

$$A(G) \subseteq B(G) \subseteq A(G)^{\rm loc}.$$

For $f : G \to \mathbb{C}$ let $Nf : G \times G \to \mathbb{C}$ be given by $Nf(s,t) = f(st^{-1})$, function of Toeplitz type.

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Theorem (Bozejko)

Let G be amenable. If $f \in L^\infty(G)$ then Nf is a Schur multiplier iff $f \in^\mu B(G) = MA(G)$, i.e. $J_f = \{h \in A(G) : fh \in^\mu A(G)\} = A(G)$.

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Lemma

Let $E_f = \{t \in G : f$ does not almost belong to $A(G)$ at $t\}$.
Then $\mathrm{null}\, D(S^*_{Nf}) = E^*_f := \{(s,t) : st^{-1} \in E_f\}$

Theorem

Let $f : G \to \mathbb{C}$ be a measurable function and $\varphi = Nf$. TFAE
(i) $f \in^\mu A(G)^{\mathrm{loc}}$;
(ii) φ is a local Schur multiplier;
(iii) φ is a w-closable multiplier.*

Example

- Let $\Delta = \{(x, y) \in \mathbb{R}^2 : x \leq y\}$. Then $\chi_\Delta(x, y) = \chi_{(-\infty,0]}(x - y)$ is not a w*-closable multiplier, since $\chi_{(-\infty,0]}$ does not almost belong to $A(\mathbb{R})$ at $x = 0$.
- There exists non-closable extremely non-Schur multiplier φ, i.e. $D(S^*_\varphi) = \{0\}$.
  In fact, $\forall S \subset \mathbb{T}$, $\mu(S) = 0$, $\exists f \in C(\mathbb{T})$, whose Fourier series diverges $\forall t \in S$. Choose S to be dense in $\mathbb{T}$. Then $\mathbb{T} = E_f$ and hence null $D(S^*_{Nf}) = \mathbb{T}^2$.

M-sets in harmonic analysis and closability of Toeplitz type multipliers

Assume G is a compact abelian, so that $\hat{G}$ is discrete. Then

$$A(G) = \{\sum_{\chi \in \Gamma} c_\chi \chi : \sum_{\chi \in \Gamma} |c_\chi| < \infty\}.$$

The space of pseudomeasures $PM(G) = A(G)^*$ identified with $\ell^\infty(\hat{G})$ via Fourier transform: $F \to \{\hat{F}(\chi)\}_{\chi \in \hat{G}}$.

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The support supp F of $F \in PM(G)$ is the set
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$$PM(E) = \{F \in PM(G) : \text{supp } F \subset E\},$$
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$PM(E)$ (resp. $N(E)$) is the largest (resp. smallest) weak* closed subspace the support of whose every element is in E.

- A closed set $E \subseteq G$ is called an M-**set** (resp. an M_1-**set**) if $PM(E)$ (resp. $N(E)$) contains a non-zero pseudofunction. $\exists$ an M-set which is not an M_1-set.
- Recall that $E \subseteq G \times G$ an *operator* M*-set* (respectively, *operator* M_1*-set*) if E supports a non-zero compact operator (resp. $\mathfrak{M}_{\min}(E)$ contains a non-zero compact operator).

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For $E \subseteq G$, we let $E^* = \{(s, t) \in G \times G : ts^{-1} \in E\}$.

Theorem

Let $E \subseteq G$ be a closed set.

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Theorem holds for amenable groups if we define M-set (resp. M_1-set) as such $E \subset G$ that E supports a non-zero $a \in C_r^*(G) \subset VN(G)$ (resp. $\exists a \neq 0$ in $C_r^*(G)$, $a \in I(E)^\perp$).

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Corollary

Let $f : G \to \mathbb{C}$ be a measurable function and $\varphi = Nf$. Then
(i) If E_f is not an M-set then φ is closable.
(ii) If E_f is an M_1-set then φ is not closable.

- Every closed subset E of an amenable group of positive Haar measure is an M_1-set, as $\chi_E d\mu$ is a non-zero measure supported in E.
- One point-set is not an M-set for non-discrete amenable group: as for $E = \{e\}$, $E^* = \{(x, x) : x \in G\}$, and the last is not an operator M-set.
- Any countable closed subset of a non-discrete amenable group is not an M-set.

THANK YOU!