Truncated Toeplitz Operators - what do they look like?

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- It is not too difficult, (and classical) to see that any isometry T on a Hilbert space $\mathcal{H}$ such that $\cap_{n\geq 0} T^n(\mathcal{H}) = \{0\}$ (i.e. pure isometry) is unitarily equivalent to a shift on $\ell^2(\mathbb{N}, (T\mathcal{H})^\perp)$ (one sees that $W(k_0, k_1, \cdots) = \sum_{n=0}^{\infty} T^n k_n$ is a unitary operator implementing this equivalence).

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- And, in fact, every isometry is (uniquely) the direct sum of a pure isometry and a unitary operator(see Fillmore AMM1974).

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- Around the same time Nagy and Foias used the operator A_z^θ in their functional calculus. In fact, contractions with defect number one and their commutants are unitarily equivalent to truncated Toeplitz operators.

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But, putting in more points makes a difference!

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- However, the best to be hoped for is that every *complex symmetric matrix or operator* be *unitarily equivalent* to a truncated Toeplitz.

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- Examples of complex symmetric operators include not only these, but also all operators of rank one, all 2×2 matrices (but not at all the 3×3) all normal matrices, all inflations of Toeplitz matrices T (an inflation of T is $T \otimes I_n$ with $n \in \mathbb{N} \cup \{\infty\}$)

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- Which leads us to the question of whether inflations of Toeplitz matrices are unitarily equivalent to a truncated Toeplitz. (Asked by CGRW).

Definitions

- We recall that the tensor product of two Hilbert spaces X and $X^{'}$ is defined by setting
$$< x \otimes x^{'}, y \otimes y^{'} >=< x, y >< x^{'}, y^{'} >$$
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- We also recall, that, if (e_i) is a basis for X and $(e_i^{'})$ a basis for $X^{'}$ then $(e_i \otimes e_j^{'})$ is a basis for $X \otimes X^{'}$ and that, if dim(X) = n, then we can think of $X \otimes X^{'}$ as n copies of X.

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- If $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(X^{'})$ then $(T \otimes S)(x \otimes x^{'}) = (Tx \otimes Sx^{'})$ and, if $T = [a_{ij}]$ then we can think of $T \otimes S$ as $[a_{ij}S]$.

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- If $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(X^{'})$ then $(T \otimes S)(x \otimes x^{'}) = (Tx \otimes Sx^{'})$ and, if $T = [a_{ij}]$ then we can think of $T \otimes S$ as $[a_{ij}S]$.
- This means that the inflation $T \otimes I_n$ can be written
$$\begin{pmatrix} T & 0 & 0 & \cdots \\ 0 & T & 0 & \cdots \\ 0 & 0 & \ddots & \cdots \end{pmatrix}$$

Our unitary operator

Proposition: Suppose B is an inner function. Then the formula

$$h \otimes f \mapsto h(f \circ B) \tag{3}$$

defined for $h \in K_B$, $f \in L^\infty$, can be extended linearly to a unitary operator Ω_B from $K_B \otimes L^2$ onto L^2. The operator Ω_B maps $K_B \otimes H^2$ onto H^2.

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$$\Omega_B(K_B \otimes \Theta H^2) = (\Theta \circ B)H^2, \qquad \Omega_B(K_B \otimes K_\Theta) = K_{\Theta \circ B}.$$

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- Suppose B, Θ are inner functions, $\psi, \phi \in L^2$, and the operators $A^B_{\bar{B}^j \psi}$ are nonzero only for a finite number of $j \in \mathbb{Z}$. Then

$$A^{\Theta \circ B}_{\psi(\phi \circ B)} \omega_B = \omega_B \left(\sum_j (A^B_{\bar{B}^j \psi} \otimes A^\Theta_{z^j \phi}) \right). \tag{4}$$

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- If Θ and B are inner functions, $\phi \in L^2$ such that A_ϕ^Θ is bounded and $\dim K_B = k$ then $A_{\phi \circ B}^{\Theta \circ B}$ is unitarily equivalent to $I_k \otimes A_\phi^\Theta$.

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We were also able to show that the analogy of a motivating result from Carl Cowen was true for truncated Toeplitz:
Suppose Θ an inner function and $\phi \in L^2$, such that A_ϕ^Θ is bounded. If B_1, B_2 are two inner functions with $\dim K_{B_1} = \dim K_{B_2}$, then $A_{\phi\circ B_1}^{\Theta\circ B_1}$ and $A_{\phi\circ B_2}^{\Theta\circ B_2}$ are bounded and unitarily equivalent.

Nice looking block Toeplitz

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Example 1

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- If θ is any inner function and $\phi_0, \phi_1, \cdots \phi_n$ are chosen so that the appropriate operators are bounded then

$$\begin{pmatrix} A^{\Theta}_{\phi_0} & A^{\Theta}_{z\phi_{n-1}} & \dots & A^{\Theta}_{z\phi_1} \\ A^{\Theta}_{\phi_1} & A^{\Theta}_{\phi_0} & \dots & A^{\Theta}_{z\phi_2} \\ \ddots & \ddots & \ddots & \ddots \\ A^{\Theta}_{\phi_{n-1}} & A^{\Theta}_{\phi_{n-2}} & \dots & A^{\Theta}_{\phi_0} \end{pmatrix}$$

  is unitarily equivalent to $A^{\theta \circ B}_{\sum_{m=0}^{n-1} z^m(\phi_m(z^n))}$

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And if B is any inner functions and $\psi_i \in K_B$ $(-n \leq i \leq n-1)$ such that $A^B_{\psi_i}$ is bounded then

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is unitarily equivalent to $A^{B^n}_{\sum_{m=-n}^{n-1}\psi_m(z)B^m}$.

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$$\begin{pmatrix} a_0 & a_{-1} & \cdots & a_{-(n-1)} & 0 & 0 & 0 & 0 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & 0 & 0 & \cdots & \cdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & a_0 & a_{-1} & \cdots & a_{-(n-1)} & 0 & \cdots \\ 0 & 0 & \cdots & 0 & a_1 & a_0 & \ddots & a_{-(n-2)} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

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and rearranging it into a Toeplitz matrix:

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by making the basis change

$(e_{11}, e_{12}, \cdots, e_{1n}, e_{21}, \cdots, e_{mn}) \rightarrow (e_{11}, e_{21}, \cdots, e_{n1}, e_{12}, e_{22}, \cdots, e_{mn})$

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$(e_{11}, e_{12}, \cdots, e_{1n}, e_{21}, \cdots, e_{mn}) \to (e_{11}, e_{21}, \cdots, e_{n1}, e_{12}, e_{22}, \cdots, e_{mn})$ which, since $e_{ij} = z^{i-1} \otimes z^{j-1}$, is indeed done by the map ω_{z^n} that sends $z^i \otimes z^j$ to $z^i(z^j \circ z^n)$

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