Homomorphisms of Convolution Algebras

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The Homomorphism Problem

Describe all (bounded) homomorphisms

$$\varphi : L^1(F) \to M(G).$$

- 1950's Helson, Rudin, Wendel, Glicksberg and many others.
- Paul Cohen, 1960 When F and G are abelian he gave a complete characterization of all homomorphisms

$$\varphi : L^1(F) \cong A(\widehat{F}) \to M(G) \cong B(\widehat{G}).$$

When F or G is nonabelian, there are two homomorphism problems:

- Describe all homoms $\varphi : L^1(F) \to M(G)$.
- Describe all homoms $\varphi : A(F) \to B(G)$.

Homomorphisms $\varphi : A(F) \to B(G)$?

- Walter 72: $A(F) \cong A(G)$, $B(F) \cong B(G)$.
- Lau-Losert 93 $B_r(F) \cong B_r(G)$, $UCB(\widehat{F})^* \cong UCB(\widehat{G})^* \Leftrightarrow F \cong G$.
- Host 86 F virtually abelian, $\varphi : A(F) \to B(G)$.
- Ilie 04, Ilie-Spronk 05 F amenable $\varphi : A(F) \to B(G)$ cb.
- Ilie-S 2008 F amenable, $\varphi : B(F) \to B(G)$ cb, wk*-continuous.
- Ilie-S 2009 F amenable, $\varphi : \mathfrak{X}(F)^* \to \mathfrak{X}_G^*$ cb, wk*-conts extensions of $\varphi : A(F) \to B(G)$.
- Pham 2010 $\varphi : A(F) \to B(G)$ contractive.

The above results all perfectly extend or complement Cohen's original theorem.

Homomorphisms $\varphi : L^1(F) \to M(G)$?

- 50's, 60's, 70's: Helson, Rudin, Glicksberg, Cohen, Greenleaf and many others.
- Wendel '52 Isomorphisms $\varphi : L^1(F) \cong L^1(G)$
- Johnson 64, Strichartz 65 Isomorphisms $\varphi : M(F) \cong M(G)$ (Determined by $\alpha \in \widehat{F}^1$ and a top isomorphism $\phi : F \cong G$).
- Gharamani, Lau, Losert, Mckennon, McClure, Dales, Strauss, 80's, 90's, 00's
  $LUC(F)^* \cong LUC(G)^*,\ L^1((F)^{**} \cong L^1(G)^{**},$
  $M(F)^{**} \cong M(G)^{**}$ if and only if $F \cong G$.
- Kalton-Wood 76 Isomorphisms $\varphi : L^1(F) \to L^1(G)$ with $\|\varphi\| \leq \sqrt{2}$. (They are isometric.)
- Wood 70's, 80's, 90's, 00's More on "small" isomorphisms.

Homomorphisms $\varphi : L^1(F) \to M(G)$?

Let F and G be arbitrary locally compact groups.

- Greenleaf 65 Contractive homomorphisms $\varphi : L^1(F) \to M(G)$.
- Greenleaf's characterization is less tractable than Cohen's.
- Since 1965, no progress has been made in the non-contractive, non-isomorphic case.

Notation **Throughout, F, G and H are lcg's.**

- The Eberlein algebra of G is

$$E(G) = \text{uniform closure of } B(G);$$

$B(G)$ is the Fourier-Stieltjes algebra of G:

$$B(G) = \{\xi *_\pi \eta : \{\pi, \mathcal{H}_\pi\} \in \Sigma(G)\ \xi, \eta \in \mathcal{H}_\pi\}$$

where

$$\xi *_\pi \eta(s) = \langle \pi(s)\xi | \eta \rangle \qquad (s \in G).$$

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- $A(G) \subseteq C_0(G) \subseteq E(G) \subseteq WAP(G) \subseteq LUC(G) \subseteq CB(G)$.
- $C_0(G)^* = M(G)$ is an involutive Banach algebra
- $E(G)^*$ is an involutive Banach algebra wrt Arens product

$$(n * m)(f) = n(m \cdot f); \;\; m \cdot f(s) = m(f \cdot s); \;\; f \cdot s = l_s f \;\; (s \in G).$$

- $E(G)^* = M(G) \oplus_1 C_0(G)^\perp$.

Basic Homomorphisms

If

$$\varphi : L^1(F) \to M(G)$$

is a contractive homom, Cohen's theorem says that φ factors into a product of four basic homomorphisms with each factor falling into one of three types:

- Let $\alpha \in \widehat{F}^1$, $M_\alpha : C_0(F) \to C_0(F) : f \mapsto \alpha f$,

  $$A_\alpha = M_\alpha^* : M(F) \to M(F) : \mu \mapsto \alpha\mu$$

  is an isometric isomorphism.

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  is an isometric isomorphism.

- For K a compact normal subgroup of H,

$$S_K : C_0(H) \to C_0(H/K), \quad S_K f(xK) = \int_K f(xk)\,dm_K(k),$$

$$S_K^* : M(H/K) \hookrightarrow M(H)$$

  is an isometric homomorphic embedding such that

$$S_K^*(\delta_{xK}) = \delta_x * m_K.$$

Basic Homomorphisms

- Let $\theta : F \to H$ be a continuous homomorphism,

$$j_\theta : C_0(H) \to E(F) : f \mapsto f \circ \theta,$$

$$j_\theta^* : E(F)^* \to M(H).$$

Recall $E(F)^* = M(F) \oplus_1 C_0(F)^\perp.$
I will abuse notation and also write

$$j_\theta^* = j_\theta^*|_{M(F)} : M(F) \to M(H).$$

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$j_\theta^* : E(F)^* \to M(H)$ and $j_\theta^* : M(F) \to M(H)$ are the unique $w^* - w^*$ conts and $so - w^*$ conts, contractive, positive homomorphisms satisfying

$$\varphi(\delta_x) = \delta_{\theta(x)} \qquad (x \in F).$$

Basic Homomorphisms

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- Let H be a closed subgroup of G,
  $R_H : C_0(G) \to C_0(H) : f \mapsto f|_H$.

$$R_H^* : M(H) \hookrightarrow M(G)$$

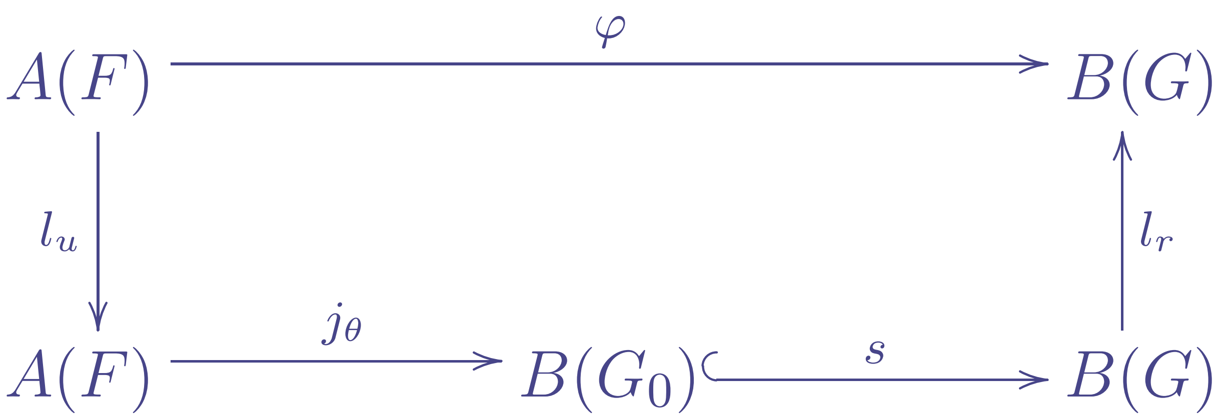
  is an isometric homomorphic embedding.

Cohen's theorem (contractive case)

Let F and G be LCA groups, $\varphi : A(F) \to B(G)$ a contractive homomorphism. Then there exist

- $u \in F$ and $r \in G$;
- an open subgroup G_0 of G ; and
- a continuous homomorphism $\theta : G_0 \to F$

such that $\varphi = l_r \circ s \circ j_\theta \circ l_u.$



$l_r u(s) = u(rs)$; $j_\theta u(s) = u(\theta(s))$; $su(s) = u(s)$ on G_0, 0 off G_0

Cohen's theorem (contractive case, dual version)

Using $A(\widehat{G}) \cong L^1(G)$ and $B(\widehat{G}) \cong M(G)$:
Let F and G be LCA groups, $\varphi : L^1(F) \to M(G)$ a contractive homom. Then there exist

- $\alpha \in \widehat{F}^1$ and $\rho \in \widehat{G}^1$;
- a compact subgroup K of G ; and
- a continuous homomorphism $\theta : F \to G/K$

such that $\quad \varphi = A_\rho \circ S_K^* \circ j_\theta^* \circ A_\alpha.$

$$\begin{array}{ccccc} L^1(F) & & \xrightarrow{\quad\varphi\quad} & & M(G) \\ \downarrow{\scriptstyle A_\alpha} & & & & \uparrow{\scriptstyle A_\rho} \\ L^1(F) & \xrightarrow{\;j_\theta^*\;} & M(G/K) & \xhookrightarrow{\;S_K^*\;} & M(G) \end{array}$$

Open Problems

Question 1 Kerlin-Pepe, Pacific JM 1975

- Does every contractive homomorphism $\varphi : L^1(F) \to M(G)$ have a Cohen factorization?

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\begin{array}{ccccc}
L^1(F) & & \xrightarrow{\quad\quad\varphi\quad\quad} & & M(G) \\
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Note Greenleaf's characterization involves non-normal subgroups, non-closed subgroups, and maps that are not homomorphisms on their domains.

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Answer to 1 Yes if G abelian (Kerlin-Pepe).

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No in general (S).

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Answer to 1 Yes if G abelian (Kerlin-Pepe).
No in general (S).
Question 2 Is there a characterization of contractive homomorphisms that shares the spirit of Cohen's theorem.
Answer to 2 Yes (S).

A counter-example

Question Kerlin-Pepe, Pacific JM 1975

Does every contractive homomorphism $\varphi : L^1(F) \to M(G)$ have a Cohen factorization?

$$\begin{array}{ccccc}
L^1(F) & & \xrightarrow{\quad\quad\quad\varphi\quad\quad\quad} & & M(H) \\
\Big\downarrow {\scriptstyle A_\alpha} & & & & \Big\uparrow {\scriptstyle A_\rho} \\
L^1(F) & \xrightarrow{j_\theta^*} & M(G/K) & \xrightarrow{S_K^*} & M(G)
\end{array}$$

A counter-example

- Let $K \lhd F$ be compact, $\rho \in \widehat{K}^1$ such that

$$\ker \rho \lhd F \quad \text{and} \quad K/\ker \rho \subseteq Z(F/\ker \rho).$$

ρm_K is a central norm one idempotent ($\sim$ Greenleaf), so

$$\varphi : L^1(F) \to M(F) : f \mapsto f * \rho m_K$$

is a contractive homomorphism.

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is a contractive homomorphism.

- If φ has a Cohen factorization

$$\varphi = A_{\rho'} \circ S_L^* \circ j_\theta^* \circ A_\alpha,$$

then $L = K$ and $\rho' \in \widehat{F}^1$ is such that $\rho'|_K = \rho$.

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- So choose K and $\rho \in \widehat{K}^1$ as above such that ρ does not extend to a character on F:

  $F = SU_2(\mathbb{C})$. Then $Z(F) = \mathbb{Z}_2$ and $\rho : \mathbb{Z}_2 \to \mathbb{T} : t \mapsto t$ does not extend continuously to F (Grosser-Moskowitz).

Main theorem

Let

$$\theta_H : \mathbb{T} \times H \to H : (\alpha, x) \mapsto x;$$

$$\alpha_{\mathbb{T}} : \mathbb{T} \times H \to \mathbb{T} : (\alpha, x) \mapsto \alpha \quad \text{so} \quad \alpha_T \in \widehat{\mathbb{T} \times H}^1.$$

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Theorem(S) Let $\varphi : L^1(F) \to M(G)$ be a contractive homomorphism. Then there exists

- a closed subgroup H of G;
- a compact normal subgroup N of $\mathbb{T} \times H$; and
- a continuous homomorphism $\theta : F \to \mathbb{T} \times H/N$

such that

$$\varphi = j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} \circ S_N^* \circ j_\theta^* .$$

$$\begin{array}{ccccccc}
L^1(F) & & \xrightarrow{\quad\quad\varphi\quad\quad} & & M(H) & \hookrightarrow & M(G) \\
\Big\downarrow j_\theta^* & & & & \Big\uparrow j_{\theta_H}^* & & \\
M(\mathbb{T} \times H/N) & \overset{S_N^*}{\hookrightarrow} & M(\mathbb{T} \times H) & \xrightarrow{A_{\alpha_{\mathbb{T}}}} & M(\mathbb{T} \times H) & &
\end{array}$$

The converse holds.

Main theorem

Thm(S) Let $\varphi : L^1(F) \to M(G)$ be a contractive homom. $\exists$

- a closed subgroup H of G;
- a compact normal subgroup N of $\mathbb{T} \times H$; and
- a continuous homomorphism $\theta : F \to \mathbb{T} \times H/N$

such that $\varphi = j^*_{\theta_H} \circ A_{\alpha_\mathbb{T}} \circ S^*_N \circ j^*_\theta$.

$$\begin{array}{ccccc}
L^1(F) & \xrightarrow{\quad\quad\varphi\quad\quad} & & M(H) & \hookrightarrow M(G) \\
\downarrow{\scriptstyle j^*_\theta} & & & \uparrow{\scriptstyle j^*_{\theta_H}} & \\
M(\mathbb{T} \times H/N) & \xhookrightarrow{S^*_N} M(\mathbb{T} \times H) & \xrightarrow{A_{\alpha_\mathbb{T}}} & M(\mathbb{T} \times H) &
\end{array}$$

We may take $N = \Omega_\rho = \{(\rho(k), k) : k \in K\}$ for some $K \lhd H$, $\rho \in \widehat{K}^1$ such that $\ker \rho \lhd H$ and $K/\ker \rho \subseteq Z(H/\ker \rho)$.

First Tool: $*$-homomorphisms (not necessarily contractive)

- so = strict top on $M(F)$ wrt $L^1(F) \lhd M(F)$

$$\mu_i \to \mu \; so \quad \Leftrightarrow \quad \|f * \mu_i - f * \mu\|_1 \to 0 \;\; (f \in L^1(F)).$$

- Let $\iota_\phi \in M(G)$ be such that $\iota_\phi^2 = \iota_\phi$ and $\iota_\phi^* = \iota_\phi$,

$$\mathbb{M}_\phi = \{\mu \in M(G) : \mu^* * \mu = \mu * \mu^* = \iota_\phi, \; \mu * \iota_\phi = \mu\}$$

with rel. wk*-topology.

$\mathbb{M}_\phi$ is a semitopological group with conts inversion.

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with rel. wk*-topology. $\mathbb{M}_\phi$ is a semitop group with conts inversion.

Theorem (S) $\exists$ a 1-1 correspondence between:

- bounded $*$-homoms $\varphi : L^1(F) \to M(G)$;
- $so - w^*$ conts bounded $*$-homoms $\varphi_m : M(F) \to M(G)$;
- $w^* - w^*$ conts $*$-homoms $\varphi_\varepsilon : E(F)^* \to M(G)$;
- continuous, bounded homoms $\phi : F \to \mathbb{M}_\phi$.

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Second Tool: Contractive subgroups of $M(G)$

Let Γ be a subgroup of $(M(G), *)_{\|\cdot\|\leq 1}$ with wk*-topology.
Greenleaf's Theorem

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Greenleaf's Theorem

- There is $K \leq G$ compact and $\rho \in \widehat{K}^1$ such that

$$e_\Gamma = \rho m_K .$$

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Greenleaf's Theorem

- There is $K \leq G$ compact and $\rho \in \widehat{K}^1$ such that

$$e_\Gamma = \rho m_K.$$

- There is a subgroup Ω of $\mathbb{T} \times G$, a compact normal subgroup Ω_ρ of Ω and a continuous group isomorphism

$$\phi : \Omega/\Omega_\rho \to \Gamma = \Gamma_\Omega : (\alpha, t)\Omega_\rho \mapsto \alpha\delta_t * \rho m_K$$

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Theorem (S) The map

$$\phi : \Omega/\Omega_\rho \to \Gamma = \Gamma_\Omega : (\alpha, t)\Omega_\rho \mapsto \alpha\delta_t * \rho m_K$$

is a topological group isomorphism.

Let $\Gamma = \Gamma_\Omega$ be a contractive subgroup of $M(G)$.

Theorem (S) $\phi : \Omega/\Omega_\rho \to \Gamma = \Gamma_\Omega : (\alpha, t)\Omega_\rho \mapsto \alpha\delta_t * \rho m_K$ is a topological group isomorphism.

Corollary (S)

- Letting

$$H = \text{support}(\Gamma) = \overline{\bigcup \{\text{support}(\mu) : \mu \in \Gamma\}},$$

$$\Gamma_{\mathbb{T}\times H} = \{\alpha\delta_t * \rho m_K : (\alpha, t) \in \mathbb{T} \times H\} \cong \mathbb{T} \times H/\Omega_\rho$$

is a locally compact contractive subgroup of $M(G)$.

First factorization theorem

Let

$$\varphi : M(F) \to M(G)$$

be a $so - w^*$ conts contractive homomorphism. Let

$$\Gamma = \{\varphi(\delta_x) : x \in F\} \leq M(G)_{\|\cdot\| \leq 1}.$$

Letting $H = \mathrm{support}(\Gamma)$, get a topological isomorphism

$$\phi : \mathbb{T} \times H/\Omega_\rho \to \Gamma_{\mathbb{T}\times H} \supseteq \Gamma.$$

First factorization theorem

Let $\varphi : M(F) \to M(G)$ be a $so - w^*$ conts contractive homom.
Let $\quad \Gamma = \{\varphi(\delta_x) : x \in F\} \leq M(G)_{\|\cdot\|\leq 1}$.
Letting $H = \mathrm{support}(\Gamma)$, get a topological isomorphism

$$\phi : \mathbb{T} \times H/\Omega_\rho \to \Gamma_{\mathbb{T}\times H} \supseteq \Gamma \subset M(H).$$

Note

$$\phi : \mathbb{T} \times H/\Omega_\rho \to \Gamma_{\mathbb{T}\times H} \subseteq \mathbb{M}_\phi$$

where $\iota_\phi = \varphi(\delta_{e_F}) = \rho m_K \in M(H)$ and

$$\mathbb{M}_\phi = \{\mu \in M(H) : \mu^* * \mu = \mu * \mu^* = \iota_\phi,\ \mu * \iota_\phi = \mu\}.$$

This gives a $w^* - w^*$ continuous homomorphism

$$\kappa_\phi^* : M(\mathbb{T} \times H/\Omega_\rho) \to M(H) \hookrightarrow M(G).$$

First factorization theorem

Let $\varphi : M(F) \to M(G)$ be a $so - w^*$ contractive homom, $\Gamma = \{\varphi(\delta_x) : x \in F\} \leq M(G)_{\|\cdot\| \leq 1}$, $H = \mathrm{support}(\Gamma)$. Get

$\phi : \mathbb{T} \times H/\Omega_\rho \to \Gamma_{\mathbb{T}\times H} \supseteq \Gamma,\ \ \kappa_\phi^* : M(\mathbb{T} \times H/\Omega_\rho) \to M(H) \hookrightarrow M(G).$

Define a continuous homomorphism

$$\theta : F \to \mathbb{T} \times H/\Omega_\rho \ \text{ by } \ \theta(x) = \phi^{-1}(\varphi(\delta_x)).$$

$$F \xrightarrow{x \mapsto \varphi(\delta_x)} \Gamma_{\mathbb{T}\times H} \xrightarrow{\phi^{-1}} \mathbb{T} \times H/\Omega_\rho$$

This gives $so - w^*$ continuous homomorphism

$$j_\theta^* : M(F) \to M(\mathbb{T} \times H/\Omega_\rho).$$

First factorization theorem

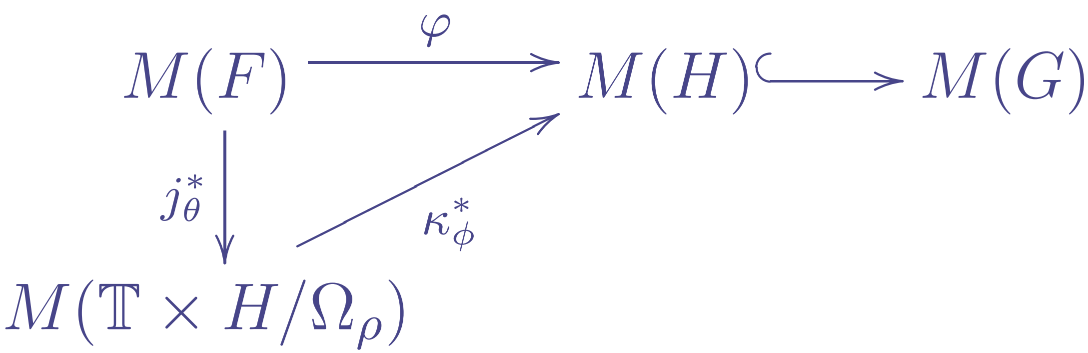
Let $\varphi : M(F) \to M(G)$ be a $so - w^*$ contractive homom.

$$\phi : \mathbb{T} \times H/\Omega_\rho \to \Gamma_{\mathbb{T}\times H}, \quad \kappa_\phi^* : M(\mathbb{T} \times H/\Omega_\rho) \to M(H) : \delta_z \mapsto \phi(z)$$

$$\theta : F \to \mathbb{T} \times H/\Omega_\rho : x \mapsto \phi^{-1}(\varphi(\delta_x))$$

$$j_\theta^* : M(F) \to M(\mathbb{T} \times H/\Omega_\rho) : \delta_x \mapsto \delta_{\theta(x)}.$$

Thm (S) $\varphi = \kappa_\phi^* \circ j_\theta^*$



Final Step: $\kappa_\phi^* = j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} \circ S_{\Omega_\rho}^*$

We have:

$$\begin{array}{ccccc}
M(F) & \xrightarrow{\ \varphi\ } & M(H) & \hookrightarrow & M(G) \\
\Big\downarrow{\scriptstyle j_\theta^*} & \nearrow{\scriptstyle \kappa_\phi^*} & & & \\
M(\mathbb{T} \times H/\Omega_\rho) & & & &
\end{array}$$

We need:

$$\begin{array}{ccccccc}
M(F) & & \xrightarrow{\qquad\qquad\ \varphi\ \qquad\qquad} & & M(H) & \hookrightarrow & M(G) \\
\Big\downarrow{\scriptstyle j_\theta^*} & & & & \Big\uparrow{\scriptstyle j_{\theta_H}^*} & & \\
M(\mathbb{T} \times H/\Omega_\rho) & \xrightarrow{S_{\Omega_\rho}^*} & M(\mathbb{T} \times H) & \xrightarrow{A_{\alpha_{\mathbb{T}}}} & M(\mathbb{T} \times H) & &
\end{array}$$

The proof can be completed by showing that

$$\kappa_\phi^* = j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} \circ S_{\Omega_\rho}^*$$

The Main Theorem

Theorem(S) Let $\varphi : L^1(F) \to M(G)$ be a contractive homomomorphism. Then there exists

- a closed subgroup H of G;
- a compact normal subgroup N of $\mathbb{T} \times H$; and
- a continuous homomorphism $\theta : F \to \mathbb{T} \times H/N$

such that $\varphi = j^*_{\theta_H} \circ A_{\alpha_{\mathbb{T}}} \circ S^*_N \circ j^*_\theta$.

$$
\begin{array}{ccccccc}
L^1(F) & & \xrightarrow{\quad\quad\varphi\quad\quad} & & M(H) & \hookrightarrow & M(G) \\
\Big\downarrow j^*_\theta & & & & \Big\uparrow j^*_{\theta_H} & & \\
M(\mathbb{T} \times H/N) & \xrightarrow{S^*_N} & M(\mathbb{T} \times H) & \xrightarrow{A_{\alpha_{\mathbb{T}}}} & M(\mathbb{T} \times H) & &
\end{array}
$$

The converse holds.

Theorem(S) Let $\varphi : M(F) \to M(G)$ be a w^* continuous contractive homomorphism. Then there exists

- a closed subgroup H of G;
- a compact normal subgroup N of $\mathbb{T} \times H$; and
- a continuous proper homomorphism $\theta : F \to \mathbb{T} \times H/N$

such that

$$\varphi = j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} \circ S_N^* \circ j_\theta^*.$$

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\end{array}$$

The converse holds.

Corollary (S) Let $\varphi : LUC(F)^* \to LUC(G)^*$. TFAE:

- φ is a $w^* - w^*$ continuous homomorphism such that
  - φ is contractive on Δ_F; and
  - $\varphi(\mu_0) \notin C_0(G)^\perp$ for some $\mu_0 \in M(F)$.
- φ has a canonical factorization

$$\varphi = j^*_{\theta_H} \circ A_{\alpha_{\mathbb{T}}} \circ S^*_N \circ j^*_\theta .$$

Recall that

$$LUC(G)^* = M(G) \oplus_1 C_0(G)^\perp$$

Some applications

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We may replace LUC by WAP or E.

Corollary (S) Let

$$\varphi : LUC(F)^* \to LUC(G)^* \quad (\text{or} \quad \varphi : M(F) \to M(G)).$$

TFAE:

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- there is a conts homom $\theta : F \to G$ and $\alpha \in \widehat{F}^1$ such that

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We may replace LUC by WAP or E.

Corollary (S) TFAE for

$$\varphi : LUC(F)^* \to LUC(G)^* :$$

- φ is a $w^* - w^*$ continuous isomorphism that is contractive on Δ_F.
- there is a topological isomorphism $\theta : F \to G$ and $\alpha \in \widehat{F}^1$ such that

$$\varphi = j_\theta^* \circ A_\alpha$$

Hence, φ is an isometric $*$-isomorphism mapping

- $M(F)$ as an isometric $*$-isomorphism onto $M(G)$;
- $L^1(F)$ as an isometric $*$-isomorphism onto $L^1(G)$.

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We may replace LUC by WAP or E.

Corollary (S) TFAE for

$$\varphi : L^1(F) \to L^1(G) :$$

- φ is a contractive epimorphism;
- there is a continuous open epimorphism $\theta : F \to G$ and $\alpha \in \widehat{F}^1$ such that

$$\varphi = j_\theta^* \circ A_\alpha$$