Noncommutative Hardy Algebras

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Outline

Introduction

We study tensor operator algebras and their ultraweak closures. Sometimes an operator algebra can be viewed usefully as an algebra of operator-valued functions on its space of representations. I claim that for the class of tensor algebras and their ultraweak closures this can be done in a way that justifies the view that the theory of these algebras is indeed "Non commutative function theory". The purpose of the talk is to try to convince you that this is indeed the case.

This raises the questions:

* What structure does the representation space (=the domain) have?
* What sort of functions are they?
* In what sense is this "Non commutative function theory"?

I will try to answer these questions in the talk.

The Setup

We begin with the following setup:

- M - a W^*-algebra.
- E - a W^*-correspondence over M. This means that E is a **bimodule** over M which is endowed with an M-**valued inner product** (making it a right-Hilbert C^*-module that is self dual). The left action of M on E is given by a unital, normal, *-homomorphism φ of M into the (W^*-) algebra of all bounded adjointable operators $\mathcal{L}(E)$ on E.

Examples

- (Basic Example) $M = \mathbb{C}$, $E = \mathbb{C}^d$, $d \geq 1$.
- $G = (G^0, G^1, r, s)$- a finite directed graph. $M = \ell^\infty(G^0)$, $E = \ell^\infty(G^1)$, $a\xi b(e) = a(r(e))\xi(e)b(s(e))$, $a, b \in M$,$\xi \in E$ $\langle \xi, \eta \rangle (v) = \sum_{s(e)=v} \overline{\xi(e)}\eta(e)$, $\xi, \eta \in E$.
- M- arbitrary , $\alpha : M \to M$ a normal unital, endomorphism. $E = M$ with right action by multiplication, left action by $\varphi = \alpha$ and inner product $\langle \xi, \eta \rangle := \xi^* \eta$. Denote it ${}_\alpha M$.
- Φ is a normal, contractive, CP map on M. $E = M \otimes_\Phi M$ is the completion of $M \otimes M$ with $\langle a \otimes b, c \otimes d \rangle = b^* \Phi(a^* c) d$ and $c(a \otimes b)d = ca \otimes bd$.

Note: If σ is a representation of M on H, $E \otimes_\sigma H$ is a Hilbert space with $\langle \xi_1 \otimes h_1, \xi_2 \otimes h_2 \rangle = \langle h_1, \sigma(\langle \xi_1, \xi_2 \rangle_E) h_2 \rangle_H$.

Similarly, given two correspondences E and F over M, we can form the (internal) tensor product $E \otimes F$ by setting

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle = \langle f_1, \varphi(\langle e_1, e_2 \rangle_E) f_2 \rangle_F$$

$$\varphi_{E\otimes F}(a)(e \otimes f)b = \varphi_E(a)e \otimes fb$$

and applying an appropriate completion.
In particular we get "tensor powers" $E^{\otimes k}$.

Also, given a sequence $\{E_k\}$ of correspondences over M, the direct sum $E_1 \oplus E_2 \oplus E_3 \oplus \cdots$ is also a correspondence (after an appropriate completion).

For a correspondence E over M we define the Fock correspondence

$$\mathcal{F}(E) := M \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \cdots$$

For every $a \in M$ define the operator $\varphi_\infty(a)$ on $\mathcal{F}(E)$ by

$$\varphi_\infty(a)(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_n$$

and $\varphi_\infty(a)b = ab$.
For $\xi \in E$, define the "shift" (or "creation") operator T_ξ by

$$T_\xi(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n.$$

and $T_\xi b = \xi b$. So that T_ξ maps $E^{\otimes k}$ into $E^{\otimes(k+1)}$.

Definition

(1) The norm-closed algebra generated by $\varphi_\infty(M)$ and $\{T_\xi : \xi \in E\}$ will be called the **tensor algebra** of E and denoted $\mathcal{T}_+(E)$.

(2) The ultra-weak closure of $\mathcal{T}_+(E)$ will be called the **Hardy algebra** of E and denoted $H^\infty(E)$.

Examples

1. If $M = E = \mathbb{C}$, $\mathcal{F}(E) = \ell^2$, $\mathcal{T}_+(E) = A(\mathbb{D})$ and $H^\infty(E) = H^\infty(\mathbb{D})$.
2. If $M = \mathbb{C}$ and $E = \mathbb{C}^d$ then $\mathcal{F}(E) = \ell^2(\mathbb{F}_d^+)$, $\mathcal{T}_+(E)$ is Popescu's $\mathcal{A}_d$ and $H^\infty(E)$ is F_d^∞ (Popescu) or $\mathcal{L}_d$ (Davidson-Pitts). These algebras are generated by d shifts.

Representations

To specify a representation π of the tensor algebra on a Hilbert space H you need to specify $\pi(\varphi_\infty(a))$ (for all $a \in M$) and $\pi(T_\xi)$ for $\xi \in E$. We do it by writing $\sigma(a)$ for $\pi(\varphi_\infty(a))$ and $T(\xi)$ for $\pi(T_\xi)$. Note that

$$\varphi_\infty(a) T_\xi \varphi_\infty(b) = T_{\varphi(a)\xi b}$$

and, thus, we should have

$$\sigma(a) T(\xi) \sigma(b) = T(\varphi(a)\xi b).$$

That means that T (or, rather, the pair (T, σ)) is a bimodule map from E to $B(H)$.
In the purely algebraic setting the representations of the Tensor algebra are indeed given by bimodule maps.
Here

Definition

Let E be a W^*-correspondence over a von Neumann algebra M. Then:

a **completely contractive covariant representation** of E on a Hilbert space H is a pair (T, σ), where

1. σ is a normal $*$-representation of M in $B(H)$.
2. T is a linear, completely contractive map from E to $B(H)$ that is continuous in the σ-topology on E and the ultraweak topology on $B(H)$.
3. T is a bimodule map in the sense that $T(\varphi(a)\xi b) = \sigma(a)T(\xi)\sigma(b)$, $\xi \in E$, and $a, b \in M$.

Thus, given a completely contractive representation π of the tensor algebra whose restriction to $\varphi_\infty(M)$ is normal, we get a c.c.c. representation of E. The converse also holds but requires more work to prove.

Theorem

Given a c.c. representation (T, σ) of E on H, there is a unique c.c. representation of $\mathcal{T}_+(E)$ on H, written $T \times \sigma$, such that

$$T \times \sigma(\varphi_\infty(a)) = \sigma(a), \quad a \in M$$

$$T \times \sigma(T_\xi) = T(\xi), \quad \xi \in E.$$

Conversely, every c.c. representation of $\mathcal{T}_+(E)$ is of this form.

Note: Given a c.c. representation (T, σ) on H, one can define a contraction

$$\tilde{T} : E \otimes_\sigma H \to H$$

by $\tilde{T}(\xi \otimes h) = T(\xi)h$ and $\tilde{T}$ has the intertwining property

$$\tilde{T}(\varphi(a) \otimes I_H) = \sigma(a)\tilde{T}, \quad a \in M.$$

Conversely: Every such map defines a representation of E whose restriction to M is σ.

The σ-dual E^σ

Recall: $\tilde{T} : E \otimes_\sigma H \to H$ and $\tilde{T}(\varphi(a) \otimes I_H) = \sigma(a)\tilde{T}$, $a \in M$.

Corollary

The c.c. representations of $\mathcal{T}_+(E)$ whose restriction to M is σ are parameterized by maps $\eta : H \to E \otimes_\sigma H$ with $\|\eta\| \leq 1$ that satisfy

$$(\varphi(a) \otimes I_H)\eta = \eta\sigma(a), \quad a \in M.$$

Note: $\eta = \tilde{T}^*$. The reason to look at the adjoint maps is:

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Lemma

*The set of all the maps $\eta : H \to E \otimes_\sigma H$ with the above intertwining property is a W^*-**correspondence** over $\sigma(M)'$ with inner product $\langle \eta, \zeta \rangle = \eta^*\zeta$ and bimodule structure $b \cdot \eta c = (I \otimes b)\eta c$ $(b, c \in \sigma(M)')$. Write E^σ for it and call it **the** σ-**dual of** E.*

Corollary

The closed unit ball, $\overline{\mathbb{D}(E^\sigma)}$ (or $\overline{\mathbb{D}(E^\sigma)^*}$) parameterizes the c.c. representations of $\mathcal{T}_+(E)$ whose restriction to $\varphi_\infty(M)$ is given by σ. The representation associated with η^* is denoted $\eta^* \times \sigma$.

Examples

(1) $M = E = \mathbb{C}$. So $\mathcal{T}_+(E) = A(\mathbb{D})$, σ is on H and $E^\sigma = B(H)$.

(2) $M = \mathbb{C}$, $E = \mathbb{C}^d$. $\mathcal{T}_+(E) = \mathcal{A}_d$ (Popescu's algebra) and $E^\sigma = (B(H))^{(d)} : H \to \mathbb{C}^d \otimes H$. c.c. representations are parameterized by row contractions $(T_1, \ldots, T_d)$.

(3) M general, $E =_\alpha M$ for an automorphism α. $\mathcal{T}_+(E) =$ the analytic crossed product.
$E^\sigma = \{X \in B(H) : \ \sigma(\alpha(T))X = X\sigma(T), T \in B(H)\}$.

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Recall :

When $M = E = \mathbb{C}$, $H^\infty(E) = H^\infty(\mathbb{D})$. If σ is on H, $E^\sigma = B(H)$ so $\eta \in B(H)$ and $\|\eta\| \leq 1$. Then η defines a c.c. representation of $A(\mathbb{D})$ and the question is: When does it have an H^∞-functional calculus? **The answer:** (well known) if and only if the spectral measure of the unitary part of the operator is absolutely continuous with respect to Lebesgue measure.

Remarks :

(i) **Induced representations**: Fix a normal representation π of M on K, let $H = \mathcal{F}(E) \otimes_\pi K$ and define the representation of E by $\sigma(a) = \varphi_\infty(a) \otimes I_K$, $T(\xi) = T_\xi \otimes I_K$. This extends to an ultra weakly continuous representation of $H^\infty(E)$ by $X \mapsto X \otimes I_K$.

Remarks :

(i) **Induced representations**: Fix a normal representation π of M on K, let $H = \mathcal{F}(E) \otimes_\pi K$ and define the representation of E by $\sigma(a) = \varphi_\infty(a) \otimes I_K$, $T(\xi) = T_\xi \otimes I_K$. This extends to an ultra weakly continuous representation of $H^\infty(E)$ by $X \mapsto X \otimes I_K$.

(ii) If $\|\tilde{T}\| = \|\eta\| < 1$, the representation can be dilated to an induced representation. (That is, it is a compression of an induced representation). Thus extends.

The problem is with η on the boundary.

Intertwiners and superharmonic elements

Fix a faithful normal representation π_0 of M of infinite multiplicity on K_0 and write (S_0, σ_0) the associated induced representation (on $H_0 = \mathcal{F}(E) \otimes_{\pi_0} K_0$). For a given $\eta \in \overline{\mathbb{D}(E^\sigma)}$ (corresponding to the representation (T, σ) on H) write $\mathcal{I}(S_0, \eta)$ for the space of intertwiners:

$$\{C : H_0 \to H : CS_0(\xi) = T(\xi)C,\ C\sigma_0(a) = \sigma(a)C, \xi \in E,\ a \in M\}.$$

Also, define the completely positive map associated with η, $\Phi_\eta : \sigma(M)' \to \sigma(M)'$ by

$$\Phi_\eta(b) = \eta^*(I_E \otimes b)\eta = \tilde{T}(I_E \otimes b)\tilde{T}^*.$$

Note: If $C \in \mathcal{I}(S_0, \eta)$, then $Q = CC^*$ lies in $\sigma(M)'$ and satisfies

* $Q \geq 0$ and $\Phi_\eta(Q) \leq Q$, and
* $\Phi_\eta^n(Q) \to 0$ ultra weakly.

Such an element will be said to be *pure superharmonic* for Φ_η.

Absolute Continuity

Theorem

Let $T \times \sigma$ be a c.c. representation of $\mathcal{T}_+(E)$ on H and write $\eta = \tilde{T}^*$ for the element of $\overline{\mathbb{D}(E^\sigma)}$ associated with it. Then the following are equivalent.

(1) *The representation $T \times \sigma$ extends to a c.c. ultra weakly continuous representation of $H^\infty(E)$.*

(2) $H = \bigvee\{Ran(C) : C \in \mathcal{I}(S_0, \eta)\}$.

(3) $H = \bigvee\{Ran(Q) : Q$ *is pure superharmonic for* $\Phi_\eta\}$

Arguments and partial results: Douglas (69), Davidson-Li-Pitts (05).

This theorem describes the representations of $H^\infty(E)$, whose restriction to M is given (equals σ), as a subset $AC(E^\sigma)$ of the ball $\overline{\mathbb{D}(E^\sigma)}$ containing $\mathbb{D}(E^\sigma)$. Recall that E^σ is a W^*-correspondence.

The functions

Conclusion: We now view the elements of $H^\infty(E)$ as functions ($B(H)$-valued) on $AC(E^\sigma)^*$ or on $\mathbb{D}(E^\sigma)^*$. For $X \in H^\infty(E)$, we write $\widehat{X}$ for the resulting function. Thus

$$\widehat{X}(\eta^*) = (\eta^* \times \sigma)(X).$$

Note: This presentation depends on σ. In fact, for every σ, X defines a function on $AC(E^\sigma)^*$. The relation between the functions (defined by the same X) for two different σ will be discussed later (if time permits). Here we deal with a fixed σ.
Note: The transform $X \mapsto \widehat{X}$ is not always injective (it depends on σ) but for σ_0 it is.
What is the image of this transform? What functions do we get?

Schur class operator functions

The classical Schur class $\mathcal{S}$ consists of the functions f in $H^\infty(\mathbb{D})$ with $\|f\| \leq 1$. The operator valued Schur class $\mathcal{S}(H)$ consists of analytic functions S on $\mathbb{D}$ with $\|S(z)\| \leq 1$ for all $z \in \mathbb{D}$. They have several characterizations. The following is well known.

Theorem

For an $B(H)$-valued function S on $\mathbb{D}$ TFAE:

(1) $S \in \mathcal{S}(H)$.

(2) *There is a Hilbert space $\mathcal{E}$ and a coisometric operator*

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{E} \\ H \end{pmatrix} \to \begin{pmatrix} \mathcal{E} \\ H \end{pmatrix}$$

so that S can be realized

$$S(z) = D + zC(I_{\mathcal{E}} - zA)^{-1}B.$$

(3)

$$K_S(z,w) = \frac{I - S(z)S(w)^*}{1 - z\overline{w}}$$

is a positive kernel on $\mathbb{D} \times \mathbb{D}$ (into $B(H)$).

Theorem

Let E be a W^*-correspondence over M, σ a faithful normal representation of M on H and $Z : \mathbb{D}(E^\sigma)^* \to B(H)$. Then $Z = \hat{X}$ for some $X \in H^\infty(E)$ with $\|X\| \leq 1$ if and only if there is a Hilbert space $\mathcal{E}$, a normal representation τ of $\sigma(M)'$ on $\mathcal{E}$ and a coisometric operator matrix

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{E} \\ H \end{pmatrix} \to \begin{pmatrix} E^\sigma \otimes_\tau \mathcal{E} \\ H \end{pmatrix}$$

(with A, B, C, D module maps) so that Z can be realized

$$Z(\eta^*) = D + C(I_{\mathcal{E}} - L_\eta^* A)^{-1} L_\eta^* B.$$

Here $L_\eta : \mathcal{E} \to E^\sigma \otimes_\tau \mathcal{E}$ is defined by $L_\eta h = \eta \otimes h$.

Theorem

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$$K_Z(\eta^*, \zeta^*) = (id - Ad(Z(\eta^*), Z(\zeta^*)) \circ (id - \theta_{\eta,\zeta})^{-1}.$$

Here $Ad(Z(\eta^*), Z(\zeta^*))(a) = Z(\eta^*)aZ(\zeta^*)^*$ and $\theta_{\eta,\zeta}(a) = \langle \eta, a\zeta \rangle$ for $a \in \sigma(M)'$. The complete positivity of K_Z means that, for every $\eta_1, \ldots, \eta_m \in \mathbb{D}(E^\sigma)$, the matrix of maps $(K_Z(\eta_i^*, \eta_j^*))$ defines a completely positive map from $M_m(\sigma(M)')$ into $M_m(B(H))$.

• Functions Z satisfying the condition above will be called **Schur class operator functions**. Thus **every $\widehat{X}$ is a multiple of a Schur class operator function**.

Noncommutative Nevanlinna-Pick interpolation

The classical NP theorem

Question: Given $z_1, \ldots, z_m$ in $\mathbb{D}$ and $w_1, \ldots, w_m$ in $\mathbb{C}$, when can one find a function $f \in H^\infty(\mathbb{D})$ with $\|f\| \leq 1$ and $f(z_i) = w_i$ for all i?

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Answer: If and only if the $m \times m$ matrix

$$\left(\frac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}} \right)$$

is positive.

Our noncommutative analogue is the following.

Theorem

Let E be a W^*-correspondence over M, σ a faithful normal representation of M on H. For $\eta_1, \ldots, \eta_m \in \mathbb{D}(E^\sigma)$ and $D_1, \ldots, D_m \in B(H)$, one can find $X \in H^\infty(E)$ with $\|X\| \leq 1$ that satisfies $\widehat{X}(\eta_i^*) = D_i$ for all $i \leq m$ if and only if the $m \times m$ matrix of maps

$$((Id - Ad(D_i, D_j)) \circ (Id - \theta_{\eta_i, \eta_j})^{-1})$$

defines a completely positive map from $M_m(\sigma(M)')$ to $M_m(B(H))$.

For $M = \mathbb{C}$, $E = \mathbb{C}^n$: Popescu, Davidson-Pitts.
Question : What about $\eta_1, \ldots, \eta_m$ in $AC(\sigma)$?
In this case $(Id - \theta_{\eta_i, \eta_j})^{-1}$ is not defined.

Simple observation : If Φ, Ψ are positive maps such that $Id - \Phi$ is invertible then $(Id - \Psi) \circ (Id - \Phi)^{-1}$ is positive if and only if :

$$\{a \geq 0 : \Phi(a) \leq a\} \subseteq \{a \geq 0 : \Psi(a) \leq a\}.$$

The last statement makes sense even if $Id - \Phi$ is not invertible. It is related to the Lyapunov preorder studied in matrix theory. This was pointed out to us by Nir Cohen.
It leads to the following definition

Definition

Let B be a W^*-algebra and suppose A is a sub-W^*-algebra of B. Suppose $\Phi : A \to A$ is a completely positive map and that $\Psi : B \to B$ is also completely positive. Then we say Ψ *completely dominates* Φ *in the sense of Lyapunov* in case every pure superharmonic element of $M_n(A)$ for Φ_n is superharmonic for Ψ_n, where Φ_n (resp. Ψ_n) is the usual promotion of Φ (resp. Ψ) to $M_n(A)$ (resp. $M_n(B)$).

Theorem

Suppose E is a W^*-correspondence over a W^*-algebra M and that σ is a faithful normal representation of M on the Hilbert space H_σ. Given $\eta_1, \eta_2, \ldots, \eta_k \in AC(E^\sigma)$ and $W_1, W_2, \ldots, W_k \in B(H)$, Define the map Φ_η on $M_n(\sigma(M)')$ by the formula $\Phi_\eta((a_{ij})) = (\langle \eta_i, a_{ij} \cdot \eta_j \rangle)$ and define the map Φ_W on $M_n(B(H_\sigma))$ by the formula $\Phi_W((T_{ij})) := \left(W_i T_{ij} W_j^* \right)$. Then there is an element X in $H^\infty(E)$, with $\|X\| \leq 1$, such that $\widehat{X}(\eta_i^*) = W_i$, $i = 1, 2, \ldots, n$, if and only if Φ_W completely dominates Φ_η in the sense of Lyapunov.

Specializing to the case $E = \mathbb{C} = M$, we get

Theorem

Suppose $Z_1, Z_2, \cdots, Z_n$ are n distinct absolutely continuous contractions on a Hilbert space H and suppose $W_1, W_2, \cdots, W_n$ are n contractions on H, then there is a function $f \in H^\infty(\mathbb{T})$, of norm at most 1, such that $f(Z_i) = W_i$, $i = 1, 2, \ldots, n$, if and only if Φ_W completely dominates Φ_Z in the sense of Lyapunov.

Here

$$\Phi_W((T_{ij})) := \left(W_i T_{ij} W_j^*\right)$$

and

$$\Phi_Z((T_{ij})) := \left(Z_i T_{ij} Z_j^*\right)$$

for $T_{i,j} \in B(H)$.

Using the characterization of $AC(E^\sigma)$, we have

Theorem

Let E be a W^*-correspondence over the von Neumann algebra M and let σ be a faithful normal representation of M on H. Given $\eta_1, \eta_2, \ldots, \eta_k \in AC(E^\sigma)$ and $W_1, W_2, \ldots, W_k \in B(H)$, the following conditions are equivalent.

(1) *There is an element $X \in H^\infty(E)$ such that $\|X\| \leq 1$ and such that*

$$\widehat{X}(\eta_i^*) = W_i \ ,$$

$i = 1, 2, \ldots, k$.

(2) *For every $m \geq 1$, $i : \{1, \ldots, m\} \to \{1, \ldots, k\}$ and $C_1, C_2, \ldots, C_m$ with $C_j \in \mathcal{I}(S_0, \eta^*_{i(j)})$, we have*

$$(W_{i(l)} C_l C_j^* W_{i(j)}^*)_{l,j} \leq (C_l C_j^*)_{l,j}.$$

About the proof of the NP Theorem : The proof follows the general lines of Sarason's proof (1967) of the classical NP Theorem. The main two ingredients used are:

(1) The Commutant Lifting Theorem for c.c. representations: Every operator that commutes with the image of the representation can be lifted to an operator that commutes with the image of the minimal isometric dilation without increasing the norm.

(2) Identification of the commutant.

The commutant of an induced representation

Let E be a W^*-correspondence over the von Neumann algebra M and let σ be a faithful normal representation of M on H. The algebra $H^\infty(E)$ can be represented (completely isometrically, ultra-weakly homeomorphically) on $\mathcal{F}(E) \otimes_\sigma H$ by $X \mapsto X \otimes I_H$. Write $H^\infty(E) \otimes I_H$ for the image of this representation. Similarly, we can write $H^\infty(E^\sigma) \otimes I_H$ for the algebra $H^\infty(E^\sigma)$ represented on $\mathcal{F}(E^\sigma) \otimes_\iota H$ (where ι is the identity representation of $\sigma(M)'$ on H).

Theorem

The commutant of $H^\infty(E) \otimes I_H$ is unitarily isomorphic to $H^\infty(E^\sigma) \otimes I_H$.
Consequently (by duality), $(H^\infty(E) \otimes I_H)'' = H^\infty(E) \otimes I_H$.

The kernel $K(\sigma)$

Recall that, in general, for a given (faithful) representation σ of M, the map

$$X \in H^\infty(E) \mapsto \widehat{X}$$

may have a kernel. Write $K(\sigma)$ for the kernel. Thus $K(\sigma)$ is the set of all elements $X \in H^\infty(E)$ such that $\pi(X) = 0$ whenever π is a c.c. representation of $H^\infty(E)$ whose restriction to $\varphi_\infty(M)$ is σ. $K(\sigma)$ is an ideal and we can view the map above as a map defined on the quotient $H^\infty(E)/K(\sigma)$.

In fact, $K(\sigma)$ is invariant under the gauge group action and, if σ is of infinite multiplicity, $K(\sigma) = 0$.

Fact (Arveson): The map $[X] \in H^\infty(E)/K(\sigma) \mapsto \widehat{X}$ is not isometric (where the norm of $\widehat{X}$ is the supremum norm).

Question : What can we say about the quotients of $H^\infty(E)$?

Quotients

Given an ideal $J \subseteq H^\infty(E)$, we write $\mathfrak{M}(J) = \overline{span}\{J(\mathcal{F}(E))\}$ (an $H^\infty(E)$-invariant sub W^*-module) and $\mathfrak{N}(J) = \mathfrak{M}(J)^\perp$ (a co-invariant sub module). Write $P_{\mathfrak{N}(J)}$ for the projection onto $\mathfrak{N}(J)$.

Theorem

(Davidson-Pitts, M. Gurevich , J. Meyer) The operator algebra $H^\infty(E)/J$ is completely isometrically isomorphic to the algebra $P_{\mathfrak{N}(J)} H^\infty(E)|\mathfrak{N}(J) \subseteq \mathcal{L}(\mathfrak{N}(J))$.

Studying the submodules $\mathfrak{N}(J)$ and these "compressed" algebras (at least if J is invariant under the gauge group action, as is the case for $K(\sigma)$) leads to the notion of **subproduct systems**. But this can be the subject of another talk.

Varying σ

In the discussion above we fixed a normal representation σ and, for $X \in H^\infty(E)$, considered the function $\widehat{X}$ defined on $\mathbb{D}(E^\sigma)^*$ or on $AC(E^\sigma)^*$. **Now we let σ vary**. We fix M and E and write Σ for the set of all normal representations σ of M on some Hilbert space H_σ. For every $X \in H^\infty(E)$ and every $\sigma \in \Sigma$, we write $\widehat{X_\sigma}$ for the (Schur class operator) function associated to X on $AC(E^\sigma)^*$. We get **a family of operator valued functions** $\{\widehat{X_\sigma} : \sigma \in \Sigma\}$. Each function $\widehat{X_\sigma}$ is defined on a different domain, $AC(E^\sigma)^*$ or $\mathbb{D}(E^\sigma)^*$. The families of domains $\{AC(E^\sigma) : \sigma \in \Sigma\}$ and $\{\mathbb{D}(E^\sigma) : \sigma \in \Sigma\}$ satisfy the conditions of the following definition.

Definition

A family $\mathcal{A} = \{\mathcal{A}(\sigma) : \sigma \in \Sigma\}$ is said to be a fully matricial E-set if

(i) for each σ, $\mathcal{A}(\sigma) \subseteq E^\sigma$,

(ii) it is closed with respect to taking direct sums; that is, $\mathcal{A}(\sigma) \oplus \mathcal{A}(\tau) \subseteq \mathcal{A}(\sigma \oplus \tau)$ and

(iii) it is closed with respect to unitary similarity; that is, if $\eta \in \mathcal{A}(\sigma)$ and $u \in \sigma(M)'$ is a unitary then $u \cdot \eta \cdot u^* \in \mathcal{A}(\sigma)$.

This is related to the notion of fully matricial sets (Voiculescu) and non-commutative sets (Helton-Klep-McCullough).

Notation:

(1) We write $\mathcal{AC}$ for the family $\{\mathcal{AC}(\sigma) = AC(E^\sigma)\}$.

(2) For a positive number R, we write $R\mathcal{D}$ for the family $\{R\mathcal{D}(\sigma) = R\mathbb{D}(E^\sigma)\}$.

(3) Write $\mathcal{B}$ for the family $\{\mathcal{B}(\sigma) = B(H_\sigma)\}$.

Note that both $\mathcal{AC}$ and $R\mathcal{D}$ are fully matricial E-sets.

The following concept is related to Voiculescu's fully matricial functions and to the notion of a free map (Helton, Klep and McCullough).

Definition

(1) Given $\eta \in E^\sigma$ and $\zeta \in E^\tau$ (for some $\sigma, \tau \in \Sigma$) and $C : H_\sigma \to H_\tau$, we say that C **intertwines** η and ζ if $C\sigma(a) = \tau(a)C$ for all $a \in M$ and $C\eta^* = \zeta^*(I_E \otimes C)$ (as maps from $E \otimes_\sigma H_\sigma$ to H_τ). We denote by $\mathcal{I}(\eta^*, \zeta^*)$ the set of all these intertwiners. Note that such an intertwiner will also satisfies $C\eta_k^* = \zeta_k^*(I_E \otimes C)$ for all $k \geq 1$.

(2) A family $f = \{f_\sigma\}$ of maps $f_\sigma : \mathcal{A}(\sigma) \to B(H_\sigma)$ is said to **preserve intertwiners** if, whenever $C \in \mathcal{I}(\eta^*, \zeta^*)$, we have $Cf_\sigma(\eta) = f_\tau(\zeta)C$.

Theorem

For a family of maps $f = \{f_\sigma : \mathcal{AC}(\sigma) \to B(H_\sigma)\}$ the following are equivalent.

(1) *f preserves intertwiners.*

(2) *There is some $X \in H^\infty(E)$ such that, for all $\sigma \in \Sigma$ and all $\eta \in \mathcal{AC}(\sigma)$, $f_\sigma(\eta) = \widehat{X_\sigma}(\eta^*)$.*

If the domain is $\mathcal{D}$, the situation can be different.

Example

Let $M = E = \mathbb{C}$. In this case $H^\infty(E)$ is the classical $H^\infty(\mathbb{D})$. Note that the representations in Σ are just the obvious representations σ_n on a space H_n of dimension $n \leq \infty$. For such σ_n, $E^{\sigma_n} = B(H_n)$ and we set $f_{\sigma_n}(A) = (I - A)^{-1}$ for $A \in B(H_n)$ with $\|A\| < 1$.
If $A \in B(H_n)$, $B \in B(H_m)$, both of norm less than 1 and if $C : H_m \to H_n$ intertwines them; that is $AC = CB$, then it also intertwines $(I - A)^{-1}$ and $(I - B)^{-1}$. Thus the map $f = \{f_\sigma\}$ preserves intertwiners. But there is no function $h \in H^\infty(\mathbb{D})$ such that $h(A) = (I - A)^{-1}$ for all A with $\|A\| < 1$.

Note that, in the example above, there was no H^∞-function that does the job but there was a function, analytic on the open disc, that did it ($h(z) = \sum_{k=0}^{\infty} z^k$).

Consider families $\theta = \{\theta_k : 0 \leq k < \infty\}$ where $\theta_k \in E^{\otimes k}$. For such a family we define (following Popescu) $R(\theta)$ by

$$\frac{1}{R(\theta)} = \overline{\lim}_k \|\theta_k\|^{1/k} \tag{1}$$

and call it *the radius of convergence* of θ.
We now have the following characterization.

Theorem

For a map $f = \{f_\sigma\} : \mathcal{D} \to \mathcal{B}$, the following are equivalent.

(1) *f preserves intertwiners.*

(2) *There exists a sequence $\theta = \{\theta_k\}$ with $R(\theta) \geq 1$ such that, for every $\sigma \in \Sigma$ and $\eta \in \mathbb{D}(E^\sigma)$,*

$$f_\sigma(\eta) = \sum_{k=0}^{\infty} \eta_k^* L_{\theta_k}.$$

Definition

Let E and F be W^*-correspondences over M, let $\mathcal{A}$ be a fully matricial E-set and let $\mathcal{A}'$ be a fully matricial F-set. A family $f = \{f_\sigma\}_{\sigma \in \Sigma}$ of maps, with $f_\sigma : \mathcal{A}(\sigma) \to \mathcal{A}'(\sigma)$, is said to **preserve intertwiners** if, for all $\eta \in \mathcal{A}(\sigma)$ and $\zeta \in \mathcal{A}(\tau)$, $\mathcal{I}(\eta^*, \zeta^*) \subseteq \mathcal{I}(f_\sigma(\eta)^*, f_\tau(\zeta)^*)$.

Theorem

Suppose $f = \{f_\sigma\}_{\sigma \in \Sigma}$ is a family of maps from $\mathcal{AC}(E)$ to $\mathcal{AC}(F)$. Then f preserves intertwiners if and only if there is a σ-weakly continuous homomorphism $\alpha : H^\infty(F) \to H^\infty(E)$ such that for all $X \in H^\infty(F)$ and all $\eta \in \mathcal{AC}(E)(\sigma)$,

$$\widehat{\alpha(X)}(\eta^*) = \widehat{X}(f_\sigma(\eta)^*).$$

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Thank You !