More on Topological Algebras of Random Elements

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Outline of the Talk

The Algebra of Random Elements

(With Maria Victoria Velasco, in progress)

Probability space $(\Omega, \mathcal{F}, \mu)$ complete, no atoms, Ban. space X
$L_0(\Omega; X) = L_0(\Omega, \mathcal{F}, \mu; X)$ = all X-valued Bochner-measurable functions on Ω, topology of convergence in probability.

$L_0(\Omega) = L_0(\Omega; \mathbb{C})$ Consider $L_0(\Omega; X)$ as a module over $L_0(\Omega)$.
$\mathbf{x} \in L_0(\Omega; X)$, $x \in X$

Convergence in probability is metrizable:

$$d_0(\mathbf{x}, \mathbf{y}) = \mathbb{E}(\min\{\|\mathbf{x} - \mathbf{y}\|, 1\})$$

The Algebra of Random Elements

Properties of $\mathbf{d_0}$ Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_0(\Omega; X)$ and $\lambda \in L_0(\Omega)$ with $0 \leq \lambda \leq 1$.

(i) (Translation Invariance) $d_0(\mathbf{x}+\mathbf{z}, \mathbf{y}+\mathbf{z}) = d_0(\mathbf{x}, \mathbf{y})$

(ii) (Subadditivity) $d_0(\mathbf{x}+\mathbf{y}, \mathbf{0}) \leq d_0(\mathbf{x}, \mathbf{0}) + d_0(\mathbf{y}, \mathbf{0})$

(iii) (Monotonicity) $d_0(\lambda\mathbf{x}, \lambda\mathbf{y}) \leq d_0(\mathbf{x}, \mathbf{y})$

A a Banach algebra with e. $L_0(\Omega; A)$ is a Fréchet algebra which is not locally convex (F-algebra). We study its properties as a topological algebra and its relationship to A.

Motivation:

Theorem (Random Johnson-Sinclair Theorem)

(Velasco/Villena, 1995) *Let A be semisimple. Then every derivation from A to $L_0(\Omega; A)$ is continuous.*

Spectrum

Two natural notions of spectrum for $L_0(\Omega; A)$:

$$\sigma_{L_0(\Omega;A)}(\mathbf{a}) = \{\lambda \in \mathbb{C} : \mathbf{a} - \lambda\mathbf{e} \notin \operatorname{Inv} L_0(\Omega; A)\}$$

(algebraic spectrum)

$$\begin{aligned}\sigma_{as}(\mathbf{a}) &= \{\lambda \in L_0(\Omega) : \mathbf{a}(\omega) - \lambda(\omega)\mathbf{e} \notin \operatorname{Inv} A \text{ a.s.}\} \\ &= \{\lambda \in L_0(\Omega) : \lambda(\omega) \in \sigma_A(\mathbf{a}(\omega)) \text{ a.s.}\}\end{aligned}$$

(almost sure spectrum)

Since inversion is continuous on $\operatorname{Inv} A$, it is easy to see that an element $\mathbf{a} \in \operatorname{Inv} L_0(\Omega; A)$ if and only if $\mathbf{a}(\omega) \in \operatorname{Inv} A$ a.s. It now follows easily from the proposition below that every element of $\sigma_{L_0(\Omega;A)}(\mathbf{a})$ agrees with an element of $\sigma_{as}(\mathbf{a})$ on a set of positive probability.

Spectrum

Easy examples show $\sigma_{L_0(\Omega;A)}(\mathbf{a})$ can be empty. But:

Proposition

For any $\mathbf{a} \in L_0(\Omega; A)$, $\underline{\sigma_{as}}(\mathbf{a}) \neq \emptyset$. *In fact, there is a countable set* $\Lambda \subset \sigma_{as}(\mathbf{a})$ *such that* $\overline{\Lambda(\omega)} = \sigma_A(\mathbf{a}(\omega))$ *a.s.*

Proof.

Since the spectrum function is u.s.c. on A, for any open U in $\mathbb{C}$ and $\mathbf{a} \in L_0(\Omega; A)$, $\{\omega : \sigma(\mathbf{a}(\omega)) \subset U\} \in \mathcal{F}$. Hence the Kuratowski/Ryll-Nardzewski selection theorem applies to give the first assertion. The second is a well-known refinement due to C. Himmelberg. □

Since we would like the natural choice for the spectrum to be nonempty, we shall call σ_{as} the *stochastic spectrum* and denote it by σ.

Ideals

In general, ideal structure of $L_0(\Omega; A)$ may be complicated.
In one case we have a complete answer, based on the following lemma.

Lemma

Suppose that A is simple, and let I be a nontrivial closed ideal in $L_0(\Omega; A)$. Then I contains an element which is invertible on a set of positive probability.

Theorem

Let A be simple and let I be a closed ideal in $L_0(\Omega; A)$. Then there exists $\Omega_I \in \mathcal{F}$ such that $I = L_0(\Omega; A)\chi_{\Omega_I} = \{\mathbf{a} \in L_0(\Omega; A) : \mathbf{a} = 0 \text{ on } X \setminus \Omega_I\}$.

The Radical

Definition

Let $\mathcal{M}_A$ denote the family of maximal left ideals in A. For each $M \in \mathcal{M}_A$, $L_0(\Omega; M)$ is a closed ideal in $L_0(\Omega; A)$. The *stochastic radical* is the ideal

$$\mathrm{Rad}_s[L_0(\Omega; A)] = \bigcap_{M \in \mathcal{M}_A} L_0(\Omega; M).$$

$L_0(\Omega; A)$ is called *stochastically semisimple* if $\mathrm{Rad}_s(L_0(\Omega; A)) = \{0\}$. Note that if $L_0(\Omega; A)$ is stochastically semisimple then A is semisimple, since

$$\mathrm{Rad}_s[L_0(\Omega; A)] \supset L_0\left(\Omega; \bigcap_{M \in \mathcal{M}_A} M\right).$$

The converse is true, for instance, if A is commutative and $\mathcal{M}_A$ is separable in the Gelfand topology.

The Radical

Definition

Let A be commutative with Gelfand space Φ_A. For $\varphi \in \Phi_A$, denote also by φ the homomorphism $\varphi : L_0(\Omega; A) \to L_0(\Omega)$ given by

$$\varphi(\mathbf{a})(\omega) = \varphi(\mathbf{a}(\omega)) \text{ a.s.}$$

We call this φ a *stochastic character*. For $\mathbf{a} \in L_0(\Omega; A)$, set $\widehat{\mathbf{a}}(\varphi) = \varphi(\mathbf{a})$, $\varphi \in \Phi_A$. Thus $\mathbf{a} \mapsto \widehat{\mathbf{a}}$ is a continuous homomorphism from $L_0(\Omega; A)$ to $L_0(\Omega; C_0(\Phi_A))$. $L_0(\Omega; A)$ is stochastically semisimple if and only if this map is injective.

Hulls and Kernels

A commutative with Gelfand space Φ_A, I a closed ideal in $L_0(\Omega; A)$.

Definition

If X is a top. space, a *closed multifunction* $F : \Omega \to 2^X$ is a mapping from Ω to closed subsets of X. The *graph* of F is $\mathrm{Gr}(F) = \{(\omega, x) : x \in F(\omega)\}$. Call F *measurable* if $\mathrm{Gr}(F)$ is $\mathcal{F} \times \mathcal{B}(X)$-measurable. For $x \in X$, let $F^x = \{\omega : x \in F(\omega)\}$.

Definition

The ideal I has *hull* the closed, measurable multifunction $F : \Omega \to \Phi_A$ if

(i) $\widehat{\mathbf{x}}(\omega) \equiv 0$ on $F(\omega)$ μ-a.s. for all $x \in I$;

(ii) (maximality) for all $\varphi \in \Phi_A$, if $\Omega_0 \in \mathcal{F}$ with $\mu(\Omega_0 \setminus F^\varphi) > 0$, then there exists $\mathbf{x} \in I$ such that $\widehat{\mathbf{x}}(\varphi)$ is not a.s. 0 on $\Omega_0 \setminus F^\varphi$.

Hulls and Kernels

Write $F = Z(I)$. It is unique up to μ-null sets.

Definition

If $F : \Omega \to 2^{\Phi_A}$ is a (closed) multifunction, the *kernel* of F is

$$I(F) = I(\mathrm{Gr}(F)) = \{\mathbf{x} \in L_0(\Omega; A) : \widehat{x}(\omega) \equiv 0 \text{ on } F(\omega) \text{ a.s.}\}.$$

Easy to see $I(F)$ is a closed ideal in $L_0(\Omega; A)$.

Hulls and Kernels

Theorem

(1) *If I is countably generated, then $Z(I)$ exists.*

(2) *Let $A = C_0(X)$, X loc. cpt. If I is a countably generated, closed ideal in $L_0(\Omega; A)$, then $I(Z(I)) = I$. If $I(F)$ is countably generated, then $Z(I(F)) = F$ a.s.*

(3) *More generally, if A is (completely) regular and $I(F)$ is countably generated, then $Z(I(F)) = F$ a.s.*

Factorization Theorem

X a left Banach A-module. Then $L_0(\Omega; X)$ is a top. left module over $L_0(\Omega; A)$. Let $\boldsymbol{\Sigma}(A, X)$ = closure in $L_0(\Omega; X)$ of all sums of the form $\sum_{i=1}^{n} \mathbf{a}_i \cdot \mathbf{x}_i$.

Definition

Boundedness of a set E in the t.v.s. $L_0(\Omega; A)$ means that it is *stochastically bounded*: For every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$\mu[\|x\| \geq M_\varepsilon] < \varepsilon \quad \forall\, x \in E.$$

The celebrated factorization theorem of Cohen, Hewitt, Allan, and Sinclair has a version in the present context.

Factorization Theorem

Theorem (Random Factorization Theorem)

Suppose that A has no identity, but $L_0(\Omega; A)$ has a stochastically bounded left approximate identity, and that X is a left Banach A-module. Let $\mathbf{x} \in \mathbf{\Sigma}(A, X)$, and let X_0 be a closed, separable subset of X such that $\mathbf{x} \in X_0$ a.s. and the values of $\mathbf{x}$ outside of some null set are dense in X_0. Then there is a separable, closed subalgebra A_0 of A with a bounded sequential left approximate identity $\{u_n\}$ such that $\mathbf{x} \in \overline{A_0 \cdot X_0}$ a.s. and $u_n y \to y, \ y \in X_0$.

Factorization Theorem

Theorem (Cont.)

Let $1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \to \infty$. *Then for some* $M > 0$ *and any* $N \geq 1$ *and* $\varepsilon > 0$, *there exist* $\mathbf{a} \in L_0(\Omega; A)$ *with* $\|\mathbf{a}\| \leq M$ *a.s. and* $\mathbf{y}_n \in \overline{L_0(\Omega; A) \cdot \mathbf{x}}$, $n = 1, 2, \ldots$ *such that:*

(i) $\mathbf{y}_n \in \overline{A_0 \cdot X_0}$ *a.s.,* $n \geq 1$;

(ii) $\mathbf{x} = \mathbf{a}^n \cdot \mathbf{y}_n$ *a.s.,* $n \geq 1$;

(iii) $\|\mathbf{x} - \mathbf{y}_n\| \leq \varepsilon$ *a.s.,* $n = 1, \ldots, N$;

(iv) $\|\mathbf{y}_n\| \leq \alpha_n^n \|\mathbf{x}\|$ *a.s.,* $n \geq 1$.

Factorization Theorem

Remarks

(1) The proof of this theorem rests on its well-known version for Banach algebras and an appropriately applied selection theorem.

(2) The assumption that $L_0(\Omega; A)$ has a stochastically bounded approximate identity is clearly satisfied if A has a bounded approximate identity. But in fact, both conditions can be proven equivalent.

Automatic Continuity

Definition

Let A and B be unital Banach algebras. We call $\theta : L_0(\Omega; B) \to L_0(\Omega; A)$ a *module homomorphism* if it is a homomorphism and a module map over $L_0(\Omega)$.

Theorem

Let A and B be (unital) Banach algebras, and let

$$\theta : L_0(\Omega; B) \to L_0(\Omega; A)$$

be a module homomorphism whose range is dense in $L_0(\Omega; A)$. If A is simple, then θ is continuous.

Automatic Continuity

Proof.

Since θ has dense range, it is unital and the separating subspace $S(\theta)$ is a closed ideal in $L_0(\Omega; A)$.
Suppose $S(\theta) \neq \{0\}$. As mentioned earlier, there exists $\Omega_0 \subset \Omega$ such that $\mu(\Omega_0) > 0$ and $\chi_{\Omega_0}\mathbf{e} \in S(\theta)$. Choose $\mathbf{b}_n \in L_0(\Omega; B)$, $n = 1, 2, \ldots$ so that $\mathbf{b}_n \to 0$ and $\theta(\mathbf{b}_n) \to \chi_{\Omega_0}\mathbf{e}$. Now, the hypotheses on θ imply that $\sigma(\theta(\mathbf{b})) \subset \sigma(\mathbf{b}),\ \mathbf{b} \in L_0(\Omega; B)$. If $\lambda_n \in \sigma(\mathbf{b}_n)$, then almost surely on Ω_0 our hypotheses imply that

$$1 - \lambda_n \in \sigma(\mathbf{e} - \theta(\mathbf{b}_n)) = \sigma(\theta(\mathbf{e} - \mathbf{b}_n)).$$

Hence $S(\theta) = \{0\}$. Since $L_0(\Omega; A)$ is a Fréchet algebra, the Closed Graph Theorem implies that θ is continuous. □

Automatic Continuity

Corollary

Let A, B, and θ be as above. If A is commutative and $L_0(\Omega; A)$ is stochastically semisimple, then θ is continuous. In particular, every stochastic character on $L_0(\Omega; A)$ is continuous.

Conjecture (Random Johnson Homomorphism Theorem)

Let A, B, and θ be as above, and assume θ is surjective. Then $S(\theta) \subset \mathrm{Rad}_s(L_0(\Omega; A))$. If $L_0(\Omega; A)$ is stochastically semisimple, then θ is continuous.

Thank you.