Some weighted groups algebras are operator algebras

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August 8, 2011

Definition 1 *Let A be a unital commutative Banach algebra. Then:*
*(i) A is a Q-***algebra** *if it is isomorphic to a quotient of a uniform algebra;*
(ii) A satisfies **multi-variable (δ, L)-von Neumann inequality** *provided that for every $n \in N$, every set of n elements $\{a_1, \ldots, a_n\} \subset A$ with $\|a_i\| \leq \delta$ $(i = 1, \ldots, n)$, and every polynomial p in n variables, we have*

$$\|p(a_1, \ldots, a_n)\| \leq L\|p\|_\infty,$$

where

$$\|p\|_\infty = \sup\{|p(z_1, \ldots, z_n) : |z_i| \leq 1, 1 \leq i \leq n\}.$$

Theorem 2 *(***Varapolous-Craw***)*
Let A be a unital commutative Banach algebra. Then the following statements are equivalent:
(i) *A is a Q-algebra;*
(ii) *A satisfies multi-variable (δ, L)-von Neumann inequality for some positive δ and M;*
(iii) *the multiplication map*

$$m : A\check{\otimes}A \to A, \quad a \times b \mapsto ab$$

is bounded. Here $\check{\otimes}$ is the injective tensor product.

Let A be an operator space which is also an algebra. We say that A is **completely isomorphic to an operator algebra** if there is an operator algebra $B \subseteq B(H)$, and a completely bounded invertible homomorphism $\rho : A \to B$ such that ρ^{-1} is completely bounded.

Theorem 3 *(***D. Blecher***)*
Let $m : A \otimes A \to A$ denote the multiplication on A. Then A is completely isomorphic to an operator algebra if and only if m extends to a completely bounded map $m : A \otimes^h A \to A$.

Note: Consider $l^1(X)$ with MAX operator space structure. Then

$$l^1(X) \otimes^h l^1(X) \cong l^1(X) \check{\otimes} l^1(X)$$

as Banach spaces.

Examples of Q-algebras:

(i) (**Varapolous/Cole**) l^p with pointwise product for $1 \leq p \leq \infty$.

(ii) (**Le Merdy/Pérez-Garcia**) S^p with Schur product if $1 \leq p \leq 4$.

(iii) (**Varapolous**) $l^1(\mathbb{Z}, (1+|n|)^\alpha)$ with convolution product if and only if $\alpha > 1/2$.

G: a countable, discrete group.

$l^1(G)$: The group algebra of G;

$$f * g(x) = \sum_{t \in G} f(t) g(t^{-1}x).$$

A **weight** on G is a function $\omega : G \to (0, \infty)$ such that

$$\omega(st) \leqslant \omega(s)\omega(t) \quad (s, t \in G).$$

Examples: If $0 \le \alpha \le 1$, $\beta \ge 0$, $C > 0$

$$n \mapsto e^{C|n|^\alpha} \quad , \quad n \mapsto (1 + |n|)^\beta.$$

The **weighted group algebra** $l^1(G, \omega)$ is all functions $f : G \to \mathbb{C}$ such that

$$\|f\|_\omega = \sum_{t \in G} |f(t)| \omega(t) dt < \infty.$$

Groups with polynomial growth:

A group G has **polynomial growth** if for every finite subset F of G such that $e \in F$, there exist a polynomial f on $\mathbb{R}$ such that

$$|F^n| \leq f(n) \quad (n \in \mathbb{N}).$$

Here $|S|$ is the cardinality of any $S \subseteq G$ and

$$F^n = \{u_1 \cdots u_n : u_i \in F, i = 1, \ldots, n\}.$$

Examples: finite groups, groups with finite conjugacy class, nilpotent groups.

M. Gromov: Every *finitely generated* group with polynomial growth is virtually nilpotent i.e. it has a nilpotent subgroup of finite index. Moreover, there is a finite symmetric subset F of G including the identity e, a polynomial f on $\mathbb{R}$ and a constant $0 < \lambda \leq 1$ such that

$$\lambda f(n) \leq |F^n| \leq f(n) \quad \text{for all} \quad n \in \mathbb{N}.$$

The least degree of any polynomial satisfying the above relation is called **the order of growth** of G and it is denoted by $d(G)$.

Bass-Guivarch formula: Let G be a finitely generated nilpotent group with lower central series

$$G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_m = \{e\}.$$

Then the order of polynomial growth of G is

$$d(G) = \sum_{k=1}^{m-1} k \ \text{rank}(G_k/G_{k+1}),$$

where rank denotes the rank of an abelian group, i.e. the largest number of independent and torsion-free elements of the abelian group.

Weights on finitely generated groups with polynomial growth:

We can define the **length function** $\tau_F : G \to [0, \infty)$ by $\tau(e) = 0$ and

$$\tau(x) = \inf\{n \in \mathbb{N} : x \in F^n\} \quad \text{for} \quad x \neq 0.$$

τ is a subadditive function on G, i.e.

$$\tau(xy) \leq \tau(x) + \tau(y) \qquad (x, y \in G).$$

and for every $x \in G$, $\tau(x) = \tau(x^{-1})$.

Let $\beta \geq 0$. We define the **polynomial weight** ω_β on G by

$$\omega_\beta(x) = (1 + \tau(x))^\beta \qquad (x \in G).$$

Theorem 4 *$l^1(G, \omega_\beta)$ is an operator algebra if one of the following condition holds:*
(i) $\lambda = 1$ and $2\beta > d$;
(ii) $0 < \lambda < 1$ and $2\beta > d + 1$.
In addition, if $m : l^1(G, \omega_\beta) \otimes^h l^1(G, \omega_\beta) \to l^1(G, \omega_\beta)$ is the multiplication map on Haagerup tensor product, then

$$\|m\|_{cb} \leq \min\{2, 2^\beta\} \left[1 + \sum_{n=1}^{\infty} \frac{f(n) - \lambda f(n-1)}{(1+n)^{2\beta}}\right.$$

For a weight ω on G, we define the center of $l^1(G,\omega)$ to be

$$Zl^1(G,\omega) := \{f \in l^1(G,\omega) : f(x^{-1}yx) = f(x)$$

$$\text{for every} \quad x, y \in G\}.$$

Theorem 5 *$Zl^1(G,\omega_\beta)$ is a Q-algebra if one of the following condition holds:*
(i) $\lambda = 1$ and $2\beta > d$;
(ii) $0 < \lambda < 1$ and $2\beta > d + 1$.
In either case, $Zl^1(G,\omega_\beta)$ satisfies the multi-variable von Neumann inequality.

Let $0 \leq \alpha \leq 1$ and $C > 0$. We define the **exponential weight** σ_α on G by

$$\sigma_\alpha(x) = e^{C\tau(x)^\alpha} \qquad (x \in G).$$

Theorem 6 *Let $0 \leq \alpha \leq 1$ and $C > 0$. Then $l^1(G, \sigma_\alpha)$ is an operator algebra if and only if $0 < \alpha < 1$.*
If $m : l^1(G, \sigma_\alpha) \otimes^h l^1(G, \sigma_\alpha) \to l^1(G, \sigma_\alpha)$ is the multiplication map on Haagerup tensor product, and

$$\beta \geq \max\left\{1, \frac{6}{C\alpha(1-\alpha)}, \frac{d + (1 - \delta_1(\lambda))}{2}\right\},$$

then

$$\|m\|_{cb} \leq M2^\beta \left[1 + \sum_{n=1}^{\infty} \frac{f(n) - \lambda f(n-1)}{(1+n)^{2\beta}}\right]^{1/2}.$$

The d-dimensional integers $\mathbb{Z}^d$:

The order of growth of $\mathbb{Z}^d$ is d.

Let

$$F = \{(x_1, \ldots, x_d) \mid x_i \in \{-1, 0, 1\}\}.$$

Then F is a finite, symmetric generating set for $\mathbb{Z}^d$ which includes the identity of $\mathbb{Z}^d$. Also, for every $n \in \mathbb{N}$,

$$F^n = \{(x_1, \ldots, x_d) \mid x_i \in \{-n, \ldots, 0, \ldots, n\}\}.$$

In particular,

$$|F^n| = (2n + 1)^d \quad (n = 0, 1, 2, \ldots).$$

If we let ω_β be the polynomial weight on $\mathbb{Z}^d$, then

$$\omega_\beta(x_1, \ldots, x_d) = (1+\max\{|x_1|, \ldots, |x_d|)^d.$$

Then $l^1(\mathbb{Z}^d, \omega_\beta)$ is a Q-algebra if

$$2\beta > d.$$

If $m : l^1(\mathbb{Z}^d, \omega_\beta) \otimes^h l^1(\mathbb{Z}^d, \omega_\beta) \to l^1(\mathbb{Z}^d, \omega_\beta)$ is the multiplication map, then

$$\|m\|_{cb} \ \leq \ \min\{2, 2^\beta\} \left[1 + \frac{d2^d}{2\beta - d}\right]^{1/2}.$$

Let $0 < \alpha < 1$, $C > 0$, and

$$\sigma_\alpha(x_1, \ldots, x_d) = e^{C \max\{|x_1|,\ldots,|x_d|\})^d}.$$

Then $l^1(\mathbb{Z}^d, \sigma_\alpha)$ is a Q-algebra.

If $m : l^1(\mathbb{Z}^d, \sigma_\alpha) \otimes^h l^1(\mathbb{Z}^d, \sigma_\alpha) \to l^1(\mathbb{Z}^d, \sigma_\alpha)$ is the multiplication map and

$$\beta \geq \max\left\{1, d, \frac{6}{C\alpha(1-\alpha)}\right\},$$

then

$$\|m\|_{cb} \leq M 2^\beta \left[1 + \frac{d 2^d}{2\beta - d}\right]^{1/2}.$$

For example, if

$$d = 1 \quad \text{and} \quad C = \frac{6}{\alpha(1-\alpha)},$$

then $\|m\|_{cb} \leq 6$.