Complete compactness in abstract harmonic analysis

Volker Runde

University of Alberta

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(Weakly) almost periodic functions...

Definition

Let G be a locally compact group. Then $f \in \mathcal{C}(G)$ is called almost periodic if its left orbit, i.e., the set of all left translates of f, is relatively compact in $\mathcal{C}(G)$ and is called weakly almost periodic if its left orbit is relatively weakly compact in in $\mathcal{C}(G)$.

Not too hard...

$$\mathcal{AP}(G) := \{f \in \mathcal{C}(G) : f \text{ is almost periodic}\}$$

and

$$\mathcal{WAP}(G) := \{f \in \mathcal{C}(G) : f \text{ is weakly almost periodic}\}$$

are C^*-subalgebras of $\mathcal{C}(G)$.

... and functionals

Definition

Let A be a Banach algebra. Then $\phi \in A^*$ is called almost periodic if the map

$$A \to A^*, \quad a \mapsto a \cdot \phi$$

is compact and is called weakly almost periodic if it is weakly compact.

Notation

$$\mathcal{AP}(A) := \{\phi \in A^* : \phi \text{ is almost periodic}\}$$

$$\mathcal{WAP}(A) := \{\phi \in A^* : \phi \text{ is weakly almost periodic}\}$$

The $L^1(G)$-case...

Theorem (partly A. Ülger, 1986)

$\mathcal{AP}(L^1(G)) = \mathcal{AP}(G)$ *and* $\mathcal{WAP}(L^1(G)) = \mathcal{WAP}(G)$.

Consequence

$\mathcal{AP}(L^1(G))$ and $\mathcal{WAP}(L^1(G))$ are C^*-subalgebras of $\mathcal{C}(G)$ and thus of $L^\infty(G)$.

. . . vs. the $A(G)$-case

Definition (C. F. Dunkl & D. E. Ramirez, 1972)

$\mathcal{AP}(\hat{G}) := \mathcal{AP}(A(G))$ and $\mathcal{WAP}(\hat{G}) := \mathcal{WAP}(A(G))$.

Question

Are $\mathcal{AP}(\hat{G})$ and $\mathcal{WAP}(\hat{G})$ C^*-subalgebras of $\mathrm{VN}(G)$?

Positive answers

- for abelian G: by Pontryagin duality;
- for discrete, amenable G:

$$\mathcal{AP}(\hat{G}) = \mathcal{WAP}(\hat{G}) = C_r^*(G)$$

(E. Granirer, 1977).

Hopf–von Neumann algebras, I

Definition

A **Hopf–von Neumann algebra** is a pair (M, Γ), where M is a von Neumann algebra and $\Gamma\colon M \to M \bar{\otimes} M$ is a **co-multiplication**, i.e., a normal, faithful, unital *-homomorphism such that

$$(\mathrm{id} \otimes \Gamma) \circ \Gamma = (\Gamma \otimes \mathrm{id}) \circ \Gamma.$$

Hopf–von Neumann algebras, II

Examples

- $M = L^\infty(G)$ and $\Gamma_G \colon L^\infty(G) \to L^\infty(G \times G)$ given by

$$(\Gamma_G \phi)(x, y) := \phi(xy) \qquad (x, y \in G, \, \phi \in L^\infty(G));$$

- $M = \mathrm{VN}(G)$ and

$$\hat{\Gamma}_G \colon VN(G) \to \mathrm{VN}(G \times G), \quad \lambda(x) \mapsto \lambda(x) \otimes \lambda(x).$$

Hopf–von Neumann algebras, III

Remember...

If (M, Γ) is a Hopf–von Neumann algebra, then M_* becomes a Banach algebra via

$$\langle f * g, x \rangle := \langle f \otimes g, \Gamma x \rangle \qquad (f, g \in M_*, \, x \in M).$$

Examples

- $(L^\infty(G), \Gamma_G)$: $L^1(G)$ with the convolution product;
- $(\mathrm{VN}(G), \hat{\Gamma}_G)$: $A(G)$ with the pointwise product.

Hopf–von Neumann algebras, IV

Question

If (M, Γ) is a Hopf–von Neumann algebra, are $\mathcal{AP}(M_*)$ and $\mathcal{WAP}(M_*)$ C^*-subalgebras of M?

Theorem (M. Daws, 2010)

$\mathcal{AP}(M_*)$ and $\mathcal{WAP}(M_*)$ are C^*-subalgebras of M if M is abelian.

Fundamental problem

For any (M, Γ), M_* is a completely contractive Banach algebra, but $\mathcal{AP}(M_*)$ and $\mathcal{WAP}(M_*)$ completely ignore the operator space structure.

A new look at compactness, I

Compactnes implies ...

Let E and F be Banach spaces, and let $T \in \mathcal{B}(E, F)$ be compact. Let $\epsilon > 0$. Then there are $y_1, \ldots, y_n \in F$ such that

$$T(\mathrm{Ball}(E)) \subset \bigcup_{j=1}^{n} y_j + \mathrm{ball}_\epsilon(F).$$

Set $Y_\epsilon := \mathrm{span}\{y_1, \ldots, y_n\}$ and let $Q_{Y_\epsilon} : F \to F / Y_\epsilon$ be the quotient map. Then $\dim Y_\epsilon < \infty$ and $\|Q_{Y_\epsilon} T\| < \epsilon$.

A new look at compactness, II

...and is implied

Let $T \in \mathcal{B}(E, F)$ be such that, for every $\epsilon > 0$, there is a subspace Y_ϵ of F with $\dim Y_\epsilon < \infty$ and $\| Q_{Y_\epsilon} T \| < \epsilon$.
Let $\epsilon > 0$, and fix $Y_{\frac{\epsilon}{3}}$. Set

$$K := \left\{ y \in Y_{\frac{\epsilon}{3}} : \text{there is } x \in \mathrm{Ball}(E) \text{ with } \| Tx - y \| < \frac{\epsilon}{3} \right\}.$$

Then K is bounded and thus totally bounded (as $\dim Y_{\frac{\epsilon}{3}} < \infty$). Let $y_1, \ldots, y_n \in K$ be such that

$$K \subset \bigcup_{j=1}^{n} y_j + \mathrm{ball}_{\frac{\epsilon}{3}}(F).$$

A new look at compactness, III

. . . and is implied (continued)

Pick $x_1, \ldots, x_n \in \operatorname{Ball}(E)$ such that

$$\|Tx_j - y_j\| < \frac{\epsilon}{3} \qquad (j = 1, \ldots, n).$$

Let $x \in \operatorname{Ball}(E)$. As $\|Q_{Y_{\frac{\epsilon}{3}}} T\| < \frac{\epsilon}{3}$, there is $y \in Y_{\frac{\epsilon}{3}}$ with $\|Tx - y\| < \frac{\epsilon}{3}$, i.e., $y \in K$. Choose $k \in \{1, \ldots, n\}$ such that $\|y - y_k\| < \frac{\epsilon}{3}$. Then

$$\|Tx - Tx_k\| \leq \|Tx - y\| + \|y - y_k\| + \|y_k - Tx_k\| < \epsilon.$$

Hence,

$$T(\operatorname{Ball}(E)) \subset \bigcup_{j=1}^{n} Tx_j + \operatorname{ball}_\epsilon(F).$$

A new look at compactness, IV

Proposition (H. Saar, 1982; H. E. Lacey, 1963)

The following are equivalent for $T \in \mathcal{B}(E, F)$:

1. *T is compact;*
2. *for every $\epsilon > 0$, there is a subspace Y_ϵ of F with $\dim Y_\epsilon < \infty$ and $\|Q_{Y_\epsilon} T\| < \epsilon$, where $Q_{Y_\epsilon} : F \to F/Y_\epsilon$ is the quotient map;*
3. *for every $\epsilon > 0$, there is a closed subspace X_ϵ of E with $\dim E/X_\epsilon < \infty$ such that $\|T|_{X_\epsilon}\| < \epsilon$.*

Completely compact maps, I

Definition (H. Saar, 1982)

Let E and F be operator spaces. Then $T \in \mathcal{CB}(E, F)$ is called completely compact if, for every $\epsilon > 0$, there is a subspace Y_ϵ of F with $\dim Y_\epsilon < \infty$ and $\|Q_{Y_\epsilon} T\|_{\mathrm{cb}} < \epsilon$.

Notation

$$\mathcal{CK}(E, F) := \{T \in \mathcal{CB}(E, F) : T \text{ is completely compact}\}$$

Completely compact maps, II

Properties

1. every finite rank operator is completely compact;
2. $\mathcal{CK}(E,F) \subset \mathcal{K}(E,F)$, but

   $$\mathcal{CK}(\mathcal{B}(\ell^2),\mathcal{K}(\ell^2)) \subset \subsetneq \mathcal{CB}(\mathcal{B}(\ell^2),\mathcal{K}(\ell^2)) \cap \mathcal{K}(\mathcal{B}(\ell^2),\mathcal{K}(\ell^2))$$

   (H. Saar, 1982);
3. $\mathcal{CK}(E,F)$ is cb-norm closed in $\mathcal{CB}(E,F)$;
4. if $S \in \mathcal{CK}(E,F)$, $T \in \mathcal{CB}(X,E)$, and $R \in \mathcal{CB}(F,Y)$, then $ST \in \mathcal{CK}(X,F)$ and $RS \in \mathcal{CK}(E,Y)$.

Failure of Schauder's theorem, I

Definition (T. Oikhberg, 2001)

$T \in \mathcal{CB}(E, F)$ is called Gelfand completely compact if, for every $\epsilon > 0$, there is a closed subspace X_ϵ of F with $\dim E/X_\epsilon < \infty$ and $\| T|_{X_\epsilon} \|_{\mathrm{cb}} < \epsilon$.

Obvious. . .

$$T \in \mathcal{CK}(E, F) \iff T^* \text{ is Gelfand completely compact}$$

Failure of Schauder's theorem, II

Recall...

F is called injective if, for every E, for every subspace X of E, and for every $T \in \mathcal{CB}(X, F)$, there is $\tilde{T} \in \mathcal{CB}(E, F)$ with $\tilde{T}|_X = T$ and $\|\tilde{T}\|_{\text{cb}} = \|T\|_{\text{cb}}$.

Example

$\mathcal{B}(H)$ is injective.

Proposition

Let F be injective. Then every Gelfand completely compact map in $\mathcal{CB}(E, F)$ is a cb*-norm limit of finite rank operators.*

Failure of Schauder's theorem, III

Example (T. Oikhberg, 2001)

$\mathcal{B}(\ell^2)$ lacks the approximation property (A. Szankowski, 1981), and thus lacks the strong operator space approximation property.

Therefore, there are

- an operator space E
- and $T \in \mathcal{CK}(E, \mathcal{B}(\ell^2))$ such that T is not a cb-norm limit of finite rank operators

(C. Webster, 1998).

As $\mathcal{B}(\ell^2)$ is injective, T cannot be Gelfand completely compact.

On the positive side...

Proposition

Suppose that E^* and F^* are injective. Then the following are equivalent for $T \in \mathcal{CB}(E, F^*)$:

1. $T \in \mathcal{CK}(E, F^*)$;
2. T *is Gelfand completely compact;*
3. T *is a* cb*-norm limit of finite rank operators.*

Completely almost periodic functionals, I

Definition

Let A be a completely contractive Banach algebra. Then $\phi \in A^*$ is called completely almost periodic if

$$(*) \qquad A \to A^*, \quad a \mapsto a \cdot \phi$$

and

$$(**) \qquad A \to A^*, \quad a \mapsto \phi \cdot a$$

are completely compact.

Notation

$$\mathcal{CAP}(A) := \{\phi \in A^* : \phi \text{ is completely almost periodic}\}$$

Completely almost periodic functionals, II

Remarks

- We have to require both $(*)$ and $(**)$ to be completely compact due to the failure of Schauder's theorem. (At least if we want a symmetric definition.)
- As $(**)$ is the adjoint of $(*)$ restricted to A (and vice versa),

$$\mathcal{CAP}(A) = \{\phi \in A^* : \\ (*), (**) \text{ are Gelfand completely compact}\}.$$

Injective Hopf–von Neumann algebras

Theorem (V. Runde, 2010)

Let (M, Γ) be a Hopf–von Neumann algebra with M injective. Then

$$\mathcal{CAP}(M_*) = \{x \in M : \Gamma x \in M \check{\otimes} M\}.$$

In particular, $\mathcal{CAP}(M_*)$ is a C^*-subalgebra of M.

Proof.

$$M \bar{\otimes} M \cong (M_* \hat{\otimes} M_*)^* \cong \mathcal{CB}(M_*, M)$$

with

$$M \check{\otimes} M \cong \text{closure of finite rank maps in } \mathcal{CB}(M_*, M).$$

□

The Fourier algebra

Recall...

$\mathrm{VN}(G)$ is injective if G is

- amenable or
- connected.

Notation

$$\mathcal{CAP}(\hat{G}) := \mathcal{CAP}(A(G)).$$

Corollary

Let G be amenable or connected. Then $\mathcal{CAP}(\hat{G})$ is a C^*-subalgebra of $\mathrm{VN}(G)$.

When is a completely bounded map weakly compact?

Theorem (W. J. Davis, et al., 1974)

The following are equivalent for $T \in \mathcal{B}(E, F)$:

1. *T is weakly compact;*
2. *T factors through a reflexive Banach space.*

Theorem (H. Pfitzner & G. Schlüchtermann, 1997; M. Daws, 2007)

The following are equivalent for $T \in \mathcal{CB}(E, F)$:

1. *T is weakly compact;*
2. *T factors through a reflexive operator space.*

A new look at $\mathcal{WAP}(M_*)$, I

A different approach

For $\mathbf{x} \in N \bar{\otimes} M$, define $\mathcal{T}(\mathbf{x}) \in \mathcal{CB}(N_*, M)$ as

$$\mathcal{T}(\mathbf{x}) \colon N_* \to M, \quad f \mapsto (f \otimes \mathrm{id})(\mathbf{x}).$$

Set

$$\mathcal{W}(N \bar{\otimes} M) = \{\mathbf{x} \in N \bar{\otimes} M : \mathcal{T}(\mathbf{x}) \text{ is weakly compact}\}.$$

If (M, Γ) is a Hopf–von Neumann algebra, then

$$\mathcal{WAP}(M_*) = \{x \in M : \Gamma x \in \mathcal{W}(M \bar{\otimes} M)\}.$$

A new look at $\mathcal{WAP}(M_*)$, II

Question

Is $\mathcal{W}(N\bar{\otimes}M)$ a C^*-subalgebra of $N\bar{\otimes}M$?

Question

Let $\mathbf{x}, \mathbf{y} \in \mathcal{W}(N\bar{\otimes}M)$. Can we give meaningful conditions making sure that $\mathbf{xy} \in \mathcal{W}(N\bar{\otimes}M)$?

A general theorem

Theorem (V. Runde, 2011)

Suppose that N and M are injective, and let $\mathbf{x}, \mathbf{y} \in N \bar{\otimes} M$ be such that:

1. $\mathcal{T}(\mathbf{x})$ *factors through an operator space* E*;*
2. $\mathcal{T}(\mathbf{y})$ *factors through an operator space* F*;*
3. $E \otimes^h F$ *is reflexive.*

Then $\mathbf{xy} \in \mathcal{W}(N \bar{\otimes} M)$.

Reflexivity of $E \otimes^h F$

Question

Is $E \otimes^h F$ always reflexive for reflexive E and F?

Partial answers...

- no: $E = H_c$ and $F = H_r$.
- yes if...
  - $\dim E < \infty$ or $\dim F < \infty$;
  - $E = F = H_c$, $E = F = H_r$, or $E = F = OH$;
  - $E = \min E$ and $F = \max E$ because $E \otimes^h F \cong E \otimes_{d_2} F$.

One final look at $\mathcal{WAP}(M_*)$

Consequences

Let (M, Γ) be a Hopf–von Neumann algebra with M injective, and let $x, y \in M$. Then:

1. if $x, y \in \mathcal{WAP}(M_*)$, and if M is subhomogeneous, then $xy \in \mathcal{WAP}(M_*)$;
2. if $x \in \mathcal{WAP}(M_*)$ and $y \in \mathcal{CAP}(M_*)$, then $xy \in \mathcal{WAP}(M_*)$;
3. if $\mathcal{T}(\Gamma x)$ and $\mathcal{T}(\Gamma y)$ each factor through minimal, reflexive operator spaces, and if (M, Γ) is co-commutative, then $xy \in \mathcal{WAP}(M_*)$.