Poisson Boundaries over Locally Compact Quantum Groups

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Harmonic Functions on Groups

Let G be a locally compact group and μ a probability measure on G.

We can define a map Φ_μ on $L_\infty(G)$ given by

$$\Phi_\mu(h)(s) = \int_G h(st)d\mu(t)$$

for all $h \in L_\infty(G)$.

It is clear that Φ_μ is a unital (completely) positive, and weak* continuous map on $L_\infty(G)$ such that

$$\Phi_\mu = m_\mu^*$$

is the adjoint of the right multiplication map $m_\mu(f) = f \star \mu$ for $f \in L_1(G)$.

In this case, we say that Φ_μ is a Markov operator on $L_\infty(G)$. A function $h \in L_\infty(G)$ is μ-harmonic (or Φ_μ-harmonic) if

$$\Phi_\mu(h) = h.$$

Motivation

Here is a reason why we use the terminology μ-harmonic functions in such a definition.

Suppode that G is a Lie group and Δ is the Laplacian operator on G. A function $f \in C^{\infty}(G)$ is harmonic if

$$\Delta(f) = 0.$$

Then we can consider the semigroup of (completely) positive maps

$$P_t = e^{t\Delta}$$

for all $t \geq 0$. Then h is harmonic if and only if $P_t(h) = h$ for all $t \geq 0$.

Poisson Boundary

We let

$$\mathcal{H}_\mu = \{h \in L_\infty(G) : \Phi_\mu(h) = h\}$$

be the space of all μ-harmonic functions on G, which is a unital weak* closed operator system in $L_\infty(G)$.

It is important to note that there is a conditional expactation

$$\mathcal{E} : L_\infty(G) \to \mathcal{H}_\mu \subseteq L_\infty(G)$$

from $L_\infty(G)$ onto $\mathcal{H}_\mu$. For instance, we can obtain a such $\mathcal{E}$ by consider the Banach limit

$$\langle \mathcal{E}(h), f\rangle = \lim_B \langle \Phi_\mu^n(h), f\rangle = \lim_B \langle h, f \star \mu^n\rangle$$

for all $h \in L_\infty(G)$ and $f \in L_1(G)$.

We could also consider $\mathcal{E}$ defined by Cesàro sums

$$\langle \mathcal{E}(h), f\rangle = \lim_{\mathcal{U}} \langle \frac{1}{n}(\Phi_\mu + \cdots + \Phi_\mu^n)(h), f\rangle = \lim_{\mathcal{U}} \langle h, \frac{1}{n} f \star (\mu + \cdots + \mu^n)\rangle$$

over any ultrafilder $\mathcal{U}$ on $\mathbb{N}$.

Then we can obtain a von Neumann algebra multiplication on $\mathcal{H}_\mu$ given by the Choi-Effros product

$$h \circ k = \mathcal{E}(hk),$$

which is unique and independent from he choice of $\mathcal{E}$.

We call $\mathcal{H}_\mu$ together with this von Neumann algebra structure is the Poisson boundary of (G, μ).

More Details about the Boundary

We note that the natural left action $\alpha : G \curvearrowright L_\infty(G)$ given by

$$\alpha_s(h)(t) = h(s^{-1}t)$$

is invariant with respct to the Markov operator Φ_μ, i.e.,

$$\Phi_\mu \circ \alpha_s = \alpha_s \circ \Phi_\mu$$

for all $s \in G$ sicne

$$\begin{aligned} \alpha_s \circ \Phi_\mu(h)(t) &= \Phi_\mu(h)(s^{-1}t) = \int_G h(s^{-1}tg)d\mu(g) \\ &= \int_G \alpha_s(h)(tg)d\mu(g) = \Phi_\mu \circ \alpha_s(h)(t) \end{aligned}$$

for all $h \in L_\infty(G)$. Therefore, α induces an action $\alpha_\mu : G \curvearrowright \mathcal{H}_\mu$.

Now it is known that there exists a (unique) measure space (Ω, ν) such that

$$(\mathcal{H}_\mu, \circ) = L_\infty(\Omega, \nu)$$

and the induced action α_μ on $\mathcal{H}_\mu$ corresponds to a measure preserving action on (Ω, ν). This space (Ω, ν) gives the Poisson boundary of (G, μ)!

Poisson Boundary for Markov Operators on von Neumann Algebras

In general, if we are given a Markov operator Φ on a von Neumann algebra M. Then we can consider Φ-operators to be elements $x \in M$ such that

$$\Phi(x) = x.$$

In this case, we can also obtain a conditional expactation

$$\mathcal{E} : M \to \mathcal{H}_\Phi$$

from M onto $\mathcal{H}_\Phi$, the space of all Φ-harmonic operators, and obtain a von Neumann algebra strucutre on $\mathcal{H}_\Phi$.

We call this von Neumann algebra $(\mathcal{H}_\Phi, \circ)$ the Poisson boundary of (M, Φ).

We are particularly Interested in the Case of LCQGs

Kustermans and Vaes' Definition of LCQG

A *LCQG* is $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ consisting of

(1) a von Neumann algebra M

(2) a *co-multiplication* $\Gamma : M \to M \bar{\otimes} M$, i.e. a unital normal *-homomorphism satisfying the co-associativity condition

$$(id \otimes \Gamma) \circ \Gamma = (\Gamma \otimes id) \circ \Gamma.$$

(3) a left Haar wight φ, i.e. a n.f.s weight φ on M satisfying

$$(\iota \otimes \varphi)\Gamma(x) = \varphi(x)1$$

(4) a right Haar weight ψ, i.e. n.f.s weight ψ on M satisfying

$$(\psi \otimes \iota)\Gamma(x) = \psi(x)1.$$

It is known that for every locally compact quantum group $\mathbb{G} = (M, \Gamma, \varphi, \psi)$, there exists a *dual quantum group* $\widehat{\mathbb{G}} = (\widehat{M}, \widehat{\Gamma}, \widehat{\varphi}, \widehat{\psi})$ such that we may obtain the perfect Pontryagin duality

$$\hat{\hat{\mathbb{G}}} = \mathbb{G}.$$

Commutative LCQGs are exactly

$$\mathbb{G}_a = (L_\infty(G), \Gamma_a, \varphi_a, \psi_a),$$

where the comultiplicaton $\Gamma_a(h)(s,t) = h(st)$, and

Co-commutative LCQGs are exactly

$$\widehat{\mathbb{G}}_a = (VN(G), \Gamma_G, \varphi_G, \psi_G),$$

where the comultiplication $\Gamma_G(\lambda_s) = \lambda_s \otimes \lambda_s$.

Banach Algebra Structure on $L_1(\mathbb{G}) = M_*$

The co-multiplication

$$\Gamma : L_\infty(\mathbb{G}) \to L_\infty(\mathbb{G}) \bar{\otimes} L_\infty(\mathbb{G})$$

induces an associative completely contractive multiplication

$$\star = \Gamma_* : f_1 \otimes f_2 \in L_1(\mathbb{G}) \hat{\otimes} L_1(\mathbb{G}) \to f_1 \star f_2 = (f_1 \otimes f_2) \circ \Gamma \in L_1(\mathbb{G})$$

on $L_1(\mathbb{G}) = M_*$ such that $A = (L_1(\mathbb{G}), \star)$ is a faithful completely contractive Banach algebra with

$$\langle L_1(\mathbb{G}) \star L_1(\mathbb{G}) \rangle = L_1(\mathbb{G}).$$

If $\mathbb{G}_a$ is a commutative LCQG, then $\star = \Gamma_{a*}$ is just the convolution on the convolution algebra

$$L_1(\mathbb{G}_a) = L_1(G).$$

If $\widehat{\mathbb{G}}_a$ is a co-commutative LCQG, then $\star = \widehat{\Gamma}_*$ is just the pointwise multiplication on the Fourier algebra

$$L_1(\widehat{\mathbb{G}}_a) = VN(G)_* = A(G).$$

Positive Definite Centralizers of $L_1(\mathbb{G})$

A map bounded m on $L_1(\mathbb{G})$ is called a (right) centralizer of $L_1(\mathbb{G})$ if

$$m(f \star g) = f \star m(g).$$

We are particularly interested in those positive definite cb-centralizers, i.e. centralizers m such that

$$\Phi_m = m^*$$

are Markov operators on $L_\infty(\mathbb{G})$. In this case, we can consider the Poisson boundary $\mathcal{H}_m$ associated with $(L_\infty(\mathbb{G}), \Phi_m)$.

• If $\mathbb{G}_a = L_\infty(G)$, we have $m = m_\mu$ for some probablity measure μ on G.

• If $\widehat{\mathbb{G}}_a = VN(G)$, we have $m = m_\varphi$ for some state φ in $B(G)$, i.e., a positive definite function φ on G with $\varphi(e) = 1$.

Extension to $B(L_2(\mathbb{G}))$

Theorem [J-N-R]: Let m be a positive definite cb-centralizer of $L_1(\mathbb{G})$, then the Markov operator $\Phi_m = m^*$ has a natural normal exntesion to a Markov operator $\Theta(m)$ on $B(L_2(\mathbb{G}))$.

Let G be a locally compact group.

• If $\mu \in M(G)$ is a probability measure on G, then the extended Markov operator $\Theta(\mu)$ on $B(L_2(G))$ is given by

$$\Theta(\mu)(x) = \int_G \rho_s x \rho_s^* d\mu(s).$$

• If φ is a state in $B(G)$, then we can write

$$\varphi(st^{-1}) = \langle \xi | \pi(st^{-1})\xi \rangle = \langle \pi(s^{-1})\xi | \pi(t^{-1})\xi \rangle$$

for some unitary representation $\pi : G \to B(\ell_2(I))$ and unit vectio $\xi \in \ell_2(I)$. In this case, we can regard $\pi(t^{-1})\xi = [\beta_i] \in M_{I,1}(\ell_\infty(G))$. It follows that $\Theta(\varphi)$ on $B(L_2(G))$ is given by

$$\Theta(\varphi)(x) = \sum_i \beta_i^* x \beta_i.$$

We can consider the Poisson boundary $\mathcal{H}_{\Theta(m)}$ of $(B(L_2(\mathbb{G})), \Theta(m))$.

We wonder what is the connection between $\mathcal{H}_m$ and $\mathcal{H}_{\Theta(m)}$.

Theorem [K-N-R]: Let $\mathbb{G}$ be a locally compact quantum group and let m be a positive definite cb-multiplier of $L_1(\mathbb{G})$. Then $\mathcal{H}_{\Theta(m)}$ is *-isomorphic to the crossed product of $\mathbb{G}$ on $\mathcal{H}_m$, i.e. we have

$$\mathcal{H}_{\Theta(m)} = \mathbb{G} \ltimes \mathcal{H}_m.$$

Remark: This result was first proved by Izumi for discrete groups. It was proved later on by Jawoski and Neufang for $L_\infty(G)$ case

$$\mathcal{H}_{\Theta(\mu)} = G \ltimes \mathcal{H}_\mu$$

and by Neufang and Runde for $VN(G)$ case

$$\mathcal{H}_{\Theta(\varphi)} = (\mathcal{H}_\varphi \cup L_\infty(G))'' = \widehat{G} \ltimes \mathcal{H}_\varphi$$

is a von Neumann subalgebra of $B(L_2(G))$, under the assumption that either $\varphi \in A(G)$, or the group G has the AP.

One Remark on the Proof :

We need to consider the induced co-action of $\mathbb{G}$ on $\mathcal{H}_m$ given by

$$\Gamma_m : \mathcal{H}_m \to L_\infty(\mathbb{G})\bar{\otimes}_{\mathcal{F}}\mathcal{H}_m \subseteq B(L_2(\mathbb{G})\bar{\otimes}B(L_2(\mathbb{G})).$$

The Fubini product is the correct tensor product to consider here when we regard $\mathcal{H}_m$ as an operator sysytem in $B(L_2(\mathbb{G}))$!

Once we regard $\mathcal{H}_m$ (with its own multiplication as we discussed above) we have

$$L_\infty(\mathbb{G})\bar{\otimes}\mathcal{H}_m = L_\infty(\mathbb{G})\bar{\otimes}_{\mathcal{F}}\mathcal{H}_m.$$

This is exactly why Neufang and Runde need Approximation Property of $\mathbb{G}$.

Summary

$$\mu - \mathbf{harmonic\ on\ L_\infty(G)} \qquad \varphi - \mathbf{harmonic\ on\ VN(G)}$$

Chu and Lau

$$\mu - \mathbf{harmonic\ on\ B(L_2(G))} \qquad \varphi - \mathbf{harmonic\ on\ B(L_2(G))}$$

Jaworski and Neufang — *Neufang and Runde*

$$\mathcal{H}_{\Theta(\mu)} = G \ltimes \mathcal{H}_\mu \qquad \mathcal{H}_{\Theta(\varphi)} = \hat{G} \ltimes \mathcal{H}_\varphi$$

In general, we have

$$\mathcal{H}_{\Theta(m)} = \mathbb{G} \ltimes \mathcal{H}_m$$

for general locally compact quantum groups.

Some Other Interesting Results

Classicial Choquet–Deny Theorem

Theorem: Let μ be a probablity measure on an abelian group G. Then

$$\mathcal{H}_\mu = \{h \in L_\infty(G) : h \text{ are constant functions on cosets of } G_\mu\},$$

where G_μ is the smallest closed subgroup generated by the support of μ.

Hence if the semigroup (resp. subgroup) generated by $supp\mu$ is dense in G, i.e. if μ is non-degenerate or adopted, then $H_\mu = \mathbb{C}1$.

Choquet-Deny theorem remains true for some nonabilian groups, but for sure fails for non-amenable groups.

A Characterization of Amenability

Theorem: Let G be a σ-compact locally compact group. Then TFAE:

1) G is amenable;

2) There exists a probability measure μ on G such that $\mathcal{H}_\mu = \mathbb{C}1$.

Therefore, Choquet-Deny theorem fals for any apoted probability measure on non-amenable groups.

Theorem [K-N-R]: Let G be a locally compact quantum group such that $L_1(\mathbb{G})$ is separable Then TFAE:

1) G is amenable

2) There exists a quantum probability measure $\mu \in M(\mathbb{G}) = C_0(\mathbb{G})^*$ such that $\mathcal{H}_\mu = \mathbb{C}1$.

Subalgebra Question

Another question is that when $\mathcal{H}_\mu$ is a subalgebra of $L_\infty(G)$?

Theorem: Let G be a locally compact group and μ a non-degerate (or an adopted) probability measure on G. Then TFAE:

1) $\mathcal{H}_\mu$ is a subalgebra of $L_\infty(G)$;

2) $\mathcal{H}_\mu = \mathbb{C}1$.

In quantum setting, we say that $\mu \in M_u(\mathbb{G})$ is non-degenerate if for any non-zero positive $x \in C_u(\mathbb{G})$, we have $\langle x, \mu^n \rangle \neq 0$ for some $n \in \mathbb{N}$.

Theorem [K-N-R]: Let $\mathbb{G}$ be a locally compact quantum group and μ a non-degerate state in $M_u(\mathbb{G})$. Then TFAE:

1) $\mathcal{H}_\mu$ is a subalgebra of $L_\infty(\mathbb{G})$;

2) $\mathcal{H}_\mu = \mathbb{C}1$.

Dual Version of Choquet–Deny Theorem

Chu and Lau have considered the dual version of Choquet-Deny theorem. In this case, we replace

$$L_\infty(G) \text{ by } VN(G),$$

replace

$$\text{probability measures } \mu \text{ on } G \text{ by states } \varphi \text{ in } B(G),$$

where states φ in $B(G)$ are exactly positive definite functions on G with $\varphi(e) = 1$.

The theory is strikingly different from the classical one.

Theorem [Chu-Lau]: For any state $\varphi \in B(G)$,

$$G_\varphi = \{g \in G : \varphi(g) = 1\}$$

is always a closed subgroup of G and we have

$$\mathcal{H}_\varphi = \lambda(G_\varphi)''$$

which is always a von Neumann subalgebra of $VN(G)$!

In this dual form, a state $\varphi \in B(G)$ is adopted if $G_\varphi = \{e\}$.

Compact Quantum Group Case

Theorem [F-S]: Let $\mathbb{G}$ be a compact quantum group and let ϕ be an idempotent state in $M_u(\mathbb{G}) = C_u(\mathbb{G})^*$. Then ϕ induces a unital completely positive projection $\tilde{\Phi}_\phi$ on $C_u(\mathbb{G})$ and

$$\tilde{\mathcal{H}}_\phi = \{x \in C_u(\mathbb{G}) : \tilde{\Phi}_\phi(x) = x\}$$

is a C*-subalgebra of $C_u(\mathbb{G})$.

Now if μ is a state in $M_u(\mathbb{G}) = C_u(\mathbb{G})^*$, then the right multiplication map $m_\mu(f) = f \star \mu$ defines a positive definite cb-centralizler of $L_1(\mathbb{G})$. $\Phi_\mu = m_\mu^*$ is a Markov operator on $L_\infty(\mathbb{G})$.

Theorem [K-N-R]: Let $\mathbb{G}$ be a compact quantum group and let μ be a state in $M_u(\mathbb{G}) = C_u(\mathbb{G})^*$. Then the Poisson boundary $\mathcal{H}_\mu$ is a von Neumann subalgebra of $L_\infty(\mathbb{G})$!

If μ is non-degenerate, we have $\mathcal{H}_\mu = \mathbb{C}1$.

- Izumi 2002: $\mathbb{G} = \widehat{SU_q(2)}$, $\mathcal{H}_\mu = L_\infty(SU_q(2)/\mathbb{T})$.

- Neshveyev-Tuset 2006: $\mathbb{G} = \widehat{SU_q(N)}$, $\mathcal{H}_\mu = L_\infty(SU_q(N)/\mathbb{T}^{N-1})$.

- Vaes-Vander Vennet 2008: $\mathbb{G} = \widehat{A_o(F)}$.

- Vaes-Vander Vennet 2010: $\mathbb{G} = \widehat{A_u(F)}$

- K-N-R 2011: $\mathbb{G} = SU_q(2)$

Thank you for your attention !